Sharp Inequalities in Harmonic Analysis

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1 Maximizers for the Strichartz inequality

after D. Foschi [2]
A summary written by David Beltran

Abstract

We give the sharp constant and a characterisation of the maximisers for the Strichartz estimates for the homogeneous free Schrödinger equation (for dimensions $d = 1, 2$) and the homogeneous wave equation (for dimension $d = 3$). In the context of Fourier restriction estimates, we obtain the sharp constant and maximisers for the Stein-Tomas theorem in the paraboloid ($d = 1, 2$) and in the cone ($d = 2, 3$).

1.1 Introduction

Consider the homogeneous free Schrödinger equation,

$$i\partial_t u - \Delta u = 0, \qquad u(0,x) = u_0(x), \tag{1}$$

where $(t,x) \in \mathbb{R}^{1+d}$. Strichartz [3] showed that there exists a constant S such that

$$\|u\|_{L^p(\mathbb{R}^{d+1})} \leq S\|u_0\|_{L^2(\mathbb{R}^d)}, \quad p = 2 + 4/d. \tag{2}$$

For $d \geq 2$, he also established a similar kind of estimate for the solution of the homogeneous wave equation, that is, if u satisfies

$$\partial_{tt}^2 u - \Delta u = 0, \qquad u(0,x) = u_0(x), \quad \partial_t u(0,x) = u_1(x), \tag{3}$$

then there exists a constant W such that

$$\|u\|_{L^p(\mathbb{R}^{1+d})} \leq W\|(u_0,u_1)\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^d)\times\dot{H}^{-\frac{1}{2}}(\mathbb{R}^d)}, \quad p = \frac{2(d+1)}{d-1}, \tag{4}$$

where

$$\|(u_0,u_1)\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^d)\times\dot{H}^{-\frac{1}{2}}(\mathbb{R}^d)} = \left(\|u_0\|^2_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^d)} + \|u_1\|^2_{\dot{H}^{-\frac{1}{2}}(\mathbb{R}^d)}\right)^{1/2}$$

and $\dot{H}^s$ denotes the homogeneous Sobolev space, with $\|f\|_{\dot{H}^s} = \|(-\Delta)^{\frac{s}{2}} f\|_{L^2}$.

Let $S(d)$ and $W(d)$ denote the best constant in (2) and (4) respectively. The main result of the article under review is to provide the values of $S(1)$, $S(2)$ and $W(3)$ and to characterise the set of extremisers in such cases.

In the case of the Schrödinger equation we have the following.

Theorem 1. *For* $(d,p) = (1,6)$ *we have* $S(1) = 12^{-1/12}$. *For* $(d,p) = (2,4)$ *we have* $S(2) = 2^{-1/2}$. *Given the initial data* $u_0^*(x) = e^{-|x|^2}$, *let* $u^*(t,x)$ *be the corresponding solution to the Schrödinger equation. Then for both* $d = 1,2$ *the set of extremisers for* (2) *is given by the initial data of solutions to* (1) *in the orbit of* u^* *under the action of the group of symmetries for the Schrödinger equation. In particular, they are given by* $L^2(\mathbb{R}^d)$ *functions of the form*

$$u_0(x) = e^{A|x|^2 + b\cdot x + C},$$

with $A, C \in \mathbb{C}$, $b \in \mathbb{C}^d$ *and* $\Re(A) < 0$.

In the case of the wave equation we have

Theorem 2. *Let* $(d,p) = (3,4)$. *Then* $W(3) = (3/(16\pi))^{1/4}$. *Given the initial data* $u_0^*(x) = (1+|x|^2)^{-(d-1)/2}$, $u_1^*(x) = 0$, *let* u^* *be the corresponding solution to the wave equation. Then the set of extremisers for* (4) *is the set of initial data of the solutions to* (3) *in the orbit of* u^* *under the action of the group of symmetries for the wave equation.*

Remark 3. *In [2] it is also stated a similar result to Theorem 2 when* $d = 2$, *but Foschi's argument turned out to be incorrect, as remarked in [1]. However, the result remains true for the wave propagator, and thus for Stein-Tomas Fourier restriction estimate in the cone - see the forthcoming section.*

1.2 Connection to Fourier restriction theorems

The Strichartz estimates for the Schrödinger and wave equation are closely related to Fourier restriction estimates. In the case of the Schrödinger equation, the solution u is given by

$$u(t,x) = e^{-it\Delta}u_0(x) = \int_{\mathbb{R}^d} \widehat{u_0}(\xi) e^{i(t|\xi|^2 + x\cdot\xi)} d\xi,$$

where $\widehat{\ }$ denotes the spatial Fourier transform [1]. Taking space-time – $(d+1)$-dimensional – Fourier transform

$$\widetilde{u}(\tau,\xi) = 2\pi \widehat{u_0}(\xi)\delta(\tau - |\xi|^2),$$

[1]The definition we take of the Fourier transform along this review is $\widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x)e^{-ix\cdot\xi}dx$. With this normalisation $\widehat{fg}(\xi) = (2\pi)^{-d}\widehat{f} * \widehat{g}(\xi)$ and $\|\widehat{f}\|_{L^2(\mathbb{R}^d)} = (2\pi)^{d/2}\|f\|_{L^2(\mathbb{R}^d)}$.

so $\widetilde{u}$ is supported in the paraboloid $\{(\tau,\xi)\in\mathbb{R}^{1+d}:\tau=|\xi|^2\}$. Thus we may write u as $\widetilde{vd\sigma}$, where $d\sigma$ is the induced Lebesgue measure on the paraboloid and v is the lift of $\widehat{u_0}$ onto the paraboloid. The Strichartz estimate (2) is then equivalent to the Fourier restriction estimate

$$\|\widetilde{vd\sigma}\|_{L^{\frac{2(d+2)}{d}}(\mathbb{R}^{d+1})}\leq(2\pi)^{-d/2}S\|v\|_{L^2(d\sigma)},$$

which is the Stein-Tomas [4] Fourier restriction estimate for the paraboloid[2]. Hence, Theorem 1 gives sharp constant and extremisers for Stein-Tomas in the paraboloid when $d=1,2$.

Similarly, one may link the solution to the wave equation with the restriction of the Fourier transform to a cone. A solution u to (3) satisfies

$$\widehat{u}(t,\xi)=\widehat{u_0}(\xi)\cos(t|\xi|)+\frac{\widehat{u_1}(\xi)}{|\xi|}\sin(t|\xi|),$$

and it may be written as $u=u_++u_-$, with

$$u_\pm(t)=e^{\pm it\sqrt{-\Delta}}(\sqrt{-\Delta})^{-\frac{1}{2}}f_\pm,\qquad f_\pm=\frac{1}{2}\Big((\sqrt{-\Delta})^{\frac{1}{2}}u_0\mp i(\sqrt{-\Delta})^{-\frac{1}{2}}u_1\Big).$$

Here

$$e^{it\sqrt{-\Delta}}f(x)=\int_{\mathbb{R}^d}\widehat{f}(\xi)e^{i(t|\xi|+x\cdot\xi)}d\xi,$$

which is typically referred to as the one-sided wave propagator. Taking space-time Fourier transform,

$$\widetilde{u_\pm}(\tau,\xi)=2\pi|\xi|^{-\frac{1}{2}}\delta(\tau\mp|\xi|)\widehat{f_\pm}(\xi)$$

and one may write $u_+(-t,-x)=\widetilde{vd\sigma}(t,x)$, where $d\sigma$ is the induced Lebesgue measure in the cone and $v(|\xi|,\xi)|\xi|^{-\frac{1}{2}}=\widehat{f_+}(\xi)$; similarly for u_-. Thus, the estimate

$$\|u_+\|_{L^{\frac{2(d+1)}{d-1}}(\mathbb{R}^{d+1})}\leq C\|f_+\|_{L^2(\mathbb{R}^d)}\tag{5}$$

is equivalent to

$$\|\widetilde{vd\sigma}\|_{L^{\frac{2(d+1)}{d-1}}(\mathbb{R}^{d+1})}\leq(2\pi)^{-d/2}C\|v\|_{L^2(d\sigma)},$$

which is the Stein-Tomas Fourier restriction estimate in the cone. Sharp constant and characterisation of extremisers for the estimates (5) in the case $d=2,3$ are established in the article under review, so one obtains the analogous result for Stein-Tomas in the cone.

[2] Of course the constant in the restriction estimate depends on the normalisation of the Fourier transform.

1.3 Scheme of the proof

We first sketch the proof of Theorem 1 and the estimates (5), which all follow the same structure. Observe that by Plancherel's theorem,

$$\|u\|_{L^p(\mathbb{R}^{d+1})}^{p/2} = \|u^{p/2}\|_{L^2(\mathbb{R}^{d+1})} = (2\pi)^{-(d+1)/2}\|\widetilde{u^{p/2}}\|_{L^2(\mathbb{R}^{d+1})},$$

and if p is an even integer, we may write $\widetilde{u^{p/2}}$ as the convolution of $\widetilde{u}$ with itself $p/2$ times.

The method presented strongly relies on the fact p is an even integer, so it may only provide results for $d = 1, 2$ in the case of the Schrödinger equation (recall that $p = 2 + 4/d$) and for $d = 2, 3$ in the case of the wave propagator (here $p = 2(d+1)/(d-1)$).

Consider the Schrödinger case for $d = 1$. Observe that

$$\widetilde{u^3}(\tau, \xi) = \frac{1}{2\pi}\int_{\mathbb{R}\times\mathbb{R}\times\mathbb{R}} \widehat{u_0}(\eta_1)\widehat{u_0}(\eta_2)\widehat{u_0}(\eta_3)\delta\begin{pmatrix}\tau - \eta_1^2 - \eta_2^2 - \eta_3^2\\ \xi - \eta_1 - \eta_2 - \eta_3\end{pmatrix}d\eta.$$

Then $\widetilde{u^3}$ is supported in the closure of the region

$$\mathcal{P}_1 = \{(\tau, \xi) \in \mathbb{R}\times\mathbb{R} : 3\tau > \xi^2\}.$$

For each $(\tau, \xi) \in \mathcal{P}_1$, we define $\langle\cdot,\cdot\rangle_{(\tau,\xi)}$ to be the L^2 inner product associated with the measure

$$\mu_{(\tau,\xi)} = \delta\begin{pmatrix}\tau - \eta_1^2 - \eta_2^2 - \eta_3^2\\ \xi - \eta_1 - \eta_2 - \eta_3\end{pmatrix}d\eta,$$

and let $\|\cdot\|_{(\tau,\xi)}$ be the associated norm. Then one may write

$$\widetilde{u^3}(\tau, \xi) = \frac{1}{2\pi}\langle\widehat{u_0}\otimes\widehat{u_0}\otimes\widehat{u_0}, 1\otimes 1\otimes 1\rangle_{(\tau,\xi)}.$$

The Cauchy-Schwarz's inequality gives

$$|\widetilde{u^3}(\tau, \xi)| \leq \frac{1}{2\pi}\|\widehat{u_0}\otimes\widehat{u_0}\otimes\widehat{u_0}\|_{(\tau,\xi)}\|1\otimes 1\otimes 1\|_{(\tau,\xi)}.$$

For each $(\tau, \xi) \in \mathcal{P}_1$, $\|1\otimes 1\otimes 1\|_{(\tau,\xi)} = (\pi/\sqrt{3})^{1/2}$. Also,

$$\begin{aligned}\int_{\mathcal{P}_1}\|\widehat{u_0}\otimes\widehat{u_0}\otimes\widehat{u_0}\|_{(\tau,\xi)}^2 d\tau d\xi &= \int_{\mathbb{R}^3}|\prod_{j=1}^{3}\widehat{u_0}(\eta_j)|^2\int_{\mathcal{P}_1}\mu_{(\tau,\xi)}d\tau d\xi\\ &= \|\widehat{u_0}\otimes\widehat{u_0}\otimes\widehat{u_0}\|_{L^2(\mathbb{R}^3)}^2 = (2\pi)^3\|u_0\|_{L^2(\mathbb{R})}^6.\end{aligned}$$

So, putting all together,

$$\|u\|_{L^6(\mathbb{R}^2)} \le 12^{-1/12}\|u_0\|_{L^2(\mathbb{R})},$$

with equality if there is equality in the application of the Cauchy-Schwarz inequality. That is, if there exists a scalar function $F : \mathcal{P}_1 \to \mathbb{C}$ such that

$$(\widehat{u_0} \otimes \widehat{u_0} \otimes \widehat{u_0})(\eta) = F(\tau,\xi)(1 \otimes 1 \otimes 1)(\eta)$$

for almost all η in the support of the measure $\mu_{(\tau,\xi)}$. Thus, we need functions u_0 and F such that

$$\widehat{u_0}(\eta_1)\widehat{u_0}(\eta_2)\widehat{u_0}(\eta_3) = F(\eta_1^2 + \eta_2^2 + \eta_3^2, \eta_1 + \eta_2 + \eta_3). \tag{6}$$

Examples of functions satisfying the above equality are given by $\widehat{u_0}(\xi) = e^{-\xi^2}$ and $F(\tau,\xi) = e^{-\tau}$.

The proofs for the Schrödinger case when $d = 2$ and the wave propagator when $d = 2, 3$ follow the same pattern; the major changes are

- Schrödinger ($d = 2$). Measure $\mu_{(\tau,\xi)} = \delta\binom{\tau-|\eta|^2-|\zeta|^2}{\xi-\eta-\zeta} d\eta d\zeta$, $\|1 \otimes 1\|_{(\tau,\xi)} = \sqrt{\pi/2}$, and equality if u_0 and F are such that

$$\widehat{u_0}(\eta)\widehat{u_0}(\zeta) = F(|\eta|^2 + |\zeta|^2, \eta + \zeta). \tag{7}$$

- Wave propagator ($d = 2$). Measure $\mu_{(\tau,\xi)} = \delta\binom{\tau-|\eta_1|-|\eta_2|-|\eta_3|}{\xi-\eta_1-\eta_2-\eta_3} d\eta_1 d\eta_2 d\eta_3$, $\||\cdot|^{-\frac{1}{2}} \otimes |\cdot|^{-\frac{1}{2}} \otimes |\cdot|^{-\frac{1}{2}}\|_{(\tau,\xi)} = 2\pi$, and equality if f_+ and F are such that

$$|\eta_1|^{\frac{1}{2}}\widehat{f_+}(\eta_1)|\eta_2|^{\frac{1}{2}}\widehat{f_+}(\eta_2)|\eta_3|^{\frac{1}{2}}\widehat{f_+}(\eta_3) = F(|\eta_1|+|\eta_2|+|\eta_3|, \eta_1+\eta_2+\eta_3). \tag{8}$$

- Wave propagator ($d = 3$). Measure $\mu_{(\tau,\xi)} = \delta\binom{\tau-|\eta|-|\zeta|}{\xi-\eta-\zeta} d\eta d\zeta$, $\||\cdot|^{-\frac{1}{2}} \otimes |\cdot|^{-\frac{1}{2}}\|_{(\tau,\xi)} = \sqrt{2\pi}$, and equality if f_+ and F are such that

$$|\eta|^{\frac{1}{2}}\widehat{f_+}(\eta)|\zeta|^{\frac{1}{2}}\widehat{f_+}(\zeta) = F(|\eta| + |\zeta|, \eta + \zeta). \tag{9}$$

A characterisation of the functions u_0, f_+ satisfying the functional equations (6), (7), (8) and (9) provide a characterisation for the extremisers for (2), $d = 1, 2$, and (5), $d = 2, 3$, respectively. Thus, the characterisation part reduces to study such functional equations.

1.3.1 Case of general wave equation

When $d = 3$, one may deduce Theorem 2 from the results previously discussed for the wave propagator. Observe that

$$\begin{aligned}\|u\|^4_{L^4(\mathbb{R}^4)} = \|u_+^2\|^2_{L^2(\mathbb{R}^4)} + \|u_-^2\|^2_{L^2(\mathbb{R}^4)} + 4\|u_+u_-\|^2_{L^2(\mathbb{R}^4)} + 2\Re\langle u_+^2, u_-^2\rangle \\ + 4\Re\langle u_+^2, u_+u_-\rangle + 4\Re\langle u_-^2, u_+u_-\rangle.\end{aligned}$$

The three last terms vanish as u_+^2, u_-^2 and u_+u_- have disjoint Fourier supports. The extremisers for the terms $\|u_\pm^2\|_2^2$ are also extremisers for the estimate $\|u_+u_-\|_2^2 \leq (2\pi)^{-1/2}\|f_+\|_2\|f_-\|_2$, which together with other elementary identities allow one to recover the right hand side of (4).

A similar argument is used in [2] to deduce an analogue for Theorem 2 in the case $d = 2$. As mentioned in the Introduction, such argument turned out to be incorrect (see [1], page 9). In this case,

$$\begin{aligned}\|u\|^6_{L^6(\mathbb{R}^3)} = \|u_+^3\|^2_{L^2(\mathbb{R}^3)} + \|u_-^3\|^2_{L^2(\mathbb{R}^3)} + 9\|u_+^2u_-\|^2_{L^2(\mathbb{R}^3)} + 9\|u_+u_-^2\|^2_{L^2(\mathbb{R}^3)} \\ + 6\Re\langle u_+^3, u_+^2u_-\rangle + 6\Re\langle u_+u_-^2, u_-^3\rangle + 18\Re\langle u_+^2u_-, u_+u_-^2\rangle \\ + 6\Re\langle u_+^3, u_+u_-^2\rangle + 2\Re\langle u_+^3, u_-^3\rangle + 6\Re\langle u_+^2u_-, u_-^3\rangle,\end{aligned}$$

but the extremisers for the estimates on the wave propagators $u_\pm$ are not extremisers for the desired estimates for other terms like $\langle u_+^3, u_+^2u_-\rangle$.

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2 A sharpened Hausdorff-Young inequality

after M. Christ [3]
A summary written by Amalia Culiuc

Abstract

We provide a sharper version of the upper bound in the Hausdorff-Young inequality and an improved estimate for functions that are close to extremizers.

2.1 Introduction

For functions $f : \mathbb{R}^d \to \mathbb{C}$ with the appropriate boundedness, let $\widehat{f}$ represent the Fourier transform ,

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i x\cdot\xi} f(x)dx.$$

Given the normalization, this operator is unitary on $L^2(\mathbb{R}^d)$, and is a contraction from L^1 to L^∞.

The Hausdorff-Young inequality in $\mathbb{R}^d$ states that if $p \in [1, 2]$ and $q = \frac{p}{p-1}$ is the conjugate exponent to p, then

$$\|\widehat{f}\|_{L^q} \leq (\mathbb{A}_p)^d \|f\|_{L^p}, \tag{1}$$

with optimal constant $\mathbb{A}_p = p^{1/2p} q^{-1/2p} < 1$. This result was proved for q being an even integer greater than 4 by Babenko [1] and for general exponents by Beckner [2]. All Gaussian functions $G(x) = ce^{Q(x)+x\cdot v}$ where Q is a negative definite real quadratic form, $c \in \mathbb{C}$ and $v \in \mathbb{C}^d$ are extremizers of inequality (1). Furthermore, it was proved by Lieb in [4] that all extremizers are Gaussians.

The current paper establishes a sharper version of the inequality above. In particular, the main result of this paper describes the dependence of the upper bound for $f \in L^p$ on the distance of f from the extremizer set, thus providing a stabler form of uniqueness for the extremizers.

2.2 Main results

Let $\mathfrak{G}$ represent the set of all Gaussians. For $f \in L^p(\mathbb{R}^d)$, define the distance from f to $\mathfrak{G}$ as

$$\operatorname{dist}_p(f, \mathfrak{G}) := \inf_{G \in \mathfrak{G}} \| |f - G\|_{L^p}.$$

With this notation, the main result of the paper is:

Theorem 1. *There exists $c > 0$ such that for every non-zero real-valued function $f \in L^p(\mathbb{R}^d)$,*

$$\|\widehat{f}\|_{L^q} \leq (\mathbb{A}_p)^d \|f\|_{L^p} - c\|f\|_{L^p}^{-1} \operatorname{dist}_p(f, \mathfrak{G})^2. \tag{2}$$

Further refinements can be formulated for functions f that are very close to Gaussians. In particular,

Theorem 2. *If $\operatorname{dist}_p(f, \mathfrak{G})/\|f\|_{L^p}$ is sufficiently small, then*

$$\begin{aligned}\|\widehat{f}\|_{L^q} \leq (\mathbb{A}_p)^d \|f\|_{L^p} &- \mathbb{B}_{p,d} \|f\|_{L^p}^{-1} \operatorname{dist}_p(f, \mathfrak{G})^2 \\ &+ o(\|f\|_{L^p}^{-1} \operatorname{dist}_p(f, \mathfrak{G})^2)\|f\|_{L^p},\end{aligned}$$

where $\mathbb{B}_{p,d} = \frac{1}{2}(p-1)(2-p)\mathbb{A}_p^d$.

The constant $\mathbb{B}_{p,d}$ is not optimal, but a restatement of the theorem with a different definition for the distance can produce an optimal constant in implicit form.

The main step in the proof of theorems 1 and 2 is the following non-quantitative result:

Proposition 3. *For every $\varepsilon > 0$ there exists $\delta > 0$ such that if*

$$\|\widehat{f}\|_{L^q} \geq (1 - \delta)(\mathbb{A}_p)^d \|f\|_{L^p},$$

then $\operatorname{dist}_p(f, \mathfrak{G}) \leq \varepsilon \|f\|_{L^p}$.

Proposition 3 is a compactness theorem. Consider a sequence of functions f_n such that $\|f_n\|_{L^p} = 1$ and $\|\widehat{f_n}\|_{L^q} \to (\mathbb{A}_p)^d$. Proposition 3 states that an appropriately renormalized subsequence $(f_{n_i}^*)$ (where the renormalization is performed using an element of the group of symmetries of the inequality) converges in $L^p(\mathbb{R}^d)$.

Assuming Proposition 3 holds, by the theorem of Lieb [4], it must be true that near-extremizers of the inequality are close to Gaussians. Therefore one can consider the linear functional Φ mapping f into $\|\widehat{f}\|_{L^q}/\|f\|_{L^p}$ and compute its Taylor expansion about an element of $\mathfrak{G}$. A slight difficulty arises by the fact that the functional in question is not twice differentiable, as its denominator $\|f\|_{L^p}$ is not C^2 if $p < 2$. To resolve it, we establish the following general lemma:

Lemma 4. *Let $1 < p < 2 < q < \infty$ and let $T : L^p \to L^q$ be a bounded linear operator. If $0 \neq G \in \mathfrak{G}$, then for any function f sufficiently small in norm and orthogonal to the tanget space of $\mathfrak{G}$ at G, there exists a decomposition*

$$f = f_\sharp + f_\flat$$

such that $f_\sharp$ and $f_\flat$ are disjointly supported, and

$$\Phi(G + f) \leq \|T\| + \mathcal{Q}(f_\sharp) - c_\varepsilon \left(\|f_\flat\|_{L^p}\right)^p + \varepsilon\|f\|_{L^p}^2,$$

where $\mathcal{Q}$ is the (formal) second variation of Φ.

By Lemma 4, the functional Φ can in fact be treated as twice continuously differentiable. Therefore, to prove the main result, the final step is to analyze the term $\mathcal{Q}$ about a Gaussian. This analysis leads to an eigenvalue problem for a specific self-adjoint compact linear operator in L^2. Computing the spectrum and eigenfunctions of this operator gives us theorems 1 and 2, as well as a further refinement of theorem 2 with a sharp constant.

2.3 Proof of Proposition 3

As stated before, although it is a non-quantitative result, Proposition 3 is the key step in the proof of the main results. We present a summary of some of the steps involved in the argument for $d = 1$.

The first step of the proof makes use of a connection with Young's convolution inequality, which states that

$$\|f * g\|_{L^r} \leq c\|f\|_{L^s}\|g\|_{L^s},$$

whenever $1 \leq r, s \leq \infty$ and $\frac{2}{s} = \frac{1}{r} + 1$.

Let $T : L^p \to L^q$ be a bounded linear operator and consider an inequality $\|Tf\|_{L^q} \leq \|T\|\|f\|_{L^p}$. A quasi-extremizer for this inequality is, by definition, a function f such that $\|Tf\|_{L^q} \geq \delta\|f\|_{L^p}$, where δ may be arbitrarily small. One can show that for any $\delta > 0$, if f is a quasi-extremizer for the Fourier transform,then a power of f is a quasi-extremizer for Young's convolution inequality. More precisely, if $\|\widehat{f}\|_{L^q} \geq \eta \|f\|_{L^p}$, then $\||f|^\gamma * |f|^\gamma\|_{L^r} \geq c\eta^2 \|f^\gamma\|_{L^s}^2$ for suitable constants γ, r, s depending on p. This observation implies that to extract information about the Hausdorff-Young inequality, we can study the quasi-extremizers of the convolution inequality instead.

We introduce the concept of multiprogressions, defined below:

Definition 5. *A discrete multiprogression P of rank r is a function*

$$P : \prod_{i=1}^{r} \{0, 1, ...N_i - 1\} \to \mathbb{R}^d,$$

$$P(n_1, n_2, ...n_r) = \left\{ a + \sum_{i=1}^{r} n_i v_i : 0 \leq n_i < N_i \right\},$$

where $a \in \mathbb{R}$ and $N_1, ...N_r$ are positive integers.

If the mapping above is injective, P is said to be proper.

Definition 6. *If Q^d represents the unit cube in $\mathbb{R}^d$, a continuum multiprogression P of rank r is a function*

$$P : \prod_{i=1}^{r} \{0, 1, ...N_i - 1\} \times Q^d \to \mathbb{R}^d,$$

$$P(n_1, n_2, ...n_r; y) = \left\{ a + \sum_{i=1}^{r} n_i v_i + sy \right\},$$

where $a, v_i \in \mathbb{R}^d$ and $s \in \mathbb{R}_+$.

Given these definitions, the next step of the proof is characterizing quasi-extremizers for Young's inequality. Suppose $\|f * f\|_{L^r} \geq \delta \|f\|_{L^p}^2$. Then we can show that there exists a decomposition $f = g + h$ and a continuum multiprogression P with the property that $\|h\|_{L^p} < (1 - c\delta^\gamma)\|f\|_{L^p}$, g is supported on P, $\|g\|_{L^\infty} \asymp_\delta |P|^{-1/p}$ when $g(x) \neq 0$, and the rank of P is controlled by C_δ.

The relationship with multiprogressions leads to a connection to results in additive combinatorics. By applying continuum analogues of Freĭman's little theorem and the result of Balog-Szemerédy to the previous step, we obtain the proposition below:

Proposition 7. *Let f satsify $\|\widehat{f}\|_{L^q} \geq (1-\delta)(\mathbb{A}_p)^d \|f\|_{L^p}$. Let $\varepsilon > 0$. Then for sufficiently small δ, there exists a decomposition $f = g + h$ with disjoint support and a continuum multiprogression P such that*

- $\|h\|_{L^p} \leq \varepsilon \|f\|_{L^p}$
- $\mathrm{supp}(g) = P$
- $\|g\|_{L^\infty} |P|^{1/p} \leq C(\varepsilon)\|f\|_{L^p}$
- $\mathrm{rank}(P) \leq C(\varepsilon)$.

The next (and decisive) step in the proof of proposition 3 is to replace P by a convex set. Once this argument is made, we prove that $\widehat{f}$ is also nearly concentrated on a convex set. It will follow that if a sequence of functions f_n satisfies $\|f_n\|_{L^p} = 1$ and $\|\widehat{f_n}\|_{L^q} \to \mathbb{A}_p$, then, through a renormalization of each f_n to F_n by the action of an element in the group of symmetries of the innequalities, the sequence $\widehat{F_n}$ is precompact in L^q. In general, precompactness of $\widehat{F_n}$ in L^q does not imply precompactness of F_n in L^p. However, we can show that the implication holds for extremizing sequences. Therefore, this final observation allows us to complete the proof of proposition 3, by showing that f_n is precompact in L^p.

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3 A note on the Sobolev inequality

after Gabriele Bianchi and Henrik Egnell [1]
A summary written by Alexis Drouot.

Abstract

Let $d \geq 3$ and $p = 2d/(d-2)$. The celebrated Sobolev inequality asserts that for an optimal constant S and every $f \in \dot{H}^1(\mathbb{R}^d)$, $|\nabla f|_2^2 - S^2|f|_p^2 \geq 0$. The set of functions $\mathcal{M}$ such that the equality holds is explicitely known. Here we address the question of the stability of the inequality: if $|f|_p$ is relatively large compared to $|\nabla f|_2$, how close from $\mathcal{M}$ does f needs to be? It is shown that if $d(f,\mathcal{M})$ is the distance from f to $\mathcal{M}$ in $\dot{H}^1(\mathbb{R}^d)$ then

$$|\nabla f|_2^2 - S^2|f|_p^2 \geq \alpha d(f,\mathcal{M})^2$$

for some positive constant α.

3.1 Introduction

In this note we are concerned with the most famous form of the Sobolev inequalities,

$$\forall f \in \dot{H}^1(\mathbb{R}^d), \quad S|f|_p \leq |\nabla f|_2.$$

Here $\dot{H}^1(\mathbb{R}^d)$ is the completion of $C_0^\infty(\mathbb{R}^d)$ with respect to the norm $|\nabla \cdot|_2$ and $|\cdot|_q$ stands for the norm on the Hölder space $L^q(\mathbb{R}^d)$. We recall its sharp form in the following:

Theorem 1. *[7][5] For every $f \in \dot{H}^1(\mathbb{R}^d)$,*

$$|\nabla f|_2^2 - S^2|f|_p^2 \geq 0, \qquad S^2 = \frac{d(d-2)}{4}|\mathbb{S}^d|^{2/d}. \tag{1}$$

Moreover the equality holds if and only if f belongs to the set

$$\mathcal{M} = \{x \mapsto c\left(a + b|x-x_0|^2\right)^{1-d/2}, \ a,b > 0, \ x_0 \in \mathbb{R}^d, \ c \in \mathbb{R}\}.$$

The functions that belong to $\mathcal{M}$ are called maximizers. In this note we explain the following refinement:

Theorem 2. *[3] [1] There exists $\alpha > 0$ such that for every $f \in \dot{H}^1(\mathbb{R}^d)$,*

$$|\nabla f|_2^2 - S^2|f|_p^2 \geq \alpha d(f,\mathcal{M})^2, \quad d(f,\mathcal{M}) = \inf_{h\in\mathcal{M}} |f-h|_{\dot{H}^1(\mathbb{R}^d)}.$$

3.2 Symmetries of the inequality

One of the interesting features of (1) is its invariance under a large group of symmetries. Let $\mathcal{G}$ be the group of affine maps on $\mathbb{R}^d$ generated by the rotations ($x \mapsto \Omega x$, Ω orthogonal matrix), the translations ($s \mapsto x + a$, $a \in \mathbb{R}^d$) and the dilations ($x \mapsto \lambda x$, $\lambda \neq 0$). This group acts by isometries on the Holdër space $L^p(\mathbb{R}^d)$, through

$$L \star f = \frac{1}{|\det L|^{1/p}} f \circ L.$$

This action preserves the inequality (1): for every $f \in L^p(\mathbb{R}^d)$ we have

$$|\nabla f|_2^2 - S^2 |f|_p^2 = |\nabla (L \star f)|_2^2 - S^2 |L \star f|_p^2.$$

It follows that $\mathcal{M}$ is the orbit of

$$\mathcal{M}_0 = \left\{ x \mapsto c \left(1 + |x|^2\right)^{1-n/2},\ c \in \mathbb{R} \right\}$$

through the action of $\mathcal{G}$. Maximizers are said to be unique modulo the set of symmetries.

The inequality (1) admits another symmetry of exceptional importance. Let π be the inverse stereographic projection on the d-dimensional sphere $\mathbb{S}^d$

$$\begin{array}{rcccc} \pi & : & \mathbb{R}^d & \to & \mathbb{S}^d \\ & & x & \mapsto & \left(\dfrac{2x}{1+|x|^2}, \dfrac{1-|x|^2}{1+|x|^2} \right). \end{array}$$

It is a conformal transformation (when the plane and the sphere are provided with their usual metric) and it induces an isometry $\mathcal{P}$ from $L^p(\mathbb{S}^d)$ to $L^p(\mathbb{R}^d)$ given by

$$\mathcal{P}F(x) = \left(\frac{2}{1+|x|^2} \right)^{n/2-1} F(\pi(x)), \quad F \in L^p(\mathbb{S}^d).$$

The exceptional feature of this map is the identity

$$|\nabla \mathcal{P} F|_2^2 = |\nabla F|_{L^2(\mathbb{S}^d)}^2 + \frac{d(d-2)}{4} |F|_{L^2(\mathbb{S}^d)}^2 =: |F|_{H^1(\mathbb{S}^d)}^2.$$

Hence,

$$I[F] := |F|_{H^1(\mathbb{S}^d)}^2 - S^2 |F|_{L^p(\mathbb{S}^d)}^2 \geq 0.$$

This is the Sobolev inequality on the sphere. The equality is realised if and only if U belongs to the set

$$\mathcal{N} = \mathcal{P}^{-1}\mathcal{M} = \left\{\omega \mapsto c(1 - \langle \xi, \omega\rangle)^{-d/2+1}, \quad |\xi| < 1,\ c \in \mathbb{R}\right\}.$$

Note that in particular the constant function 1 is an extremizer.

3.3 Local version of theorem 2

We now perform a local study of $I[F]$ for F near a non-zero element $H \in \mathcal{N}$. For that we note that $\mathcal{N} \setminus \{0\}$ is a smooth $n+2$ dimensional manifold and we define $T_H\mathcal{N}$ the tangent space of $\mathcal{N} \setminus \{0\}$ at H. Since $\mathcal{N} \setminus \{0\} \subset H^1(\mathbb{S}^d)$ it has a natural identification with a subspace of $H^1(\mathbb{S}^d)$. We define then the normal space of $\mathcal{N}$ at H as

$$(T_H\mathcal{N})^\perp = \{V \in H^1(\mathbb{S}^d),\ \forall U \in T_H\mathcal{N},\ \langle U, V\rangle_{H^1(\mathbb{S}^d)} = 0\}.$$

The following lemma is the key for theorem 2.

Lemma 3. *Let $A > 0$. For $H \in \mathcal{N} \setminus \{0\}, V \in (T_H\mathcal{N})^\perp$ with $|H|_{H^1(\mathbb{S}^d)} \leq A$, $|V|_{L^p(\mathbb{S}^d)} \leq A$ we have*

$$I[H + tV] \geq t^2 \frac{4}{d+6}\left(|V|^2_{H^1(\mathbb{S}^d)} + o(1)\right)$$

uniformly as t goes to 0.

Proof. Since the maximizers are unique modulo the set of symmetries we can assume without loss of generality that H and is constant and even $H = 1$. Let $V \in (T_1\mathcal{N})^\perp$. Since $1 \in T_1\mathcal{N}$ we have $V \perp 1$ which implies $\int_{\mathbb{S}^d} V = 0$. This yields

$$|\nabla(1 + tV)|^2_{H^1(\mathbb{S}^d)} = t^2|\nabla V|^2_{L^2(\mathbb{S}^d)},$$

$$|1 + tV|^2_{L^2(\mathbb{S}^d)} = |\mathbb{S}^d| + t^2|V|^2_{L^2(\mathbb{S}^d)},$$

$$\begin{aligned}|1 + tV|^2_{L^p(\mathbb{S}^d)} &= \left(|\mathbb{S}^d| + t^2\frac{p(p-1)}{2}|V|^2_{L^2(\mathbb{S}^d)}\right)^{2/p} + o(t^2)\\ &= |\mathbb{S}^d|^{2/p} + t^2(p-1)|\mathbb{S}^d|^{2/p-1}|V|^2_{L^2(\mathbb{S}^d)} + o(t^2),\end{aligned}$$

and this holds uniformly as $t \to 0$ as long as $|V|_{L^p(\mathcal{S}^d)}$ is bounded. Therefore we obtain

$$\begin{aligned}
I[1+tV] &= t^2|\nabla V|^2_{L^2(\mathbb{S}^d)} + t^2\frac{d(d-2)}{4}|V|^2_{L^2(\mathbb{S}^d)} - t^2(p-1)S^2|\mathbb{S}^d|^{2/p-1}|V|^2_{L^2(\mathbb{S}^d)} + o(t^2) \\
&= t^2|\nabla V|^2_{L^2(\mathbb{S}^d)} + t^2\frac{d(d-2)}{4}|V|^2_{L^2(\mathbb{S}^d)} - t^2\frac{n(n+2)}{4}|V|^2_{L^2(\mathbb{S}^d)} + o(t^2) \\
&= t^2|\nabla V|^2_{L^2(\mathbb{S}^d)} - dt^2|V|^2_{L^2(\mathbb{S}^d)} + o(t^2).
\end{aligned}$$

Using the explicit characterization of $\mathcal{N}$, the space $(T_1\mathcal{N})^\perp$ is generated by the constant function and the first order spherical harmonics. Thus by the minmax characterization of eigenvalues, every V in $(T_1\mathcal{N})^\perp$ satisfies the inequality

$$|\nabla V|^2_{L^2(\mathbb{S}^d)} \geq \lambda_3|V|^2_{L^2(\mathbb{S}^d)}.$$

where $\lambda_3 = 2(2+d-2) = 2d$ is the third eigenvalue of the Laplacian on the sphere. It follows that for all $\theta \in [0,1]$,

$$I[1+tV] \geq t^2\left(\frac{\theta}{2}|\nabla V|^2_{L^2(\mathbb{S}^d)} + d(1-\theta)|V|^2_{L^2(\mathbb{S}^d)} + o(t^2)\right).$$

Take $\theta = 8/(d+6)$ to conclude:

$$I[1+tV] \geq t^2\frac{4}{d+6}\left(|V|^2_{H^1(\mathbb{S}^d)} + o(1)\right).$$

This ends the proof. $\square$

Remark. The proof of this lemma contains many geometric aspects. For every $H \in \mathcal{N}$, $I[H] = 0$. Consequently if $\alpha > 0$ the inequality

$$I[H+tV] \geq t^2\alpha\left(|V|^2_{H^1(\mathbb{S}^d)} + o(1)\right)$$

cannot hold for $V \in T_H\mathcal{N}$. This is why we must restrict our attention to the normal bundle of $\mathcal{N} \setminus \{0\}$. At first it might seem surprising that the spectral study that is required to complete the proof can be realized explicitly. However there is a very simple explanation of this fact. Since I is invariant through the action of rotation on the sphere, so is its Hessian. Therefore it induces a selfadjoint operator L on $H^1(\mathbb{S}^d)$ that is invariant through the action of rotations of the sphere. Since the Laplacian operator can be seen formally as a combination of infinitesimal rotations, L must commute with the Laplacian. We can then apply what is maybe the most fundamental principle of harmonic analysis: if two selfadjoint operators commute then we can diagonalize them in the same basis. The basis of spherical harmonics diagonalizes the operator L and thus one can perform an explicit spectral study on the Hessian of I.

3.4 Proof of theorem 2 and comments.

Let us now prove theorem 2. Assume that it does not hold. Then there exists a sequence of functions $f_n \in \dot{H}^1(\mathbb{R}^d)$ such that

$$\frac{1}{n} d(f_n, \mathcal{M})^2 \geq |\nabla f_n|_2^2 - S^2 |f_n|_p^2 \geq 0.$$

We can assume without loss of generality that $|\nabla f_n|_2 = 1$. Consequently, $d(f_n, \mathcal{M}) \leq |\nabla(f_n - 0)|_2 = 1$ and therefore

$$|\nabla f_n|_2^2 - S^2 |f_n|_p^2 \to 0, \quad n \to \infty.$$

Such functions are called extremizing sequences. It is then known (see [6]) that there exists $L_n \in \mathcal{G}$ such that $L_n \star f_n$ converges to a (non-zero) element of $\mathcal{M}$. It yields

$$d(f_n, \mathcal{M}) = d(L_n \star f_n, \mathcal{M}) \to 0.$$

Let $F_n = \mathcal{P}^{-1} L_n \star f_n$. Because of the conformal invariance mentioned above F_n satisfies the same properties as f_n, that is

$$\frac{1}{n} d(F_n, \mathcal{N})^2 \geq I[F_n] \geq 0 \quad \text{and} \quad d(F_n, \mathcal{N}) \to 0.$$

Here $d(F_n, \mathcal{N})$ is the distance from F_n to $\mathcal{N}$ measured in $H^1(\mathbb{S}^d)$. Write $F_n = H_n + d(F_n, \mathcal{N}) V_n$ where $H_n \in \mathcal{N}$ is non-zero (at least for n large enough), $V_n \in (T_{H_n}\mathcal{N})^\perp$ and $|V_n|_{H^1(\mathbb{S}^d)} = 1$. Then both $|H_n|_{H^1(\mathbb{S}^d)}$ and $|V_n|_{L^p(\mathbb{S}^d)}$ must be bounded as $n \to \infty$. Moreover as $d(F_n, \mathcal{N}) \to 0$,

$$\frac{1}{n} d(F_n, \mathcal{N})^2 \geq I[F_n] \geq \frac{4}{d+6} d(F_n, \mathcal{N})^2 \left(1 + o(1)\right).$$

Taking $n \to +\infty$ we obtain

$$0 \geq \frac{4}{d+6}$$

which is a contradiction. This ends the proof of theorem 2.

We end up this note by a few comments. Conformal invariance has proved to be very useful to give sharper form of inequalities since the seminal work of Lieb [5]. The proof given here for the Sobolev inequalities can be found in substance in [1] although its more modern formulation using the conformal invariance of the inequality goes back to [3]. The method applies for many inequalities, including some without a Hilbertian framework and a larger group of symmetries, see for instance [4]. It was also applied to treat the Hausdorff-Young inequality in [2], which is not known to be conformally invariant.

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4 Gaussian kernels have only Gaussian maximizers

after E. H. Lieb [3]
A summary written by Polona Durcik

Abstract

A Gaussian integral kernel $G(x, y)$ on $\mathbb{R}^n \times \mathbb{R}^n$ is the exponential of a quadratic form in x and y. The examined paper addresses the question of finding the sharp bound of G as an operator from L^p to L^q and showing that the functions which satisfy the bound are necessarily Gaussians. This is achieved generally for $1 < p \leq q < \infty$ and also for $p > q$ in certain cases.

4.1 Introduction

A *Gaussian kernel* on $\mathbb{R}^n \times \mathbb{R}^n$ is

$$G(x, y) = \exp\{-(x, Ax) - (y, By) - 2(x, Dy) + 2(L, (x, y))\}$$

where A, B and D are (complex) $n \times n$ matrices with A and B being symmetric while L is a vector in $\mathbb{C}^{2n}$. We shall also write

$$\begin{pmatrix} A & D \\ D^T & B \end{pmatrix} = M + iN \tag{1}$$

for M, N real, symmetric $2n \times 2n$ matrices. The only condition imposed on M is that it is positive semidefinite. If M is positive definite, then G is called *non-degenerate*. If M has a zero eigenvalue, then G is called *degenerate*.

The action of G on complex valued, measurable functions $f : \mathbb{R}^n \to \mathbb{C}$ is formally given by

$$(\mathcal{G}f)(x) = \int_{\mathbb{R}^n} G(x, y) f(y) dy. \tag{2}$$

The linear operator $\mathcal{G}$ will be studied as an operator from L^p to L^q for $1 < p, q < \infty$. When G is non-degenerate, (2) makes sense by Hölder's inequality. In the degenerate case one needs $f \in L^p \cap L^1$. Assuming that $\mathcal{G}|_{L^p \cap L^1}$ is bounded from L^p to L^q, then for any $f \in L^p$, $\mathcal{G}f \in L^q$ is uniquely defined as $\mathcal{G}f = \lim_{j \to \infty} \mathcal{G}f_j$, where $f_j \in L^p \cap L^1$ is any sequence that converges to f in L^p. Then $\mathcal{G}f$ is well defined as $\mathcal{G}f_j$ is a Cauchy sequence in L^q.

Example 1. *The Fourier transform*

$$\widehat{f}(x) = \int_{\mathbb{R}^n} \exp\{-2i(x,y)\} f(y) dy$$

is associated with the degenerate kernel corresponding to $A = B = 0$, $L = 0$ *and* $D = iI$, *where* I *is the identity matrix.*

The norm of $\mathcal{G}$ from L^p to L^q is defined to be

$$C_{p \to q} = \sup_f \frac{\|\mathcal{G}f\|_q}{\|f\|_p} \tag{3}$$

where sup is taken over $f \in L^p, f \neq 0$, and, if G is degenerate, $f \in L^1$ as well. A function $0 \neq f \in L^p$ is a *maximizer* for $\mathcal{G}$ (or G) if $C_{p \to q} < \infty$ and

$$\|\mathcal{G}f\|_q = C_{p \to q} \|f\|_p.$$

A *Gaussian function* is a function from $\mathbb{R}^n$ to $\mathbb{C}$ of the form

$$g(x) = \mu \exp\{-(x, Jx) + (l, x)\} \tag{4}$$

where $0 \neq \mu \in \mathbb{C}, l \in \mathbb{C}$ and J is a symmetric $n \times n$ matrix with $\mathrm{Re}(J)$ positive definite. If $l = 0$, then g is called *centered*.

The presented article [3] investigates existence and uniqueness of maximizers for $\mathcal{G}$. The results of [3] are discussed in the following two sections and can be summarized as follows. In the non-degenerate case, $\mathcal{G}$ has a unique maximizer which is a centered Gaussian function. The precise result depends on the exponents p and q. The degenerate situation is much more subtle. In the degenerate case, the supremum (3) can be assumed to be over centered Gaussians. If the supremum is achieved for some Gaussian function then, when $p < q$, every maximizer is a Gaussian function.

Before proceeding we turn our attention to some known results for the Fourier transform.

Example 2. *The Fourier transform is bounded from* L^p *to* L^q *if and only if* $q = p' \geq 2$. *The Hausdorff-Young inequality states that for* $1 \leq p \leq 2$, *the Fourier transform is a bounded map from* L^p *to* $L^{p'}$ *with norm at most* 1, *i.e.*

$$\|\widehat{f}\|_{p'} \leq \|f\|_p \;\text{ for }\; 1 \leq p \leq 2, \;\; 1/p + 1/p' = 1.$$

Plancherel's theorem states that for $p = 2$ we have the equality

$$\|\widehat{f}\|_2 = \|f\|_2.$$

However, when $p < 2$, the bound is actually less than one. It is shown by Babenko [1] and Beckner [2] that for $1 < p < 2$ the optimal constant is

$$C_{p\to p'} = \left(\frac{p^{1/p}}{p'^{1/p'}}\right)^{n/2}.$$

The sharp bound is achieved for Gaussians if and only if they are of the form (4) *with J real. It is due to Beckner that Gaussian functions are the only maximizers when $p' \geq 4$. Lieb [3] shows that this is the case for all $p' > 2$. Of course, when $p = 2$, every L^2 function is a maximizer.*

The following reduction is made. Studying maximizers for G, without loss of generality it suffices to consider only G's for which

- A and B are real symmetric $n \times n$ matrices
- $L = 0$, i.e. G is centered.

The first fact follows as the imaginary parts of A and B can be omitted without changing $\|\mathcal{G}f\|_q$ and $\|f\|_p$, respectively. The second claim follows by a suitable change of variables.

4.2 Non-degenerate Gaussian kernels

Let G be a non-degenerate, centered Gaussian kernel. Then $\mathcal{G}$ is a compact operator from L^p to L^q and there exists at least one maximizer $f \in L^p$. There exists a unique Gaussian maximizer in each of the following cases.

(A) Real case, $1 < p, q < \infty$

By "real" we mean that the matrix N is zero.

Theorem 3. *Let G be a non-degenerate centered Gaussian kernel with $N = 0$. Let $1 < p, q < \infty$. Then, $\mathcal{G}$ has exactly one maximizer, f, (up to a multiplicative constant) from L^p to L^q and f is a real, centered Gaussian, i.e. $f(x) = \exp\{-(x, Jx)\}$ where J is a real, positive definite matrix.*

(B) Imaginary case, $1 < p \le 2$ and $1 < q < \infty$ or $1 < p < \infty$ and $2 \le q < \infty$

By "imaginary" we mean that the kernel has a real diagonal part and a purely imaginary off-diagonal part, i.e.

$$G(x, y) = \exp\{-(x, Ax) - (y, By) - 2i(x, Dy)\} \tag{5}$$

where A, B, D are real $n \times n$ matrices and A, B are positive definite.

Theorem 4. *Let G be as in (5) and let either $1 < p \le 2$ and $1 < q < \infty$ or else $1 < p < \infty$ and $2 \le q < \infty$. Then, $\mathcal{G}$ has exactly one maximizer, f, (up to a multiplicative constant) from L^p to L^q and f is a real, centered Gaussian, i.e. $f(x) = \exp\{-(x, Jx)\}$ where J is a real, positive definite matrix.*

(C) Complex case, $1 < p \le q < \infty$

This is the general case with M and N as in (1).

Theorem 5. *Let G be a non-degenerate centered Gaussian kernel and let $1 < p \le q < \infty$. Then $\mathcal{G}$ has exactly one maximizer (up to a multiplicative constant) from L^p to L^q which is a centered Gaussian function.*

4.2.1 Sketch of proofs

The main idea of the proofs of (A) and (C) is to study $\mathcal{G} \otimes \mathcal{G}$ from $L^p(\mathbb{R}^{2n})$ to $L^q(\mathbb{R}^{2n})$ and use Minkowski's integral inequality. Considering the $\mathcal{G} \otimes \mathcal{G}$ maximizer

$$F(y_1, y_2) = f\Big(\frac{y_1 + y_2}{\sqrt{2}}\Big)\Big(\frac{y_1 - y_2}{\sqrt{2}}\Big)$$

where f is a maximizer for $\mathcal{G}$, it is possible to deduce f must be a Gaussian. For instance, in (A) this follows by an analiticity result for the maximizer f and using that the equality in Minkowski's inequality implies existence of positive functions α and β such that F is an elementary tensor

$$F(y_1, y_2) = \alpha(y_1)\beta(y_2).$$

Uniqueness is obtained from the tensor structure of F as well.

The technicalities in (C) are different since the argument in (A) relies on $G > 0$. This is also the reason for different ranges of exponents. Namely, lack of positivity in (C) forces the integration to be in a different order than in (A). This results in applying Minkowski's inequality for all exponents $r > 1$ in the first case and for all $r = q/p \geq 1$ in the third case.

To prove (B) one performs a change of variables which turns (at least for non-singular D) the kernel G into a canonical form for which $(\mathcal{G}f)(x) = \mu(\widetilde{\mathcal{G}}\widehat{f})(x)$ where $\widetilde{G}$ is the real, centered, non-degenerate Gaussian

$$\widetilde{G}(x,y) = \exp\{-(x,Ax) - (y,Ay) - (x-y, A^{-1}(x-y))\}$$

and $\mu > 0$ is a constant depending only on A. Then the trick is to apply (A) to $\widetilde{\mathcal{G}}\widehat{f}$ and use the sharp Hausdorff-Young inequality. This gives the desired result for any pair of exponents satisfying $1 < p \leq 2$ and $1 < q < \infty$. The second claimed range follows by duality.

4.3 Degenerate Gaussian kernels

The following formula for the $L^p \to L^q$ norm of degenerate kernels shows that it is determined by examining only Gaussian functions.

Theorem 6. *Let G be a centered Gaussian kernel and let p and q satisfy the conditions of (A), (B) or (C) of Section 4.2 according to the properties of G. Then $\mathcal{G}$ is bounded from L^p to L^q if and only if the following supremum is finite, in which case the supremum is equal to $C_{p\to q}$.*

$$\sup_g \frac{\|\mathcal{G}g\|_q}{\|g\|_p} = C_{p\to q},$$

where the supremum is taken over all centered Gaussian functions, and in cases (A) and (B) they can be assumed to be real.

Of course, the same formula holds for non-degenerate kernels, because in that case there is a uniquely determined Gaussian maximizer. In the degenerate case a maximizer may not exist even if $\mathcal{G}$ is bounded. An example is the convolution operator with $G(x,y) = \exp\{-\lambda(x-y, x-y)\}$, which is bounded when $p \leq q$ but has no maximizers when $p = q$. Also, a Gaussian maximizer may not be centered, even if G is. This is the case for the Fourier transform.

The following theorem gives a sufficient condition for Gaussian maximizers. In the real case (A) it is also necessary.

Theorem 7. *Let G be a degenerate Gaussian kernel with the property that the real, symmetric matrices A and B in* (1) *are positive definite. If* $1 < p \leq q < \infty$*, then $\mathcal{G}$ is bounded from L^p to L^q. If, additionally, $p < q$, $\mathcal{G}$ has a maximizer which is a Gaussian function.*

If G is real, then A and B must be positive definite if $\mathcal{G}$ is bounded at all. In this real, degenerate case $\mathcal{G}$ is unbounded when $1 < q < p < \infty$ and $\mathcal{G}$ has no maximizer of any kind when $1 < p = q < \infty$.

In the degenerate case maximizers need not be unique. However, if there is any Gaussian maximizer for $p < q$, then every maximizer is a Gaussian.

Theorem 8. *Let G be a degenerate Gaussian kernel and let $1 < p < q < \infty$. Assume that $\mathcal{G}$ is bounded from L^p to L^q and that g is a Gaussian function that is a maximizer for $\mathcal{G}$. If f is another maximizer for $\mathcal{G}$ then f is also Gaussian (but not necessarily proportional to g).*

Note that the last theorem completely settles the Fourier transform case. By this result Gaussians are the only maximizers for all $1 \leq p < 2$. It also completely settles the real case (A) as by Theorem 7, no maximizer exists if $p \geq q$ and a Gaussian maximizer exists if $p < q$.

4.4 Closing remarks

All theorems have extensions to more general Gaussian kernels on $\mathbb{R}^m \times \mathbb{R}^n$ for $m \neq n$. Moreover, the same methods yield similar results for real multilinear forms. These results can be used to derive sharp constants in the fully multidimensional, multilinear generalization of Young's inequality.

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5 A New, Rearrangement-free Proof of the Sharp Hardy-Littlewood-Sobolev Inequality

after Frank and Lieb [1]
A summary written by Taryn C. Flock

Abstract

Frank and Lieb [1] give characterization of extremizers in a particular case of the Hardy-Littlewood-Sobolev inequality using conformal symmetry and spherical harmonics (but not rearrangement inequalities).

5.1 Introduction

5.1.1 The Hardy-Littlewood-Sobolev Inequality

Theorem 1. *Let $p, r > 1$ and $0 < \lambda < n$ such that $\frac{1}{p} + \frac{\lambda}{n} + \frac{1}{r} = 2$. Then there exists $C > 0$ such that for all $f \in L^p$ and $h \in L^r$,*

$$HLS_\lambda(f,h) = \left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x)|x-y|^{-\lambda} h(y)dxdy \right| \leq C\|f\|_{L^p}\|h\|_{L^r} \qquad (1)$$

The basic motivating questions are:

- What is the best constant C?
- Are there (pairs of) functions, called extremizers (or extremizing pairs) which achieve the best constant?

Theorem 2 ([2])**.** *Let $0 < \lambda < n$ and $p = r = 2n/(2n-\lambda)$. Then equality holds in* (1) *if and only if there exists $c, c' \in \mathbb{C}$, $\delta > 0$ and $a \in \mathbb{R}^n$ such that*

$$f(x) = cH(\delta(x-a)) \; g(x) = c'H(\delta(x-a))$$

where

$$H(x) = (1+|x|^2)^{-(2n-\lambda)/2}$$

A few preliminary observations are in order. First, when $p = r$, any if an extremizing pair exists it has form (f, cf) for some $f \in L^p$ and $c \in \mathbb{C}$. This can be seen by viewing (HLS) as a quadratic form. Secondly, if f is an extremizer then for any $c \in \mathbb{C}$, cf is an extremizer as well.

5.1.2 Outline of Proof

The basic idea is process of elimination. More specifically, the proof can be divided into 5 steps.

1. Extremizers exist.
2. The problem on $\mathbb{R}^n$ can be stated equivalently on $\mathbb{S}^N$ using conformal symmetry. (Here $N = n + 1$)
3. Spherical extremizers with an additional property exist (the center of mass of h^p vanishes).
4. No function with this additional property can be an extremizer, except perhaps constant functions. From which we may immediately conclude that all constant functions are extremizers.
5. Constants functions are the unique extremizers up to the natural family of symmetries.

The main work goes into the proof of Step 4, which separates into proving two "nearly contradictory" inequalities. The first will be specific to extremizers with the additional property. The second, the opposite of the first, will hold for all functions and be an equality only for constants. The proof of the first inequality uses calculus of variations; the proof of the second uses spherical harmonics. The two inequalities are:

Lemma 3. *Let* $h \in L^p(\mathbb{S}^N)$ *be a nonnegative extremizer for* (1) *such that the center of mass of* h^p *vanishes then*

$$\int\int \frac{\bar{h}(\omega)h(\eta)\omega\cdot\eta}{|\omega-\eta|^\lambda}d\omega d\eta - (p-1)\int\int \frac{h(\omega)h(\eta)}{|\omega-\eta|^\lambda}d\omega d\eta \leq 0 \qquad (2)$$

Lemma 4. *Let* $f \in L^p(\mathbb{S}^N)$ *then*

$$\int\int \frac{\bar{f}(\omega)f(\eta)\omega\cdot\eta}{|\omega-\eta|^\lambda}d\omega d\eta - (p-1)\int\int \frac{\bar{f}(\omega)f(\eta)}{|\omega-\eta|^\lambda}d\omega d\eta \geq 0 \qquad (3)$$

with equality if and only if f *is a constant function.*

5.2 Sketch of Proof

5.2.1 Existence of Extremizers

A rearrangement-free proof for the existence of extremizers is given in [4] using concentration compactness.

5.2.2 Uplifting the problem

When $p = r$, the Hardy-Littlewood-Sobolev inequality on $\mathbb{R}^n$ is equivalent to the Hardy-Littlewood-Sobolev inequality on $\mathbb{S}^{n+1}$. This is seen by lifting the problem from $\mathbb{R}^n$ to $\mathbb{S}^{n+1}$ using Stereographic projection.

5.2.3 Vanishing center of mass

Definition 5. *Let $h : \mathbb{S}^N \to \mathbb{C}$. We say the center of mass of h^p vanishes if*

$$\int_{\mathbb{S}^N} \omega_j |h(\omega)|^p d\omega = 0 \text{ for all } j \in [1, \ldots, N+1].$$

Lemma 6. *Let $h \in L^1(\mathbb{S}^N)$ be an extremizer of (1) then there exists h_0 an extremizer of (1) such that the center of mass of h_0^p vanishes.*

The Hardy-Littlewood-Sobolev inequality enjoys a large family of symmetries (operations like $f \mapsto cf$ which preserve extremizers). The proof shows that given any function on $\mathbb{S}^N$, we may transform it to a function for which the center of mass of h^p vanishes using only these symmetries.

5.2.4 Two Inequalities

The inequality in Lemma 3 is proved using the calculus of variations. As the function h is an extremizer, the second variation is at most zero. Testing with functions f such that $\int h^{p-1} f \, d\omega = 0$ simplifies the expression leaving:

Lemma 7. *Let $h \in L^p(\mathbb{S}^N)$ be a nonnegative extremizer for (1) then for any f satisfying $\int h^{p-1} f \, d\omega = 0$,*

$$\int \int \frac{\bar{f}(\omega) f(\eta)}{|\omega - \eta|^\lambda} d\omega d\eta \int h^p d\omega - (p-1) \int \int \frac{h(\omega) h(\eta)}{|\omega - \eta|^\lambda} d\omega d\eta \int h^{p-2} |f|^2 d\omega \leq 0.$$

Given an extremizer h such that center of mass vanishes, for each $j \in [1,\ldots,n{+}1]$, $f_j = \omega_j h(\omega)$ is a function satisfying the condition $\int h^{p-1} f_j \, d\omega = 0$. Summing the inequality with these f_j in j produces the inequality in Lemma 3 .

The inequality in Lemma 4 is a special case of a more general inequality proved using spherical harmonics. The main tool is the Funk-Heck theorem which tells us that an operator on $\mathbb{S}^N$ with kernel $K(\omega \cdot \eta)$ is diagonal with respect to the decomposition

$$L^2(\mathbb{S}^N) = \bigoplus_{\ell \geq 0} \mathcal{H}_\ell$$

where $\mathcal{H}_\ell$ is the space of harmonic polynomials on $\mathbb{R}^{N+1}$ which are homogeneous of degree ℓ. The Funk-Hecke theorem further gives an explicit formula for the eigenvalues of such an operator.

Proposition 8. *Let $0 < \alpha < N/2$. For any f on $\mathbb{S}^N$,*

$$\int\int \frac{\bar{f}(\omega) f(\eta) \omega \cdot \eta}{|\omega - \eta|^{2\alpha}} d\omega d\eta \geq \frac{\alpha}{N-\alpha} \int\int \frac{\bar{f}(\omega) f(\eta)}{|\omega - \eta|^{2\alpha}} d\omega d\eta$$

with equality if and only if f is constant.

Lemma 4 is the case where $2\alpha = \lambda$. The condition $\frac{2}{p} + \frac{\lambda}{n} = 2$, insures that $\frac{\alpha}{N-\alpha} = p - 1$.

Proof. Writing $|\omega - \eta|^2 = 2(1 - \omega \cdot \eta)$. The claimed inequality is equivalent to

$$\int\int \frac{\bar{f}(\omega) f(\eta)(\omega \cdot \eta - 1)}{(1 - \omega \cdot \eta)^{\alpha}} d\omega d\eta + (1 - \frac{\alpha}{N-\alpha}) \int\int \frac{\bar{f}(\omega) f(\eta)}{(1 - \omega \cdot \eta)^{\alpha}} d\omega d\eta > 0.$$

Taking this one step further it is enough to show that

$$\int\int \frac{\bar{f}(\omega) f(\eta)}{(1 - \omega \cdot \eta)^{\alpha - 1}} d\omega d\eta \leq (1 - \frac{N-2\alpha}{N-\alpha}) \int\int \frac{\bar{f}(\omega) f(\eta)}{(1 - \omega \cdot \eta)^{\alpha}} d\omega d\eta > 0.$$

View both the expressions on the right hand side and the expression on the left hand side as quadratic forms indexed by α. These operators maybe simmultaneously diagonalized and their eigenvalues explicitly computed using the Funk-Hecke theorem. In this setting, it is enough to show that for each eigenvalue $E_\ell(\alpha - 1) \leq \frac{N-2\alpha}{N-\alpha} E_\ell(\alpha)$.

Using the Funk-Hecke theorem there is a dimensional constant K_N such that

$$E_\ell(\alpha) = K_N 2^{-\alpha}(-1)^\ell \frac{\Gamma(1-\alpha)\Gamma(N/2-\alpha)}{\Gamma(-\ell+1-\alpha)\Gamma(\ell+N-\alpha)}$$

Using that $\Gamma(t+1) = t\Gamma(t)$, we have

$$E_\ell(\alpha+1) = \frac{2E_\ell(\alpha)(1-\alpha)(N/2-\alpha)}{(-\ell+1-\alpha)(\ell+N-\alpha)}.$$

Now

$$\frac{(\alpha-1)}{(\alpha-1+\ell)}\frac{1}{(N-\alpha+\ell)} \leq \frac{1}{N-\alpha}$$

Further, this inequality is strict unless $\ell = 0$. □

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6 Maximizers for the adjoint Fourier restriction inequality on the sphere

after D. Foschi
A summary written by Marius Lemm

Abstract

We present a recent paper of Foschi [4] which proves that the constant functions are the global maximizers for the adjoint Fourier restriction inequality of Stein and Tomas on the sphere.

6.1 Introduction and main result

We consider the adjoint Fourier restriction inequality of Stein and Tomas [6] on the sphere. Let $\mathbb{S}^2$ denote the unit sphere in $\mathbb{R}^3$ an let σ denote the standard surface measure on $\mathbb{S}^2$, induced by the Lebesgue measure on $\mathbb{R}^3$. Stein and Tomas [6] proved that there exists a constant $C > 0$ such that

$$\|\widehat{f\sigma}\|_{L^4(\mathbb{R}^3)} \leq C\|f\|_{L^2(\mathbb{S}^2)} \tag{1}$$

holds for all $f \in L^2(\mathbb{S}^2)$. Here we denoted the Fourier transform of an integrable function on the sphere by

$$\widehat{f\sigma}(x) := \int_{\mathbb{S}^2} e^{-ix\omega} f(\omega) \mathrm{d}\sigma_\omega,$$

for all $x \in \mathbb{R}^3$. We are interested in the *optimal constant* in (1), defined as

$$\mathcal{R} := \sup_{f\in L^2(\mathbb{S}^2), f\not\equiv 0} \frac{\|\widehat{f\sigma}\|_{L^4(\mathbb{R}^3)}}{\|f\|_{L^2(\mathbb{S}^2)}}. \tag{2}$$

A natural questions to ask for this variational problem (aside from the numerical value of $\mathcal{R}$) is whether the supremum is achieved, i.e. whether there exists a function $f \in L^2(\mathbb{S}^2)$ which is not identically zero and satisfies

$$\|\widehat{f\sigma}\|_{L^4(\mathbb{R}^3)} = \mathcal{R}\|f\|_{L^2(\mathbb{S}^2)}.$$

Such a function is called a *maximizer* for the variational problem (2).

In 2012, Christ and Shao [1] proved that maximizers for (2) exist, using a concentration compactness argument. They also show that constants are *local* maximizers. A substantial part of their analysis is to exclude that a maximizing sequence weakly converges to a Dirac mass at a single point or two Dirac masses at two antipodal points. Since the sphere locally looks like a paraboloid, [1] achieve this by proving that $\mathcal{R}$ well exceeds the optimal constant for the analogous inequality on the paraboloid.

Maximizers do not exist for the analogous inequality on the paraboloid [5], showing that their existence is a rather subtle phenomenon.

In 2015, Foschi [4] gave a remarkably short proof of the fact that *constants are the global maximizers* for (2) (up to a simple "gauge" freedom, see Remark 2 below). A straightforward calculation then yields $\mathcal{R} = 2\pi$.

Here we present the ideas of [4] and sketch the proof of its main result

Theorem 1 ([4])**.** *A nonnegative $f \in L^2(\mathbb{S}^2)$ satisfies*

$$\|\widehat{f\sigma}\|_{L^4(\mathbb{R}^3)} = \mathcal{R}\|f\|_{L^2(\mathbb{S}^2)} \tag{3}$$

iff f is equal to a nonzero constant. Moreover, $\mathcal{R} = 2\pi$.

Remark 2. *Theorem 1.2 in [2] then implies that the set of all (complex-valued) global maximizers for* (1) *is equal to*

$$\left\{ke^{i\theta e^{i\xi\cdot\omega}} \,:\, k > 0,\, \theta \in [0, 2\pi),\, \xi \in \mathbb{R}^3\right\}.$$

Remark 3. *A direct consequence of Theorem 1 and Hölder's inequality is that for the family of non-endpoint adjoint Fourier restriction inequalities*

$$\mathcal{R}_p := \sup_{f \in L^p(\mathbb{S}^2), f \not\equiv 0} \frac{\|\widehat{f\sigma}\|_{L^4(\mathbb{R}^3)}}{\|f\|_{L^p(\mathbb{S}^2)}},$$

indexed by $2 < p \leq \infty$, constants are again the global maximizers. It follows that $\mathcal{R}_p = 2\pi(4\pi)^{\frac{1}{2}-\frac{1}{p}}$.

6.2 Sketch of the proof

The *general strategy* is to prove a series of upper bounds on $\|\widehat{f\sigma}\|_{L^4(\mathbb{R}^3)}$ which are equalities for the constant functions, until one arrives at a quantity which is strictly maximized by the constants.

Writing the L^4 norm as a convolution One makes the standard but essential observation that the L^4 norm (or more generally any L^p norm for p an even integer) can be realized as the L^2 *norm of a convolution*, by Parseval's theorem. That is

$$\|\widehat{f\sigma}\|^2_{L^4(\mathbb{R}^3)} = \|\widehat{f\sigma}\widehat{f_\star\sigma}\|_{L^2(\mathbb{R}^3)} = \|\widehat{f\sigma f_\star\sigma}\|_{L^2(\mathbb{R}^3)} = (2\pi)^{3/2}\|f\sigma * f_\star\sigma\|_{L^2(\mathbb{R}^3)}, \quad (4)$$

where we introdcued $f_\star(\omega) := \overline{f(-\omega)}$, the "antipodal conjugate" of f.

Reductions By the triangle inequality and a rearrangement argument (a clever application of Cauchy-Schwarz), one can reduce to the case of nonnegative, antipodially symmetric functions f. For such functions, the quantity of interest simply reads

$$\|f\sigma * f\sigma\|_{L^2(\mathbb{R}^3)} = \int_{(\mathbb{S}^2)^4} f(\omega_1)f(\omega_2)f(\omega_3)f(\omega_4)\mathrm{d}\Sigma_\omega, \quad (5)$$

where we introduced the following measure $\mathrm{d}\Sigma$ on $(\mathbb{S}^2)^4$

$$\mathrm{d}\Sigma_\omega := \delta(\omega_1+\omega_2+\omega_3+\omega_4)\mathrm{d}\sigma_{\omega_1}\mathrm{d}\sigma_{\omega_2}\mathrm{d}\sigma_{\omega_3}\mathrm{d}\sigma_{\omega_4}.$$

Cauchy-Schwarz in $(\mathbb{S}^2)^4$ This part contains the central idea. One proves

Lemma 4. *We have*

$$\|f\sigma * f\sigma\|_{L^2(\mathbb{R}^3)} \le \frac{6\pi}{4} \iint_{\mathbb{S}^2\times\mathbb{S}^2} f(\omega_1)^2 f(\omega_2)^2 |\omega_1+\omega_2| \mathrm{d}\sigma_{\omega_1}\mathrm{d}\sigma_{\omega_2} \quad (6)$$

and equality holds iff there exists a measurable function h such that $f(\omega_1)f(\omega_2) = h(\omega_1+\omega_2)$.

Remark 5. *Observe that equality holds in particular for constants.*

Before the proof, we discuss the naive approach. In the spirit of [3], one would like to interpret (5) as an instance of the bilinear form

$$B(F,G) := \int_{(\mathbb{S}^2)^4} F(\omega_1,\omega_2)G(\omega_3,\omega_4)\mathrm{d}\sigma_\omega \quad (7)$$

and then apply the Cauchy-Schwarz inequality for $B(\cdot,\cdot)$. If one does this naively, one obtains the bound

$$\begin{aligned}
&\|f\sigma * f\sigma\|_{L^2(\mathbb{R}^3)} \\
&\leq \iint\limits_{\mathbb{S}^2\times\mathbb{S}^2} f(\omega_1)^2 f(\omega_2)^2 \left(\iint\limits_{\mathbb{S}^2\times\mathbb{S}^2} \delta(\omega_1+\omega_2+\omega_3+\omega_4)\mathrm{d}\sigma_{\omega_3}\mathrm{d}\sigma_{\omega_4} \right) \mathrm{d}\sigma_{\omega_1}\mathrm{d}\sigma_{\omega_2} \\
&= 2\pi \iint\limits_{\mathbb{S}^2\times\mathbb{S}^2} \frac{f(\omega_1)^2 f(\omega_2)^2}{|\omega_1-\omega_2|}\mathrm{d}\sigma_{\omega_1}\mathrm{d}\sigma_{\omega_2} \qquad (8)
\end{aligned}$$

where the second line follows by an explicit computation of the integral in parentheses, see Lemma 2.2 in [4]. The problem with the last line is that, for the purpose of proving Theorem 1, the integral is *too singular*.

We come to the

Proof of Lemma 4. The *key input* is the *"algebraic geometric identity"*

$$|\omega_1+\omega_2||\omega_3+\omega_4| + |\omega_1+\omega_3||\omega_2+\omega_4| + |\omega_1+\omega_4||\omega_2+\omega_3| = 4$$

for all $\omega_1,\omega_2,\omega_3,\omega_4 \in \mathbb{S}^2$ satisfying $\omega_1+\omega_2+\omega_3+\omega_4 = 0$. Using this on (5), as well as symmetry, one gets

$$\int\limits_{(\mathbb{S}^2)^4} f(\omega_1)f(\omega_2)f(\omega_3)f(\omega_4)\mathrm{d}\sigma_\omega = \frac{3}{4}B(F,F)$$

with $F(\omega,\nu) := f(\omega)f(\nu)|\omega+\nu|$. Now we use Cauchy-Schwarz for $B(\cdot,\cdot)$,

$$\frac{3}{4}B(F,F) \leq \frac{6\pi}{4} \iint\limits_{\mathbb{S}^2\times\mathbb{S}^2} f(\omega_1)^2 f(\omega_2)^2 |\omega_1+\omega_2| \mathrm{d}\sigma_{\omega_1}\mathrm{d}\sigma_{\omega_2} \qquad (9)$$

and the singularity from (8) has disappeared! Equality in (9) holds iff there exists $\lambda > 0$ such that $F(\omega_1,\omega_2) = \lambda F(\omega_3,\omega_4)$ holds Σ-almost everywhere. This condition is equivalent to $f(\omega_1)f(\omega_2) = h(\omega_1+\omega_2)$ for some h. □

Explicit computations with spherical harmonics Finally, the integral on the right hand side of (9) is analyzed by decomposing the function f^2 into spherical harmonics (plus a denseness argument, since only functions in

$L^2(\mathbb{S}^2)$ have such a decomposition). One can compute the resulting integral explicitly in this basis using known properties of special functions.

The upshot is that only the zeroth order spherical harmonic (i.e. the constant functions) can contribute non-negatively to the expression.

We now sketch the ideas. We rewrite the right hand side in (6) as

$$H(g) := \iint_{(\mathbb{S}^2)^2} g(\omega)g(\nu)\sqrt{2-2\omega\cdot\nu}\,\mathrm{d}\sigma_\omega \mathrm{d}\sigma_\nu \tag{10}$$

where $g \in L^1(\mathbb{S}^2)$ represents f^2. Here we used that $|\omega - \nu| = \sqrt{2-2\omega\cdot\nu}$.

For L^2 functions, we have a decomposition into the basis of spherical harmonics [7] and it turns out that H is diagonal in this basis:

Lemma 6. *For $g \in L^2(\mathbb{S}^2)$, write*

$$g = \sum_{l=0}^{\infty} Y_l$$

where Y_l is an l-th order spherical harmonic. Then

$$H(g) = 2\pi \sum_{l=0}^{\infty} \lambda_l \|Y_l\|^2_{L^2(\mathbb{S}^2)} \tag{11}$$

with $\lambda_0 > 0$ and $\lambda_l < 0$ for all $l \geq 1$.

Corollary 7. *For $g \in L^2(\mathbb{S}^2)$,*

$$H(g) \leq H(\langle g \rangle), \tag{12}$$

where $\langle g \rangle := \frac{1}{4\pi}\int_{\mathbb{S}^2} g(\omega)\mathrm{d}\sigma_\omega$ is the average value of g.

In fact, Lemma 6 implies the main result by a standard denseness argument (the map $g \mapsto H(g)$ is L^1 continuous).

The Funk-Hecke formula We discuss the key ingredient for the proof of Lemma 6. We write P_l for the Legendre polynomial of order l.

Lemma 8 (Funk-Hecke formula)**.** *Let $\phi : [-1,1] \to \mathbb{R}_+$ be a function such that the integral*

$$\Lambda_l := \int_{-1}^{1} \phi(t)P_l(t)\mathrm{d}t \tag{13}$$

makes sense. Then, for every $\omega \in \mathbb{S}^2$,

$$\int_{\mathbb{S}^2} \phi(\omega \cdot \nu) Y_l(\nu) \mathrm{d}\sigma_\nu = 2\pi \lambda_l Y_l(\omega). \tag{14}$$

Recall that $g = \sum_{l=0}^{\infty} Y_l$. Plugging this decomposition into (10), applying the Funk-Hecke formula and the orthogonality of the spherical harmonics, we find

$$H(g) = 2\pi \sum_{l,l'} \Lambda_l \int_{(\mathbb{S}^2)^2} Y_l(\omega) Y_{l'}(\omega) \mathrm{d}\sigma_\omega = 2\pi \sum_{l=0}^{\infty} \Lambda_l \|Y_l\|^2_{L^2(\mathbb{S}^2)}.$$

To prove Lemma 6, it remains to show that $\Lambda_0 > 0$ and $\Lambda_l < 0$ for all $l \geq 1$. This follows from properties of Legendre polynomials, for the details we refer to section 5 in [4].

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7 Extremals of functionals with competing symmetries

after E. A. Carlen and M. Loss [1]
A summary written by Teresa Luque

Abstract

We describe a method for generating extremals of functionals with high symmetry. We apply this method to prove the sharp version of certain geometric inequalities, like the Hardy-Littlewood-Sobolev inequality and the logarithmic Sobolev inequality.

7.1 Introduction

The Hardy-Littlewood-Sobolev inequality (HLS inequality) for functions f, g on $\mathbb{R}^d$

$$|H(f,g)| \leq C(d,\lambda,p)||f||_p||g||_q \tag{1}$$

where

$$H(f,g) \equiv \int_{\mathbb{R}^d} f(x) \int_{\mathbb{R}^d} \frac{g(y)}{|x-y|^\lambda} dydx \tag{2}$$

holds for all $0 < \lambda < d$ and $p, q > 1$ with $1/p + 1/q + \lambda/d = 2$. It is relevant to establish the value of the best constant $C(d, \lambda, p)$ and if it is achieved, to describe the family of extremizers; that is, the functions f and g that turn (1) into an equality with the smallest constant. Lieb [3] prove that for any admissible choice of d, p and q there exist extremizers. Moreover, for the diagonal case $p = q = 2d/(2d - \lambda)$, Lieb also identified such extremizers and computed the best constant $C(d, \lambda)$. In this particular case, (1) is invariant by conformal transformations and this will play an important role in the Lieb's proof. These invariants will be referred as symmetries of the HLS. Concretely, the proof of Lieb proceeds in two steps:

- The existence of the extremals, where he shows that there is a *maximizing* sequence for H whose limit is an extremal. In this part, the many symmetries of the HLS are an obstacle, since they make easy for a maximizing sequence to converge to zero.

- The identification of the optimizers for the diagonal case where he crucially uses the symmetries.

The main achievement of [1] is to present an unified approach to these two problems utilizing the many symmentries of the functional under study as an advantage for the purpose of optimality. A very rough idea of the method is the following. In order to maximize a certain functional I over a Banach function space X, we apply two transformations that both improve the functional. The first operation, R is a symmetrization that satisfies $R^2 = R$ and that improves I; that is:

$$I(f) \leq I(Rf).$$

The second operation D is a transformation, usually an isometry of X, that leaves the functional I invariant. In the case we detail here D is a rotation.

Both transformations R and D somewhat contradict each other because every time we make the function with R symmetrical, we destroy the symmetry applying D. In this sense, both operations *compete* with each other to produce a strongly convergent sequence. The result, as the theoren below shows, is that such sequence

$$f^k = (RD)^k f$$

converges to a function that is indeed a maximizer of the functional I.

Theorem 1 (Competing simmetries)**.** [3] *For $1 < p < \infty$, let $f \in L^p(\mathbb{R}^d)$ be any nonnegative function. Then the sequence $f^k := (RD)^k f$ converges in L^p as $k \to \infty$ to the function $h_f := ||f||_p h$, where*

$$h(x) = |\mathbb{S}^d|^{-1/p} \left(\frac{2}{1 + |x|^2} \right)^{d/p}, \tag{3}$$

and $|\mathbb{S}^d|$ is the area of the sphere of radius 1 in $\mathbb{R}^{d+1}$.

We will illustrate the technique for two examples, the conformally invariant case of the HLS inequality and the logarithmic Sobolev inequality:

Theorem 2 (Hardy-Littlewood-Sobolev inequality)**.** *For every $0 < \lambda < d$ and for every $f, g \in L^p(\mathbb{R}^d)$ with $p = 2d/(2d - \lambda)$ (1) holds with*

$$C(d, \lambda, p) = C(d, \lambda) = \pi^{\lambda/2} \frac{\Gamma(d/2 - \lambda/2)}{\Gamma(d - \lambda/2)} \left(\frac{\Gamma(d/2)}{\Gamma(d)} \right)^{-1+\lambda/d}. \tag{4}$$

[3] [1, Theorem 2.1] is a more abstract version, for general Banach spaces. For the clearity in the summary we focus our attention in the case of $L^p(\mathbb{R}^d)$.

Moreover (1) *gives equality if and only if* $f(x) = c_1 h(x/\mu^2 - a)$ *and* $g(x) = c_2 f(x)$, *where* h *is the function* (3), $a \in \mathbb{R}^d$ *and* $c_1, c_2, \mu \in \mathbb{R} \setminus \{0\}$.

Theorem 3 (Logarithmic Sobolev inequality)**.** *Let* $d > 2$ *and fix* $p = 2d/(d-2)$. *Then for every* $f \in L^p(\mathbb{R}^d)$

$$||\nabla f||_2 \geq C(d)||f||_p$$

holds for

$$C(d) = [\pi d(d-2)]^{1/2} \frac{[d(d-2)]^{1/2}}{2}.$$

Moreover the equality is reached if and only if f *is (up to a multiple) a conformal transformation of the function* (3).

7.2 The competing transformations

To understand the operations involved in [1], a few considerations related to the conformal group and the symmetric rearrangements need to be made.

A conformal transformation $\gamma : \mathbb{R}^d \to \mathbb{R}^d$ is a function that preserves angles between any two curves. Translations, rotations, reflections, scaling, inversion on the unit sphere and its combinations build the whole group of the conformal transformations of $\mathbb{R}^d$ (see [2, Theorem 15.2]).

As we pointed out, we are going to work with two transformations: R and D. Let's specify them. For $f \in L^p(\mathbb{R}^d)$, Rf denotes the spherical decreasing rearrangement of f:

$$Rf(x) := \int_0^\infty \chi_{A^*}(x)dt,$$

where $A = \{y \in \mathbb{R}^d : |f(y)| > t\}$ and A^* is the open ball centered at the origin whose volume is that of A. See [4, Section 3.3] for a more complete description of R and its main properties.

The operation D is a rotation by 90°, mapping the north pole e_{d+1} into $e_d = (0, \ldots, 1, 0)$; that is:

$$D : \mathbb{S}^d \to \mathbb{S}^d, \quad D(s) := (s_1, \ldots, s_{d+1}, -s_d).$$

We define the action of D on $f \in L^p(\mathbb{S}^d)$ as

$$D^* f(s) := f(D^{-1}s), \quad s \in \mathbb{S}^d.$$

Since we want to apply D to a function in $L^p(\mathbb{R}^d)$, we use the steriographic projection[4] S and its inverse S^{-1} to *lift* functions on $\mathbb{R}^d$ to the sphere (S^{-1}) and the other way around (S). More preciseley, the sequence is the following:

$$L^p(\mathbb{R}^d) \xrightarrow{S^{*-1}} L^p(\mathbb{S}^d) \xrightarrow{D^*} L^p(\mathbb{S}^d) \xrightarrow{S^*} L^p(\mathbb{R}^d),$$

where

$$(S^* f)(x) := \left(\frac{2}{1+|x|^2}\right)^{d/p} f(S^{-1}(x)), \quad x \in \mathbb{R}^d, \quad f \in L^p(\mathbb{S}^d)$$

and

$$(S^{*-1})f(s) := (1+s_{d+1})^{-d/p} f(S(s)), \quad s \in \mathbb{S}^d, \quad f \in L^p(\mathbb{R}^d).$$

For simplicity the function $S^{*-1}D^*S^*f$ will be denoted again by D and it is the other *competing* operation. The *good* properties of D and R with respect to the L^p-norm will be fundamental to prove Theorem 1.

7.3 Competing symmetries

We sketch the proof of Theorem 1. Let $f \in L^p(\mathbb{R}^d)$ be bounded and vanishing outside a bounded set (by density the result will be extended to $L^p(\mathbb{R}^d)$). Then, with this assumption and (3), it is not difficult to see that there exist a constant C such that

$$f(x) \leq C h_f(x) \quad \text{for a.e } x \in \mathbb{R}^d.$$

Using the order preserving properties of R (see [4, pp.81, property (vi)]) and D, the same relation with the same constant C holds for every $f^k = (RD)^k f$. Thus, the f^k's are uniformly bounded and by Helly's selection principle there exists a subsequence f^{k_l} that converges pointwise a.e. to a certain symmetric decreasing function g. By the dominated convergence theorem, $g \in L^p(\mathbb{R}^d)$.

To show that $g = h_f$ we proceed as follows. Using the fact that $||Rf - Rg||_p \leq ||f-g||_p$ (see [4, Theorem 3.5,pp.83]), $||Df - Dg||_p = ||f-g||_p$ and the invariance of h_f under both R and D:

$$\begin{aligned} \inf_k ||h_f - f^k||_p &= \lim_{k\to\infty} ||h_f - f^k||_p = \lim_{l\to\infty} ||h_f - f^{k_l+1}||_p \\ &= ||h_f - RDg||_p \leq ||Dh_f - Dg||_p = \inf_k ||h_f - f^k||_p. \end{aligned} \tag{5}$$

[4]The definition we take for the steriographic projection is the map $S : \mathbb{S}^d \to R^d \cup \{\infty\}$ is given by $x_i := s_i/(1+s_{d+1})$ for $i = 1, \ldots, d$ and $S(e^{d+1}) = \infty$. The inverse map S^{-1} : $R^d \cup \{\infty\} \to \mathbb{S}^d$ with $s_i := 2x_i/(1+|x|^2)$ for $i = 1, \ldots, d$ and $s_{d+1} := (1-|x|^2)/(1+|x|^2)$.

So it must hold that $||h_f - RDg||_p = ||Dh_f - Dg||_p = ||h_f - g||_p$. By the non-expansivity rearrangement theorem ([4, Theorem 3.5, pp.83]), this equality only holds if $RDg = Dg$. Thus, both Dg and g are symmetric-decreasing functions, what is possible if and only if $g = Ch$. Since $||g||_p = \lim_{l\to\infty} ||f^{k_l}||_p$, then $C = ||f||_p$ and $g = h_f$.

Finally, by (5), f^k converges in $L^p(\mathbb{R}^d)$ to h_f as $k \to \infty$.

7.4 Applications

7.4.1 The Hardy-Littlewood Sobolev inequality

In this section we prove Theorem 2. Our first goal is to maximize the functional H defined in (2). Since H is positive-definite, in order to prove Theorem 2 it is enough to consider those functionals where $g(x) \equiv f(x)$ (in this case we will use the shorthand notation $H(f, f) := H(f)$). Moreover we can assume without loss of generality that $f \geq 0$, real-valued and $||f||_p = 1$. Observe that H has two interesting properties, which are the key aspects to prove inequality (1) with the sharp constant (4):

- H is invariant under any conformal transformation γ; In particular, $H(Df) = H(f)$.
- As an inmediate consequence of the Riesz's rearrangement inequality (see [4, Theorem 3.7 (pp.87)]) and $||f||_p = ||Rf||_p = 1$, we have $H(f) \leq H(Rf)$. Thus, it is enough to look for the upper bound in the class of symmetric-decreasing functions.

By monote convergence, it is sufficient to prove (1) for those f such that $f \leq Ch$. Now, we can apply Theorem 1 to obtain a sequence f^k that converges to h in $L^p(\mathbb{R}^d)$ as $k \to \infty$. Moreover, there exists a subsequence f^{k_l} that converges to h pointwise and such that $f^{k_l} \leq C(1+|x|^2)^{-d/p}$. Hence, by the dominated convergence theorem, the nondecreasing sequence $H(f^{k_l})$ converges to $H(h)$ from bellow and then,

$$H(f) \leq H(h)$$

for every f considered in here. One gets the specific constant (4) after computing $H(h)$.

It remains to discuss the case of equality. By the result above and the properties of the functional H, we have that h and any conformal transformation of h is an optimizer. Moreover, they are the only ones. Indeed, if f is an optimizer and $H(Rf) = H(f)$, then by Riesz's rearrangement inequality, f must be a translate of some symmetric-decreasing function and R (it only translates f to the origin) acts as a conformal transformation too. Then, f^k is a conformal image of f (we denote $f^k := \gamma^k f$) and, in particular, an isometry ($||f_k||_p = ||f||_p$). Therefore by Theorem 1 and the special nature of h_f

$$\lim_{k\to\infty} ||f - (\gamma^{-1})^k h_f||_p = 0$$

for any optimizer f, with $(\gamma^{-1})^k h_f = ch_f(x/\mu_k^2 - a_k)$ (it is not difficult to check the action of the conformal group on h_f). The strong converge assures that μ_k and a_k converge as $k \to \infty$ to some $\mu > 0$ and $a \in \mathbb{R}^d$ respectively.

7.4.2 The logarithmic Sobolev inequality

Theorem 3 is, by a duality argument, equivalent to the special case $\lambda = d-2$ of Theorem 2, but [1] presents a direct proof using the competing technique. It follows the same lines of the proof of HLS inequality, using the conformal invariance of $||\nabla f||_2$ and the rearrangement inequality

$$||\nabla f||_2 \geq ||\nabla Rf||_2.$$

The remaining details will be presented in the Summer School.

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8 Extremals for the Tomas-Stein inequality

after M. Christ and S. Shao [1]
A summary written by Dominique Maldague

Abstract

The adjoint Fourier restriction inequality of Tomas and Stein states that the mapping $f \mapsto \widehat{f\sigma}$ is bounded from $L^2(S^2)$ to $L^4(\mathbb{R}^3)$, where σ denotes surface measure on S^2. The authors prove that there exist functions which extremize the inequality, and that any extremizing sequence of nonnegative functions has a subsequence which converges to an extremizer. We summarize their results.

8.1 Introduction

We define the operator R on Schwartz functions $g \in \mathcal{S}(\mathbb{R}^d)$ by $Rg = \check{g}|_{S^{d-1}}$. That is, take the inverse Fourier transform of g and restrict it to the sphere. For $f, g \in \mathcal{S}(\mathbb{R}^d)$, using Fubini's theorem we have

$$\begin{aligned}\int_{\mathbb{R}^d} Rg(x)\overline{f(x)}dx &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{ix\cdot y}g(y)dy\sigma(x)\overline{f(x)}dx \\ &= \int_{\mathcal{S}^{d-1}}\int_{\mathbb{R}^d} e^{ix\cdot y}g(y)dy\overline{f(x)}d\sigma(x) \\ &= \int_{\mathbb{R}^d} g(y)\overline{\int_{\mathcal{S}^{d-1}} e^{-ix\cdot y}f(x)d\sigma(x)}dy.\end{aligned}$$

Thus the formal adjoint T of the Fourier restriction operator R is $Tf(x) = \int_{S^{d-1}} e^{-ix\cdot y}f(x)d\sigma(x)$, where σ is surface measure on S^{d-1}. The Tomas-Stein inequality states that T is bounded from $L^2(S^2)$ to $L^4(\mathbb{R}^3)$, and moreover that 4 is the smallest p so that this is true.

The main results are the following:

Theorem 1 (Christ–Shao '10)**.** *There exists an extremizer in $L^2(S^2)$ for the inequality*

$$\|\widehat{f\sigma}\|_{L^4(\mathbb{R}^3)} \leq C\|f\|_{L^2(S^2,\sigma)}. \tag{1}$$

Theorem 2. *Any extremizing sequence of nonnegative functions in $L^2(S^2)$ for the inequality is precompact; that is, any subsequence has a sub-subsequence that is Cauchy in $L^2(S^2)$.*

To avoid dealing with the Fourier transform, it is useful to work with the equivalent (by Plancherel's theorem) inequality

$$\|f\sigma * f\sigma\|_{L^2(\mathbb{R}^3)} \leq C\|f\|^2_{L^2(S^2)} \tag{2}$$

Henceforth, let $\mathbf{S}$ denote the optimal constant satisfying

$$\|f\sigma * f\sigma\|_{L^2(\mathbb{R}^3)} \leq \mathbf{S}^2\|f\|^2_{L^2(S^2)} \quad \text{for all } f \in L^2(S^2).$$

8.2 Two Principles

Definition 3. *We call f a δ-nearly extremal if $\|f\sigma * f\sigma\|_2^2 \geq (1-\delta)\mathcal{S}^4\|f\|_2^4$. We call f δ-quasiextremal if $\|f\sigma * f\sigma\|_2^{1/2} \geq \delta\|f\|_2$.*

Suppose that $f + g$ is a δ-nearly extremal, $f \perp g$, and $\|f+g\|_2 = 1$.

1. If $\|g\|_2 \geq \varepsilon$ then g is an $\eta(\varepsilon)$-quasiextremal, provided $\delta \leq \delta(\varepsilon)$.
2. If $\min(\|f\|_2, \|g\|_2) \geq \varepsilon$ then $\|f\sigma * g\sigma\|_2 \geq \eta(\varepsilon)$, provided $\delta \leq \delta(\varepsilon)$.

The first principle above means that if g is a component of a near extremizer, then it may be far from extremizing, but in a manner controlled by $\|g\|_2$. The second principle is used to rule out certain structural properties of near extremals. Two examples are when the supports of f and g are too dissimilar, or when f has high frequencies and g has low frequencies. Then we can obtain upper bounds on $\|f * g\|_2$ that contradict the second principle. These themes are used repeatedly in the technically intensive proof that follows concentration compactness ideas.

We first make the observation that from every extremizing sequence, we can obtain a nonnegative, symmetric extremizing sequence. Then, using principles (1) and (2), we are able to show that any near extremizer must concentrate on a certain cap (to be defined in the technical tools section) up to a small error. In an extremizing sequence, up to taking subsequences, there are then two cases. The caps have radii shrinking to 0, or the radii are uniformly bounded below. In either case, we are able to obtain a certain compactness for the sequence. In the former case, we have to pull-back the extremizing sequence to $\mathbb{R}^2$ to do the analysis and in the latter, the precompactness follows for the original sequence. The last task then is to rule out that the radii shrink to zero. Since these functions now concentrate in an arbitrarily small neighborhood of a point, we are able to use some results of Foschi [3] on the analogous problem for the paraboloid to eliminate this case.

8.3 Sufficient properties for extremizing sequences

8.3.1 Nonnegativity

By the pointwise inequality $|f\sigma * f\sigma| \le |f|\sigma * |f|\sigma$, the relation $\widehat{\mu * \nu} = \widehat{\mu}\widehat{\nu}$, and Plancherel's theorem, we have

Lemma 4. *For any complex-valued function $f \in L^2(S^2)$,*

$$\|\widehat{f\sigma}\|_{L^4(\mathbb{R}^3)} \le \|\widehat{|f|\sigma}\|_{L^4(\mathbb{R}^3)}.$$

In particular, if f is an extremizer for inequality (1)*, then so is $|f|$; if $\{f_\nu\}$ is an extremizing sequence, so is $\{|f_\nu|\}$.*

8.3.2 Symmetrization

Definition 5. *Let $f \in L^2(S^2)$ be nonnegative. The antipodally symmetric rearrangment $f_\star$ is the unique nonnegative element of $L^2(S^2)$ which satisfies*

$$f_\star(x) = f_\star(-x) \qquad \text{for all } x \in S^2$$

$$f_\star(x)^2 + f_\star(-x)^2 = f(x)^2 + f(-x)^2 \quad \text{for all } x \in S^2.$$

Proposition 6. *Let $f \in L^2(S^2)$ be nonnegative. Then $\|f\sigma * f\sigma\|_{L^2(\mathbb{R}^3)} \le \|f_\star\sigma * f_\star\sigma\|_{L^2(\mathbb{R}^3)}$ with strict inequality unless $f = f_\star$ a.e. Thus any extremizer of the inequality* (1) *satisfies $|f(-x)| = |f(x)|$ for a.e. $x \in S^2$.*

Sketch of proof: first note that for $h \ge 0$ in $L^2(S^2)$, we can write

$$\|h\sigma * h\sigma\|^2_{L^2(\mathbb{R}^3)} = \int h(a)h(b)h(c)h(d)d\lambda(a,b,c,d) \tag{3}$$

where λ is a nonnegative measure supported on the set where $a + b = c + d$ and which is invariant under $(a,b,c,d) \mapsto (b,a,c,d)$, $(a,b,c,d) \mapsto (c,d,a,b)$, and $(a,b,c,d) \mapsto (a,-c,-b,d)$.

The invariance follows from the identities $f\sigma * g\sigma = g\sigma * f\sigma$, $\langle f\sigma * g\sigma, h\sigma * k\sigma\rangle = \langle h\sigma * k\sigma, f\sigma * g\sigma\rangle$, and $\langle f\sigma * g\sigma, h\sigma * k\sigma\rangle = \langle f\sigma * \tilde{h}\sigma, \tilde{g}\sigma * k\sigma\rangle$ (here $\tilde{F}(x) = F(-x)$). Let G be the group generated by the symmetries of $(\mathbb{R}^3)^4$ which they generate. The size of G is 48. By a *generic point*, we mean one whose orbit under G has cardinality 48. In (3), it suffices to integrate over all generic 4-tuples (a,b,c,d) satisfying $a + b = c + d$, since they form a set of full λ-measure.

Letting Ω denote the set of full orbits $\mathcal{O}$, we have for a certain nonnegative measure $\tilde{\mathcal{O}}$,

$$\int h(a)h(b)h(c)h(d)d\lambda(a,b,c,d) = \int_{\Omega} \sum_{(a,b,c,d)\in\mathcal{O}} h(a)h(b)h(c)h(d)d\tilde{\lambda}(\mathcal{O}).$$

Thus, it suffices to show that

$$\sum_{(a,b,c,d)\in\mathcal{O}} h(a)h(b)h(c)h(d) \leq \sum_{(a,b,c,d)\in\mathcal{O}} h_\star(a)h_\star(b)h_\star(c)h_\star(d).$$

Without loss of generality, we can assume $f(a)^2 + f(-a)^2 = 1$ and that the same holds simultaneously for b, c, d. Note that this means $f_\star(x) = 2^{-1/2}$ for each $x \in \{\pm a, \pm b, \pm c, \pm d\}$. By writing the $f(x)$ as the cosine of some angle and $f(-x)$ as the sine, we can write $\frac{1}{8}\sum_{(a,b,c,d)\in\mathcal{O}} h(a)h(b)h(c)h(d)$ as a relatively simple trigonometric expression. The expression has a maximum of exactly $\frac{3}{2}$, i.e. $\frac{1}{8}$ of the right hand side of the inequality above.

8.4 Technical tools: Caps and gauge functions

Define a cap $\mathcal{C}(z,r) \subset S^2$ along the lines of Moyua, Vargas, and Vega [2].

Definition 7. *The cap $\mathcal{C} = \mathcal{C}(z,r)$ with center $z \in S^2$ and radius $r \in (0,1]$ is the set of all points $y \in S^2$ which lie in the same hemisphere, centered at z, as z itself, and which satisfy $|\pi_{H_z}(y)| < r$, where the subspace $H_z \subset \mathbb{R}^3$ is the orthogonal complement of z and π_{H_z} denotes the orthogonal projection onto H_z.*

Definition 8. *An even function $f \in L^2(S^2)$ is said to be upper even-normalized with respect to Θ, $\mathcal{C} = \mathcal{C}(z,r)$ if f can be decomposed as $f = f_+ + f_-$ where $f_-(x) = \overline{f_+(-x)}$ and f_+ satisfies $\|f_+\|_2 \leq C < \infty$,*

$$\int_{|f_+(x)|\geq Rr^{-1}} |f_+(x)|^2 d\sigma(x) \leq \Theta(R) \quad \textit{for all } R \geq 1,$$

$$\int_{|x-z|\geq Rr} |f_+(x)|^2 d\sigma(x) \leq \Theta(R) \quad \textit{for all } R \geq 1.$$

Here $\Theta(R) : [1,\infty) \to (0,\infty)$ satisfies $\Theta(R) \to 0$ as $R \to \infty$.

This means that f cannot be too large in magnitude, and that to a degree, f is localized.

8.5 Almost upper even-normalization

Theorem 9. *For any $\varepsilon > 0$ there exists $\delta > 0$ such that any nonnegative even function $f \in L^2(S^2)$ satisfying $\|f\|_2 = 1$ which is δ-nearly extremal may be decomposed as $f = F + G$, where F, G are even and nonnegative with disjoint supports, $\|G\|_2 < \varepsilon$, and F is upper even-normalized with respect to some cap.*

This result relies on the first principle mentioned above, as well as the repeated application of a technical lemma of Moyua, Vargas, and Vega [2].

8.6 Resulting concentration

Supposing that the caps $\mathcal{C}_\nu$ associated to the f_ν in an extremizing sequence have radii less than or equal to some $r_0 < 1$, we can obtain a sequence of pullbacks $\phi_\nu^*(f_\nu) \in L^2(\mathbb{R}^2)$ such that $\|\phi_\nu^*(f_\nu)\|_{L^2(\mathbb{R}^2)} \asymp \|f_\nu\|_{L^2(S^2)}$ and the $\phi_\nu^*(f_\nu)$ are uniformly upper even-normalized with respect to the unit ball in $\mathbb{R}^2$.

Theorem 10 (Christ–Shao '10)**.** *Let $\{f_\nu\} \subset L^2(S^2)$ be an extremizing sequence of nonnegative even functions for the inequality* (2)*, satisfying $\|f_\nu\|_2 \equiv 1$. Suppose that each f_ν is upper even-normalized with respect to a cap $\mathcal{C}_\nu = \mathcal{C}(z_\nu, r_\nu)$, with constants uniform in ν. Then for any $\varepsilon > 0$ there exists $C_\varepsilon < \infty$ with the following property.*

(i) For every ν, if $r_\nu \le \frac{1}{2}$ then $\phi_\nu^*(f_\nu)$ may be decomposed as $\phi_\nu^*(f_\nu) = G_\nu + H_\nu$ where $\|H_\nu\|_2 < \varepsilon$, G_ν is supported where $|x| \le C_\varepsilon$, and $\|G_\nu\|_{C^1} \le C_\varepsilon$.

(ii) If $r_\nu \ge \frac{1}{2}$ then f_ν itself may be decomposed as $f_\nu = g_\nu + h_\nu$ where $\|h\|_2 < \varepsilon$ and $\|g_\nu\|_{C^1} \le C_\varepsilon$.

Note that $1/2$ is not special, but could be any $1 > r_0 > 0$.

The idea of the argument is that the nonnegativity and uniform upper even-normalization of the $\{\phi_\nu^*(f_\nu)\}$ ($L^2(\mathbb{R}^2)$ normalized to be ≈ 1) lead to a lower bound $\int_{|\xi| \lesssim 1} |\widehat{\phi_\nu^*(f_\nu)}(\xi)|^2 d\xi \ge \alpha$ uniform in ν. If the $\widehat{\phi_\nu^*(f_\nu)}$ also have high frequency components, then an upper bound on $\|(\text{hi freq}) * (\text{lo freq})\|_{L^2(\mathbb{R}^3)}$ (the resulting cross-term in the bilinear expression $f\sigma * f\sigma$) leads to a contradiction of extremality. Thus the $\{\phi_\nu^*(f_\nu)\}$ are composed of a slowly-varying (G_ν) plus perhaps an intermediately-varying (H_ν) portion.

By decomposing the extremizing sequence as compact plus small, applying Rellich's lemma gives the following corollary.

Corollary 11. *Let* $\{f_\nu\}$ *be as above. If* $r_\nu \to 0$ *then* $\{\phi^*_\nu(f_\nu)\}$ *is precompact in* $L^2(\mathbb{R}^2)$*. If* $\liminf_{\nu\to\infty} r_\nu > 0$ *then* $\{f_\nu\}$ *is precompact in* $L^2(S^2)$*.*

8.6.1 Ruling out concentration to a δ-function

In order to rule out the case where $r_\nu \to 0$, we make a comparison to the paraboloid $\mathbb{P}^2 = \{(x_1, x_2, x_3) : x_3 = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2\}$ case studied by Foschi [3]. Define

$$\mathbf{P} := \sup_{0 \neq g \in L^2(\mathbb{P}^2)} \frac{\|g\sigma_P * g\sigma_P\|^{1/2}_{L^2(\mathbb{R}^3)}}{\|g\|_{L^2(\mathbb{P}^2)}}$$

where the measure on $\mathbb{P}^2$ is given by $d\sigma_P = dx_1 dx_2$. In [3], Foschi calculated $\mathbf{P}$. Thus, by calculating $\|f * f\|^{1/2}_{L^2(|R^3)}/\|f\|_{L^2(S^2)}$ for $f \equiv 1$, we observe that

$$\mathbf{S} \geq 2^{1/4}\mathbf{P}. \tag{4}$$

The contradiction to $r_\nu \to 0$ arises by a rescaling and transference argument leading to a sequence $\{\tilde{f}_\nu\}$ on $\mathbb{P}^2$. Using Foschi's precompactness result for the paraboloid, we have a limit $F \in L^2(\mathbb{P}^2)$ which satisfies $\|\widehat{F\sigma_P}\|_4/\|F\|_2 = (3/2)^{-1/4}\lim_{\nu\to\infty}\|\widehat{f_\nu\sigma}\|_4/\|f_\nu\|_2$. It follows that $\mathbf{P} \geq (3/2)^{-1/4}\mathbf{S}$, which contradicts (4).

We note that (4) relies on a calculation that exploits the particular symmetries of S^2. To generalize this approach, we may instead do a variational calculation giving a sufficient inequality for the purposes of this proof.

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9 Heat-flow monotonicity of Strichartz norms

after J. Bennett, N. Bez, A. Carbery and D. Hundertmark [1]
A summary written by Lisa Onkes

Abstract

We prove that in some low dimensional cases the Strichartz norm $\|e^{it\Delta}u_0\|_{L^p_t L^q_x}$ is nondecreasing if the initial datum u_0 evolves under a certain quadratic heat flow. In the second part we extend this result to higher dimensions for a closely related norm.

9.1 Introduction: Strichartz estimates for the free Schrödinger equation

The free Schrödinger equation in dimension $d \in \mathbb{N}$ is given by

$$\begin{cases} iu_t = -\Delta_x u, & (t,x) \in \mathbb{R} \times \mathbb{R}^d \\ u(0,x) = u_0(x) \end{cases} . \tag{1}$$

Taking the Fourier transform (for $u_0 \in \mathcal{S}(\mathbb{R}^d)$) of this equation shows that a solution is given by $u(t,x) := e^{it\Delta}u_0$, where we define the operator $e^{it\Delta}$ via the Fourier multiplier $e^{-it|\xi|^2}$

$$\widehat{e^{it\Delta}u_0}(\xi) := e^{-it|\xi|^2}\widehat{u_0}(\xi) = \widehat{K_t}(\xi)\widehat{u_0}(\xi), \tag{2}$$

and the Schrödinger kernel K_t is defined as $K_t(x) := \frac{1}{(4\pi it)^{\frac{d}{2}}} e^{-\frac{|x|^2}{4t}}$.
Applying the inverse Fourier transform for $u_0 \in \mathcal{S}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$ to (2) we get

$$e^{it\Delta}u_0 = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{i(x\cdot\xi - t|\xi|^2)}\widehat{u_0}(\xi)\mathrm{d}\xi = K_t * u_0.$$

The work of Strichartz in 1977 (case $p = q$, [6]) and Keel and Tao ($p \neq q$, [4], 1998) shows that the solution operator $e^{it\Delta}$ extends to a bounded operator from $L^2(\mathbb{R}^d)$ to $L^p_t L^q_x(\mathbb{R} \times \mathbb{R}^d) := L^p_t(\mathbb{R}, L^q_x(\mathbb{R}^d))$ iff (p,q,d) is Schrödinger-admissible that is

$$p, q \geq 2, \qquad (p,q,d) \neq (2,\infty,2) \qquad \text{and} \qquad \frac{2}{p} + \frac{d}{q} = \frac{d}{2}. \tag{3}$$

In this case there exists a finite constant $C_{p,q}$ such that

$$\|e^{it\Delta}u_0\|_{L_t^p L_x^q(\mathbb{R}\times\mathbb{R}^d)} \leq C_{p,q}\|u_0\|_{L^2(\mathbb{R}^d)}. \tag{4}$$

Hundertmark and Zharnitsky [3] found the here best possible constants in the cases $(p,q,d) = (6,6,1)$, $(4,4,2)$ and showed that those are attained iff u_0 is a Gaussian. As we will later see, one can estimate the best constant in the case $(8,4,1)$ from the two-dimensional case. Notice, that these are exactly all the cases, where q is an even integer which divides p!

9.2 The heat-flow monotonicity property

We want to prove the following monotonicity property which immediately yields the optimal constants in (4) as a corollary:

Theorem 1. *Let $f \in L^2(\mathbb{R}^d)$, (p,q,d) Schrödinger-admissible and let q be an even integer which divides p. Then*

$$Q_{p,q}(s) := \|e^{it\Delta}\left(e^{s\Delta}|f|^2\right)^{\frac{1}{2}}\|_{L_t^p L_x^q(\mathbb{R}\times\mathbb{R}^d)}$$

is nondecreasing for all $s > 0$.

The here appearing heat operator $e^{t\Delta}$ is defined as the Fourier multiplier operator with multiplier $e^{-t|\xi|^2}$. Direct computations show

$$e^{t\Delta}f = H_t * f,$$

where $H_t(x) := \frac{1}{(4\pi t)^{\frac{d}{2}}}e^{-\frac{|x|^2}{4t}}$, called the heat kernel.
To find the sharp constants in equation (4) one shows

$$\begin{aligned}\lim_{s\to\infty} Q_{p,q}(s) &= \|e^{it\Delta}(H_1(x)\cdot\|f\|_{L^2}^2)^{\frac{1}{2}}\|_{L_t^p L_x^q(\mathbb{R}\times\mathbb{R}^d)} \\ &= \|e^{it\Delta}(H_1(x))^{\frac{1}{2}}\|_{L_t^p L_x^q(\mathbb{R}\times\mathbb{R}^d)}\|f\|_{L^2}\end{aligned}$$

and

$$\lim_{s\to 0} Q_{p,q}(s) = \|e^{it\Delta}|f|\|_{L_t^p L_x^q(\mathbb{R}\times\mathbb{R}^d)} \geq \|e^{it\Delta}f\|_{L_t^p L_x^q(\mathbb{R}\times\mathbb{R}^d)},$$

where in the last inequality we used that q is an even integer, which divides p.

From the monotonicity of $Q_{p,q}$ it now immediately follows, that the sharp constants are at most $\|e^{it\Delta}(H_1(x))^{\frac{1}{2}}\|_{L_t^p L_x^q(\mathbb{R}\times\mathbb{R}^d)}$ as well as that this constant is attained for the Gaussian $H_1^{\frac{1}{2}}$. One can further show, that all Gaussians are extremisers, see [3].

9.2.1 Proof sketch of the monotonicity property

By using that the heat kernel combined with the convolution builds a semi-group, the Cauchy-Schwarz inequality and $\int H_t * f = \int f$ one can show:

Lemma 2. *For nonnegative functions $f_1, f_2 \in L^1(\mathbb{R}^d)$, the quantity*

$$\Lambda_{f_1,f_2}(s) := \int_{\mathbb{R}^d} (e^{s\Delta} f_1)^{\frac{1}{2}} (e^{s\Delta} f_2)^{\frac{1}{2}}$$

is nondecreasing for all $s > 0$.

To shorten our notation we will from now on only look at dimension 1. The proof in 2 dimensions works in the same way.
The next step is to prove the following representation of Strichartz-norms, which was shown in [3].

Lemma 3. *For $f \in L^2(\mathbb{R})$ we have*

$$\|e^{it\Delta} f\|_{L^6_{t,x}(\mathbb{R}\times\mathbb{R})} = \frac{1}{2\sqrt{3}} \langle f \otimes f \otimes f, P(f \otimes f \otimes f) \rangle,$$

where $P : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ is the projection operator onto the subspace of functions on $\mathbb{R}^3$ invariant under the isometries that fix the direction $(1, 1, 1)$.

Here we defined $(f \otimes f \otimes f)((x_1, x_2, x_3)) := f(x_1)f(x_2)f(x_3)$.

Proof sketch. To shorten our notation lets define $F(X) := (f \otimes f \otimes f)(X)$. We expand

$$|e^{it\Delta} f|^6 = \frac{1}{(4\pi t)^3} \int_{\mathbb{R}^3} e^{i\frac{3|x|^2 - 2x(1,1,1)\cdot\eta + |\eta|^2}{4t}} F(\eta) \mathrm{d}\eta \cdot \int_{\mathbb{R}^3} e^{-i\frac{3|x|^2 - 2x(1,1,1)\cdot\xi + |\xi|^2}{4t}} \overline{F(\xi)} \mathrm{d}\xi,$$

integrate with respect to x and carry out a change of variables to obtain for $F \in C_0^\infty(\mathbb{R}^3)$

$$\begin{aligned}\int_{\mathbb{R}} |e^{it\Delta} f|^6 \mathrm{d}x &= \frac{2t}{(4\pi t)^3} \int_{\mathbb{R}} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} e^{-ix(1,1,1)\cdot(\eta-\xi)} \quad e^{i\frac{|\eta|^2 - |\xi|^2}{4t}} F(\eta) \overline{F(\xi)} \mathrm{d}\eta \mathrm{d}\xi \mathrm{d}x \\ &= \frac{2t}{(4\pi t)^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} 2\pi \cdot \delta((1,1,1)\cdot(\eta - \xi)) e^{i\frac{|\eta|^2 - |\xi|^2}{4t}} F(\eta) \overline{F(\xi)} \mathrm{d}\eta \mathrm{d}\xi.\end{aligned}$$

Doing the same in t leads to

$$\begin{aligned}\int_{\mathbb{R}} \int_{\mathbb{R}} |e^{it\Delta} f|^6 \mathrm{d}x \mathrm{d}t &= \frac{1}{2\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \delta((1,1,1)\cdot(\eta - \xi)) \delta(|\eta|^2 - |\xi|^2) F(\eta) \overline{F(\xi)} \mathrm{d}\eta \mathrm{d}\xi \\ &=: \langle F, AF \rangle_{L^2_\eta(\mathbb{R}^3)}.\end{aligned}$$

To finish the proof the following two steps are left to do:

1.) Show that $A : C_0^\infty \to L^2$ and extends to an operator from L^2 into L^2.

2.) Show that $A = \frac{1}{2\sqrt{3}} P$.

Comment on step 2: A change of variables shows that AF (for $F \in C_0^\infty(\mathbb{R}^3)$) stays invariant under isometries that fix the direction $(1,1,1)$, because those isometries don't change the input of the δ-functions. By step 1 this extends to all $F \in L^2(\mathbb{R}^3)$ and thus A maps into the range of P. Because A is symmetric it follows that A maps the complement of the range of P to $\{0\}$ and it remains to check, that A acts as the wanted multiple on the range of P. □

We can now conclude the monotonicity property:
By using that every isometry ρ on $\mathbb{R}^3$ has the form $\rho(x) = \rho(0) + \tilde{\rho}x$, where $\tilde{\rho}$ is an orthogonal matrix, a change of variables shows that $(e^{t\Delta}|F|^2)(\rho\cdot) = (e^{t\Delta}|F_\rho|^2)$ where $F_\rho = F(\rho\cdot)$. Therefore we get

$$(e^{s\Delta}|f|^2 \otimes e^{s\Delta}|f|^2 \otimes e^{s\Delta}|f|^2)(\rho\cdot) = (e^{s\Delta}|F|^2)(\rho\cdot) = (e^{s\Delta}|F_\rho|^2),$$

because $e^{s\Delta}$ commutes with tensor products.
Let $\mathcal{O}$ be the group of isometries on $\mathbb{R}^3$ that coincide with the identity on the span of $(1,1,1)$. One can than calculate, that for $G \in L^2(\mathbb{R}^3)$ the projection P is given by

$$PG(\cdot) = \int_{\mathcal{O}} G(\rho\cdot)\mathrm{d}\mathscr{H}(\rho),$$

where $\mathrm{d}\mathscr{H}$ denotes the right-invariant Haar probability measure on $\mathcal{O}$. Therefore we have by the previous lemma and the above statements

$$Q_{6,6}(s) = \frac{1}{2\sqrt{3}} \int_{\mathbb{R}^3} (e^{s\Delta}|F|^2)^{\frac{1}{2}} \int_{\mathcal{O}} (e^{s\Delta}|F_\rho|^2)^{\frac{1}{2}} \mathrm{d}\mathscr{H} = \frac{1}{2\sqrt{3}} \int_{\mathcal{O}} \Lambda_{|F|^2,|F_\rho|^2}(s)\mathrm{d}\mathscr{H},$$

which is by lemma 2 and the positivity of $\mathrm{d}\mathscr{H}$ nondecreasing. As previously mentioned the proof for the case $(p,q,d) = (4,4,2)$ works analogous. For $(p,q,d) = (8,4,1)$ we use again, that the operators $e^{it\Delta}$ and $e^{s\Delta}$ commute with tensor products and therefore

$$\|e^{it\Delta}(e^{s\Delta}|f|^2)^{\frac{1}{2}}\|^2_{L_t^8 L_x^4(\mathbb{R}\times\mathbb{R})} = \|e^{it\Delta}(e^{s\Delta}(|f|^2 \otimes |f|^2))^{\frac{1}{2}}\|_{L_t^4 L_x^4(\mathbb{R}\times\mathbb{R}^2)},$$

which is nondecreasing by the 2-dimensional case.

Remark 4. *One can use the same strategy as above to proof the monotonicity of the Strichartz norm for input which evolves according to a quadratic Mehler flow. That is, for a bounded and compactly supported function f which fulfils the assumptions of theorem 1 the quantity $\|e^{it\Delta}(e^{-|\cdot|^2}e^{tL}|f|^2)^{\frac{1}{2}}\|_{L_t^pL_x^q(\mathbb{R}\times\mathbb{R}^d)}$, where $L = \Delta - \langle x, \nabla\rangle$, is nondecreasing for all $s > 0$.*

9.3 Higher dimensions

It is already proven that for all nonendpoint ($p \neq 2$) Schrödinger-admissible (p, q, d) the best possible constant is attained (see Shao [5]), however the extremisers are in general not known. We would like to generalise theorem 1 to all dimensions. Therefore we consider the case $p = q = 2 + \frac{4}{d} =: p(d)$. Unfortunately $p(d)$ is not an even integer for $d \geq 3$ (then we would be finished). Our approach is to embed the Strichartz norm into a family of norms $\|\!|\cdot|\!\|_p$ which fulfil the statement of theorem 1 for even integers p. For this procedure we choose for $f \in \mathcal{S}(\mathbb{R}^d)$ the norm

$$\|\!|f|\!\|_p^p := \left(\frac{p(d)}{\pi}\right)^{\frac{d}{2}} \frac{1}{(2\pi)^{d+2}} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_0^\infty\int_{\mathbb{R}} \left|\int_{\mathbb{R}^d} e^{-|z-\sqrt{\zeta}\xi|^2} e^{i(x\cdot\xi - t|\xi|^2)}\widehat{f}(\xi)\mathrm{d}\xi\right|^p \frac{\zeta^{\nu-1}}{\Gamma(\nu)}\mathrm{d}t\ \mathrm{d}\zeta\ \mathrm{d}z\ \mathrm{d}x,$$

where $\nu := \frac{d(p-p(d))}{4}$, and proof the following:

Theorem 5. *As p tends to $p(d)$, the norm $\|\!|f|\!\|_p$ converges to the Strichartz norm $\|e^{it\Delta}f\|_{L_{t,x}^{p(d)}}$ for all $f \in \mathcal{S}(\mathbb{R}^d)$. If p is an even integer, then*

$$Q_p(s) := \|\!|(e^{s\Delta}|f|^2)^{\frac{1}{2}}|\!\|_p$$

is nondecreasing for all $s > 0$.

Comment on the proof. The proof of this theorem turns out to be very similar to our proof of theorem 1. We again (compare to lemma 3) first show that

$$\|\!|f|\!\|_p^p = C_{d,p}\int_{\mathbb{R}^{\frac{pd}{2}}} F(X)PF(X)\mathrm{d}X,$$

where P is the orthogonal projection onto functions on $\mathbb{R}^{\frac{pd}{2}}$ which are invariant under the action of $\mathcal{O}$, the group of isometries that coincide with the identity on W, the span of $\mathbf{1}_1, \ldots, \mathbf{1}_d \in \mathbb{R}^{\frac{pd}{2}}$ where for each $1 \leq j \leq d$ $\mathbf{1}_j := (e_j, \ldots, e_j)$ and e_j denotes the jth standard basis vector of $\mathbb{R}^d$. We than again express P through the right-invariant Haar probability measure on $\mathcal{O}$ and use lemma 2 to prove the nondecreasingness. □

Remark 6. *It is possible to generalise lemma 2 to higher exponents then* $1/2$ *by multiplying with a polynomially growing factor. Bennett, Carbery, Christ and Tao prove in [2], that for all* $\alpha \in [1/2, 1]$ *the quantity*

$$s^{d(\alpha-1/2)/2} \int_{\mathbb{R}^d} (e^{s\Delta} f_1)^\alpha (e^{s\Delta} f_2)^\alpha,$$

where f_1, f_2 *are functions which satisfies the assumptions of lemma 2, is nondecreasing for all* $s > 0$.
By replacing lemma 2 with this statement in the proof of theorem 1 and 5 one also obtains these results for higher exponents.

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10 Multidimensional van der Corpus and sublevel set estimates

after A. Carbery, M. Christ, and J. Wright [2]
A summary written by Guillermo Rey

10.1 Introduction

In this article the authors consider the principle:

> If a function has a large derivative, then it changes rapidly and hence cannot spend too much time near any particular value.

Their main goal is to quantify this principle.

Suppose u is a smooth real valued function on $\mathbb{R}$ and assume that $u^{(k)} \geq 1$ everywhere. According to the above principle the sublevel set $\{x : |u(x)| \leq \alpha\}$ should be decrease as $\alpha \to 0$, and the rate of decay should be faster for smaller k.

One could also consider the oscillatory integral

$$I(\lambda) := \int_a^b e^{i\lambda u(t)}\, dt.$$

If the derivative of u is very small then we would expect the phase to be essentially constant, and hence $e^{i\lambda u(t)}$ should not introduce much cancellation, at least for small λ. But when the derivative of u is large then we should expect cancellation.

The authors give quantitative versions of these intuitive statements, furthermore their results will be *uniform* in a sense which we'll describe later.

10.2 The one-dimensional case

Much more is known about the problems discussed in the introduction if we restrict the dimension to one. In fact, a very precise estimate is known for both questions. The following estimates can be found in an article by Arhipov, Karacuba and Čubarikov in [1].

Proposition 1. *If u is a smooth function satisfying $u^k(t) \geq 1$ on $\mathbb{R}$, then*

$$|\{t : |u(t)| \leq \alpha\}| \leq (2e)((k+1)!)^{1/k}\alpha^{1/k}.$$

For the oscillatory integral estimate we have

Proposition 2. *There exists an absolute constant C such that for any $a < b$, any $k \geq 2$, and any smooth function u satisfying $u^{(k)} \geq 1$*

$$\left| \int_a^b e^{i\lambda u(t)}\, dt \right| \leq Ck|\lambda|^{\frac{-1}{k}}.$$

Let us give the main ideas of the proof of the first proposition:

1. First we let
$$E = \{t : |u(t)| \leq \alpha\}.$$
One can show that it is possible to find $k+1$ points: $a_0, \ldots, a_k$ which are well-spaced in the sense that
$$|E|^k \leq (2e)^k \prod_{j \neq l} |a_j - a_l| \quad \forall l. \tag{1}$$

2. Next, one can give a higher-order version of the Mean Value Theorem. It essentially says that one can find a point ζ such that
$$u^{(k)}(\zeta) = k! \sum_j \pm u(a_j) \prod_{j \neq l} |a_j - a_l|^{-1} \tag{2}$$

3. Finally, since $u^{(k)} \geq 1$, we can reorganize (2) and use the estimate from (1) to conclude.

Proposition 2 follows from Proposition 1. Indeed, one can write

$$\begin{aligned} \int_a^b e^{i\lambda u(t)}\, dt &= \int_{\{t \in (a,b):\, |u'(t)| \leq \beta\}} e^{i\lambda u(t)}\, dt + \int_{\{t \in (a,b):\, |u'(t)| > \beta\}} e^{i\lambda u(t)}\, dt \\ &=: I + II. \end{aligned}$$

For I we can use Proposition 1 with $v = u'$, while for II one should expploit the large cancellation arising from the fact that u' is large on the set where we are integrating. In particular, the set $\{t : |u'(t)| > \beta\}$ is a union of $O(k)$ intervals, and for each interval one can integrate by parts to exploit the cancellation. To conclude one optimises β and the result follows.

10.3 Higher dimensions

Section 3 of the article gives versions of the results from the one-dimensional case which work in higher dimensions. Let us fix notation first.

We will assume throughout that u is a smooth function on $Q^n = [0,1]^n$. Let

$$E_\alpha = \{x \in Q : |u(x)| \leq \alpha\}.$$

If $n_1 + n_2 = n$ one can define an operator S_α which takes functions on Q^{n_2} to functions on Q^{n_1} as follows:

$$S_\alpha f(x) = \int_{Q^{n_2}} \mathbb{1}_{E_\alpha}(x,y) f(y)\, dy,$$

where $\mathbb{1}_E$ denotes the characteristic function of the set E, and where by (x,y) we understand the point in Q^n whose first n_1 coordinates are x and the next n_2 coordinates are y.

The main use of the operator S_α is that $L^p \to L^q$ bounds for this operator imply estimates on the size of E_α. For example if $\|S_\alpha\|_{L^\infty \to L^1} \leq C_0$ then

$$\begin{aligned} |E_\alpha| &= \int_{Q^{n_1}} \int_{Q^{n_2}} \mathbb{1}_{E_\alpha}(x,y)\, dy\, dx \\ &= \left\| S_\alpha(\mathbb{1}_{Q^{n_2}}) \right\|_{L^1(Q^{n_1})} \\ &\leq C_0. \end{aligned}$$

However, having an operator, instead of just a sublevel set at our disposal gives us more felxibility (and in particular will allow us to use the T^*T method in a streamlined fashion).

10.3.1 The two-dimensional case

The two-dimensional case already has most of the ideas, so we will describe this case first.

Suppose for the sake of exposition that $0 \leq j, k \leq d$ and that

$$\frac{\partial^d u}{\partial x^j \partial x^k} \geq 1 \quad \text{on } Q = [0,1]^2, \tag{3}$$

where $j + k = d$.

To see which kind of estimates to expect, the authors include the following examples:

Proposition 3. *For each j, k there exists a $C < \infty$ such that the following hold:*

1. *If $u(x,y) = -(x-y)^d$ with $d \geq 2$, then $\|S_\alpha\|_{L^p \to L^q} \geq C\alpha^{1/d}$.*
2. *If $u(x,y) = x^j y^k$, then $\|S_\alpha\|_{L^p \to L^q} \geq C\alpha^{1/jq}$.*
3. *If $u(x,y) = x^k y^k$, then $\|S_\alpha\|_{L^p \to L^q} \geq C\alpha^{1/kp'}$.*

Proof. If $f = \mathbb{1}_{(0,1)}$ then $S_\alpha f \simeq \alpha^{1/d}$, and hence the first result follows.

The second result follows from noting that

$$S_\alpha f(x) = \int_0^1 f(y)\, dy$$

for $x \in (0, \alpha^{1/j})$.

The second and third results are equivalent by duality so this finishes the proof. ☐

As a corollary, we have

$$\sup \|S_\alpha\|_{L^p \to L^q} \gtrsim \alpha^{\min(\frac{1}{d}, \frac{1}{jq}, \frac{1}{kp'})},$$

where the supremum is taken over all functions u which satisfy (3), so this is the best we can expect for upper estimates.

The main result in the two-dimensional case is the following:

Theorem 4. *There exists an absolute constant C so that if u satisfies* (3) *with $j = k = 1$, then*

$$\|S_\alpha\|_{L^2 \to L^2} \leq C\big(\alpha \log(1/\alpha)\big)^{1/2} \quad \forall \alpha \in (0, 1/2).$$

Let us sketch the main ideas of the proof. We begin with the lemma

Lemma 5. *For u as in the previous theorem, set $E = E_\alpha$ and let*

$$E(y) = \{x : (x,y) \in E\}.$$

Then

$$|E(y_1) \cap E(y_2)| \leq \frac{4\alpha}{|y_1 - y_2|}. \tag{4}$$

This can be seen as a two-dimensional version of 1

Now the estimate follows from a T^*T argument:

$$\begin{aligned}
\|S_\alpha f\|^2 &= \int |S_\alpha f(x)|^2\,dx \\
&= \int \Big(\int \mathbb{1}_{E_\alpha}(x,y) f(y)\,dy\Big)^2\,dx \\
&= \int\int\int \mathbb{1}_{E_\alpha}(x,y_1) f(y_1) \mathbb{1}_{E_\alpha}(x,y_2) f(y_2)\,dy_1\,dy_2\,dx \\
&= \int\int |E(y_1)\cap E(y_2)| f(y_1) f(y_2)\,dy_1\,dy_2 \\
&\le \iint_{|y_1-y_2|\le 4\alpha} f(y_1)f(y_2)\,dy_1\,dy_2 + 4\alpha \iint_{|y_1-y_2|>4\alpha} \frac{f(y_1)f(y_2)}{|y_1-y_2|}\,dy_1\,dy_2,
\end{aligned}$$

where we have used the previous lemma in the last line.

The first summand is bounded by $4\alpha\|f\|_2^2$, while for the second one we can use Cauchy-Schwarz to conclude

$$4\alpha \iint_{|y_1-y_2|>4\alpha} \frac{f(y_1)f(y_2)}{|y_1-y_2|}\,dy_1\,dy_2 \lesssim 4\alpha \log(1/4\alpha)\|f\|_2^2$$

and the proof follows.

For dimensions larger than two the authors use an induction argument, but then the exponents behave much worse:

Theorem 6. *For each $n \ge 1$ and each multiindex β there exists an $\varepsilon > 0$ and $C < \infty$ such that for any real-valued smooth function satisfying $\partial^\beta u \ge 1$ we have*

$$|\{x \in Q : |u(x)| \le \alpha\}| \le C\alpha^\varepsilon.$$

A similar version for the oscillatory integral estimate follows from this last theorem.

10.4 Applications and further remarks

The results of this paper have certain combinatorial applications. Consider the following problem:

Conjecture 7. *There exists a constant $\varepsilon_0 > 0$ such that for any $E \subset Q = [0,1]^2$ with $|E| > 0$ one can always find four points $A, B, C, D \in E$ which are* *the vertices of a rectangle with sides parallel to the axes and whose area is at least* $\varepsilon_0|E|^2$*.*

If the conjecture is true then Theorem 4 would be true without the logarithmic term. On the other hand Theorem 4 implies that there exists such a rectangle but with area at least

$$\varepsilon_0 \frac{1}{\log |E|^{-1}} |E|^2.$$

The authors describe further applications, we refer the reader to the original article.

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11 Optimal Young's inequality and its converse: a simple proof

after F. Barthe [1]
A summary written by Johanna Richter

Abstract

The sharp form of Young's inequality for convolutions was first proven by Beckner [2] and Brascamp and Lieb [3]. The first known proof of the sharp reverse inequality for exponents less than 1 can also be found in [3]. We summarize Barthes [1] proof of Young's inequality, which is rather elementary and gives the inequality and its converse at one time.

11.1 Introduction and main result

The classical Young's inequality for convolutions states that

$$\|f * g\|_r \leq \|f\|_p \|g\|_q, \tag{1}$$

if $f \in L^p(\mathbb{R}), g \in L^q(\mathbb{R})$ and $p, q, r \geq 1$ satisfy $1/p + 1/q = 1 + 1/r$. For exponents $0 < p, q, r \leq 1$ and $f, g \geq 0$ Leindler [4] found the reverse form

$$\|f * g\|_r \geq \|f\|_p \|g\|_q. \tag{2}$$

Both inequalities are sharp, only if $p = 1$ or $q = 1$. In the other cases the best constants were found by Beckner [2] for (1) and Brascamp and Lieb [3] for (1) and (2). They are attained when f and g are gaussian functions, $f(x) = \exp(-|p'|x^2)$ and $g(x) = \exp(-|q'|x^2)$. Throughout this summary we use the convention $1/p + 1/p' = 1$. Note that $p' < 0$, if $0 < p < 1$.

We will present an argument, which proves the following sharp, multidimensional Young's inequality and its converse at one go.

Theorem 1. *Let $p, q, r > 0$ satisfy $1/p + 1/q = 1 + 1/r$, let $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$ be non-negative and define $C_t = \frac{t^{1/2t}}{|t'|^{1/2t'}}$ for every $t > 0$.*

(i) If $p, q, r \geq 1$, then

$$\|f * g\|_r \leq \left(\frac{C_p C_q}{C_r}\right)^n \|f\|_p \|g\|_q. \tag{3}$$

(ii) If $p, q, r \leq 1$*, then*

$$\|f * g\|_r \geq \left(\frac{C_p C_q}{C_r}\right)^n \|f\|_p \|g\|_q. \tag{4}$$

It is enough to prove Theorem 1 for $n = 1$. Via tensorisation arguments, it is possible to deduce the multidimensional case from the one-dimensional: if C is the best constant for $\mathbb{R}$, then C^n is the best constant for $\mathbb{R}^n$. This is done for example in Beckner [2] (see Topic 2).

11.2 Reformulation of the problem

For $n = 1$ and $p, q, r \neq 1$ we obtain by a change of variables the following, equivalent form of Theorem 1:

Theorem 2. *Let* $p, q, r > 0$ *satisfy* $1/p + 1/q = 1 + 1/r$*, let* $f, g \in L^1(\mathbb{R})$ *be non-negative and define* $c = \sqrt{r'/q'}$, $s = \sqrt{r'/p'}$ *and* $K(p, q, r) = \frac{p^{1/2p} q^{1/2q}}{r^{1/2r}}$.
(i) If $p, q, r > 1$*, then*

$$\left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(cx - sy)^{1/p} g(sx + cy)^{1/q} dx\right)^r dy\right)^{1/r} \leq K(p, q, r) \|f\|_1^{1/p} \|g\|_1^{1/q}.$$

(ii) If $p, q, r < 1$*, then*

$$\left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(cx - sy)^{1/p} g(sx + cy)^{1/q} dx\right)^r dy\right)^{1/r} \geq K(p, q, r) \|f\|_1^{1/p} \|g\|_1^{1/q}.$$

It is enough to prove Theorem 2 for positive, continuous functions $f, g \in L^1(\mathbb{R})$.

Indeed, for all $0 \leq f \in L^1(\mathbb{R})$ we find $0 \leq f_n \in L^1(\mathbb{R}), n \in \mathbb{N}$, such that $f_n \uparrow f$ pointwise and $f_n(x) \leq M_n \exp(-\varepsilon_n x^2) =: F_n(x)$ for some $M_n, \varepsilon_n > 0$ and all $n \in \mathbb{N}$. For $m \in \mathbb{N}$ let $\varphi_m(x) = \frac{m}{\sqrt{2\pi}} \exp(-\frac{(mx)^2}{2})$ and define $f_{n,m}(x) = \min\{(f_n * \varphi_m)(x), F_n(x)\}$. Note that the functions $f_{n,m}$ are positive and continuous and that $\lim_{m\to\infty} \|f_{n,m} - f_n\|_p = 0$ for all $p \geq 1$. Do the same construction for $0 \leq g \in L^1(\mathbb{R})$.

If Theorem 2 holds for all $f_{n,m}$ and $g_{n,m}$, then by the dominated convergence theorem, it is true for all f_n and g_n and by the monotone convergence theorem, it is true for f and g.

11.3 The key element of the proof

Lemma 3. *Let $p,q,r > 1$ satisfy $1/p + 1/q = 1 + 1/r$ and let $f, F, g, G \in L^1(\mathbb{R})$ be positive, continuous functions, such that $\int_{\mathbb{R}} f\,dx = \int_{\mathbb{R}} F\,dx$ and $\int_{\mathbb{R}} g\,dx = \int_{\mathbb{R}} G\,dx$. Then*

$$\left(\int_{\mathbb{R}}\left(\int_{\mathbb{R}} f(cx-sy)^{1/p} g(sx+cy)^{1/q} dx\right)^r dy\right)^{1/r} \\ \le \int_{\mathbb{R}}\left(\int_{\mathbb{R}} F(cx-sy)^{r/p} G(sx+cy)^{r/q} dy\right)^{1/r} dx. \quad (5)$$

Note that if $p,q,r > 1$ then $P = p/r, Q = q/r, R = 1/r$ satisfy $0 < P,Q,R < 1$ and $1/P + 1/Q = 1 + 1/R$. Further, it is $C = \sqrt{R'/P'} = s, S = \sqrt{R'/Q'} = c$. By applying Lemma 3 to $p,q,r > 1$ with suitable f, F, g, G, we obtain for $P,Q,R < 1$ and $\tilde{f}, \tilde{F}, g, G$, where $\tilde{f}(x) = f(-x), \tilde{F}(x) = F(-x)$

$$\left(\int_{\mathbb{R}}\left(\int_{\mathbb{R}} \tilde{F}(Cx-Sy)^{1/P} G(Sx+Cy)^{1/Q} dx\right)^R dy\right)^{1/R} \\ = \left(\int_{\mathbb{R}}\left(\int_{\mathbb{R}} F(cx-sy)^{r/p} G(sx+cy)^{r/q} dy\right)^{1/r} dx\right)^r \\ \ge \int_{\mathbb{R}}\left(\int_{\mathbb{R}} f(cx-sy)^{1/p} g(sx+cy)^{1/q} dx\right)^r dy \\ = \int_{\mathbb{R}}\left(\int_{\mathbb{R}} \tilde{f}(Cx-Sy)^{R/P} g(Sx+Cy)^{R/Q} dy\right)^{1/R} dx. \quad (6)$$

Proof of Lemma 3. The proof is based on a parametrization of functions. Due to the assumptions on f, F, g, G, there are two increasing bijections $u, v \in C^1(\mathbb{R})$ such that for all t

$$\int_{-\infty}^{u(t)} f(x)\,dx = \int_{-\infty}^{t} F(x)\,dx, \quad \int_{-\infty}^{v(t)} g(x)\,dx = \int_{-\infty}^{t} G(x)\,dx \quad \text{and} \\ u'(t)f(u(t)) = F(t), \quad v'(t)g(v(t)) = G(t). \quad (7)$$

Construct another differentiable bijection $\Theta = R^t T R : \mathbb{R}^2 \to \mathbb{R}^2$, using the rotation R with matrix $\begin{pmatrix} c & -s \\ s & c \end{pmatrix}$ and the bijection $T(x,y) = (u(x), v(y))$. Its

Jacobian $J\Theta$ at a point (X,Y) is equal to $J\Theta(X,Y) = u'(cX-sY)v'(sX+cY)$.

Let I denote the left-hand side of inequality (5). We may assume $I<\infty$, using the classical Young's inequality (1). Thus, the duality between L^r and $L^{r'}$ implies the existence of a positive function $h \in L^{r'}(\mathbb{R}), \|h\|_{r'} = 1$ such that

$$I = \int_{\mathbb{R}}\int_{\mathbb{R}} f(cx-sy)^{1/p} g(sx+cy)^{1/q} h(y)\,dx\,dy.$$

By the change of variables $(x,y) = \Theta(X,Y)$ and an application of Hölder's inequality, we find

$$I \le \int_{\mathbb{R}} \left(\int_{\mathbb{R}} F(cX-sY)^{r/p} G(sX+cY)^{r/q} dY\right)^{1/r} H(X)^{1/r'}\,dX,$$

where

$$H(X) = \int_{\mathbb{R}} h(a(X,Y))^{r'} u'(cX-sY)^{s^2} v'(sX+cY)^{c^2} dY \quad \text{and}$$
$$a(X,Y) = -su(cX-sY) + cv(sX+cY).$$

Using the arithmetic-geometric inequality, we obtain

$$u'(cX-sY)^{s^2} v'(sX+cY)^{c^2} \le s^2 u'(cX-sY) + c^2 v'(sX+cY) = \partial_Y a(X,Y).$$

Thus,

$$H(X) \le \int_{\mathbb{R}} h(a(X,Y))^{r'} \partial_Y a(X,Y)\,dY = \int_{\mathbb{R}} h(z)^{r'} dz = 1.$$

This proves the lemma. □

11.4 Proof of Theorem 2

For the functions

$$f(x) = F(x) = \sqrt{p/\pi}\exp(-px^2), \quad g(x) = G(x) = \sqrt{q/\pi}\exp(-qx^2),$$

we obtain equality in Lemma 3 and both sides of (5) are equal to $K(p,q,r)$. Substituting f,g by any two functions $0 < f,g \in C(\mathbb{R})$ such that $\int_{\mathbb{R}} f\,dx = \int_{\mathbb{R}} g\,dx = 1$, immediately leads to Theorem 2 for $p,q,r>1$.

As we showed in (6), Lemma 3 is valid for $0<P,Q,R<1$, but with the reverse inequality sign. Therefore, the assertion for exponents less than one follows from (6), applyed to $0<\tilde{F},G \in C(\mathbb{R})$ such that $\int_{\mathbb{R}} \tilde{F}\,dx = \int_{\mathbb{R}} G\,dx = 1$ and $\tilde{f}(x) = \sqrt{P/\pi}\exp(-Px^2), g(x) = \sqrt{Q/\pi}\exp(-Qx^2)$. □

11.5 Extremizers

Theorem 4. *Let $p, q, r > 0$ satisfy $1/p + 1/q = 1 + 1/r$ and either $p, q, r > 1$ or $p, q, r < 1$. Let further $f, g \in L^1(\mathbb{R})$ be non-negative and c, s and $K = K(p, q, r)$ defined as above. Then*

$$\left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(cx - sy)^{1/p} g(sx + cy)^{1/q} dx\right)^r dy\right)^{1/r} = K \, \|f\|_1^{1/p} \|g\|_1^{1/q} \tag{8}$$

if and only if there exist $a, b \geq 0, \lambda > 0$ and $y, z \in \mathbb{R}$ such that for all $x \in \mathbb{R}$

$$f(x) = a \exp(-\lambda p(x - y)^2), \quad g(x) = b \exp(-\lambda q(x - z)^2). \tag{9}$$

For the proof of Theorem 4 we need the following Lemma.

Lemma 5. *Let $m, n \in \mathbb{Z}, m \geq n$, let $\alpha_i \in \mathbb{R}$ be positive and $u_i \in \mathbb{R}^n$ for $i = 1, \dots, m$. Assume that $M > 0$ ist the smallest possible constant such that for all non- negative functions $f_i \in L^1(\mathbb{R}), i = 1, \dots, m$, one has*

$$\int_{\mathbb{R}^n} \prod_{i=1}^m f_i(\langle x, u_i \rangle)^{\alpha_i} \, dx \leq M \prod_{i=1}^m \|f_i\|_1^{\alpha_i}. \tag{10}$$

If there ist equality in (10) for the functions $f_1, \dots, f_m$ and $g_1, \dots, g_m$, then there ist equality for $f_1 * g_1, \dots, f_m * g_m$.

Sketch of the proof. Assume there is equality in (10) for $f_1, \dots, f_m$ and $g_1, \dots, g_m$, with $\|f_i\|_1 = \|g_i\|_1 = 1$ for all $i = 1, \dots, m$. A straightforward computation shows that

$$\begin{aligned} M^2 &= \left(\int_{\mathbb{R}^n} \prod_{i=1}^m f_i(\langle x, u_i \rangle)^{\alpha_i} \, dx\right) \left(\int_{\mathbb{R}^n} \prod_{i=1}^m g_i(\langle x, u_i \rangle)^{\alpha_i} \, dx\right) \\ &\leq M \int_{\mathbb{R}^n} \prod_{i=1}^m (f_i * g_i)(\langle x, u_i \rangle)^{\alpha_i} \, dx \leq M^2. \end{aligned}$$

This implies the assertion. □

Proof of Theorem 4. A simple calculation shows that functions of the form (9) satisfy (8).

Let $p, q, r > 1$ and $0 < f, g \in C(\mathbb{R}) \cap L^1(\mathbb{R})$ satisfy (8) and $\int_{\mathbb{R}} f \, dx = \int_{\mathbb{R}} g \, dx = 1$. If we choose $F(x) = \sqrt{p/\pi} \exp(-px^2), G(x) = \sqrt{q/\pi} \exp(-qx^2)$, we obtain equality in (5) and therefore everywhere in the proof of Lemma 3, especially in the arithmetic-geometric inequality. This leads us to the condition $u'(cx - sy) = v'(sx + cy) = \mu$ for all $x, y \in \mathbb{R}$ and a constant $\mu > 0$. Thus, $u(t) = \mu(t - y_0), v(t) = \mu(t - z_0)$ for some $y_0, z_0 \in \mathbb{R}$ and by (7) we end up with $\mu f(\mu(t - y_0)) = \sqrt{p/\pi}\exp(-pt^2)$ and $\mu g(\mu(t - z_0)) = \sqrt{q/\pi}\exp(-qt^2)$.

Now let $f, g \in L^1(\mathbb{R})$ be arbitrary, non-negative functions, satisfying (8). By the $(L^r, L^{r'})$-duality, (8) is equivalent to the existence of a non-negative function $\tilde{h} \in L^{r'}(\mathbb{R})$ such that f, g, and $h = \tilde{h}^{r'}$ satisfy

$$\int_{\mathbb{R}}\int_{\mathbb{R}} f(cx - sy)^{1/p} g(sx + cy)^{1/q} h(y)^{1/r'}\, dx\, dy = K\, \|f\|_1^{1/p} \|g\|_1^{1/q} \|h\|_1^{1/r'}.$$

Brascamp and Lieb [3] showed that $F(x) = \exp(-px^2), G(x) = \exp(-qx^2)$ and $H(x) = \exp(-r'x^2)$ have the same extremal property as f, g, h. Thus, by Lemma 5, the positive and continuous functions $f * F$ and $g * G$ satisfy (8). As we showed above, they are of the form (9) and so are f and g, by properties of the Fourier transform.

The case $p, q, r < 1$ is very similar and is omitted here. □

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12 Extremizers of a Radon transform inequality

after M. Christ[4]
A summary written by Joris Roos

Abstract

The Radon transform maps functions defined on $\mathbb{R}^d$ to functions on the affine Grassmann manifold $\mathfrak{G}_d$ of affine hyperplanes in $\mathbb{R}^d$. It satisfies a $L^{(d+1)/d}(\mathbb{R}^d) \to L^{d+1}(\mathfrak{G}_d)$ endpoint estimate. In the presented article [4], all the extremizers of this inequality are identified.

12.1 Introduction

Let $d \geq 2$. By $\mathcal{G}_d$ we denote the affine Grassmann manifold of all affine hyperplanes in $\mathbb{R}^d$. Define the measure μ on $\mathcal{G}_d$ to be the pushforward of $drd\theta$ along the canonical two-to-one map

$$\mathbb{R} \times S^{d-1} \to \mathcal{G}_d, (r, \theta) \mapsto \{x : x \cdot \theta = r\}.$$

Here $d\theta$ denotes the surface measure on S^{d-1}. The Radon transform is defined by integrating over affine hyperplanes as follows,

$$\mathcal{R}f(r, \theta) = \int_{x \cdot \theta = r} f(x) d\sigma_{r,\theta}(x),$$

where $d\sigma_{r,\theta}$ is the surface measure of the affine hyperplane $\{x : x \cdot \theta = r\}$.

The point of departure in the presented article is the endpoint inequality

$$\|\mathcal{R}f\|_{L^{d+1}(\mathcal{G}_d,\mu)} \leq A\|f\|_{L^{(d+1)/d}(\mathbb{R}^d)}, \tag{1}$$

where A denotes the optimal constant of this estimate, $A = \sup_{f \neq 0} \frac{\|\mathcal{R}f\|_{L^{d+1}(\mathfrak{G}_d,\mu)}}{\|f\|_{L^{(d+1)/d}(\mathbb{R}^d)}}$. A function $0 \neq f \in L^{(d+1)/d}(\mathbb{R}^d)$ is said to be an *extremizer* in the inequality (1), if $\|\mathcal{R}f\|_{L^{d+1}(\mathfrak{G}_d,\mu)} = A\|f\|_{L^{(d+1)/d}(\mathbb{R}^d)}$. Let $\mathfrak{A}(d)$ denote the group of affine, invertible maps $\phi : \mathbb{R}^d \to \mathbb{R}^d$. Also, let us write $\langle x \rangle = (1 + |x|^2)^{1/2}$ for convenience.

The main result of [4] is the following.

Theorem 1. *Every extremizer of* (1) *is of the form*

$$f(x) = c\langle \phi(x)\rangle^{-d},$$

where $c \in \mathbb{C}\backslash\{0\}$ *and* $\phi \in \mathfrak{A}(d)$.

This solves a special case of a conjecture of Baernstein and Loss [1]. The results of [4] also encompass two related operators, $\mathcal{R}^\sharp$ and $\mathcal{C}$. Let us view $\mathbb{R}^d$ as $\mathbb{R}^{d-1} \times \mathbb{R}^1$ with coordinates $x = (x', x_d) \in \mathbb{R}^{d-1} \times \mathbb{R}^1$. Then we define

$$\mathcal{R}^\sharp f(x) = \int_{\mathbb{R}^{d-1}} f(y', x_d + y' \cdot x')dy',$$

$$\mathcal{C} f(x) = \int_{\mathbb{R}^{d-1}} f(x' - y', x_d - \frac{1}{2}|y'|^2)dy'.$$

The operators $\mathcal{R}^\sharp$, $\mathcal{C}$ satisfy endpoint inequalities of the same form as (1) and the corresponding optimal constants are in fact equal to A and there is a corresponding theorem for the extremizers of these inequalites.
The general outline for the proof of the main result is as follows.

(1) Show existence of radial extremizers. This follows by prior results of the author [5], [6] and the connection to the operator $\mathcal{C}$.

(2) Show that every extremizer has the form $f \circ \phi$ with $\phi \in \mathfrak{A}(d)$. This is done using an array of tools from symmetrization and inverse symmetrization theory.

(3) Exhibit an additional symmetry of the inequality (1) under which the set of radial functions composed with affine, invertible transformations is not invariant.

(4) Prove that every radial extremizer has the form $c\langle ax\rangle^{-d}$ with $c \in \mathbb{C}\backslash\{0\}$, $a > 0$. This is done by exploiting the additional symmetry from the previous step.

In the following we will briefly describe some of the main ingredients of the proof.

12.2 Some ingredients of the proof

12.2.1 Connection of $\mathcal{R}, \mathcal{R}^\sharp, \mathcal{C}$.

The operators $\mathcal{R}$ and $\mathcal{R}^\sharp$ are connected by a change of variables leading up to the identity

$$\langle x' \rangle \mathcal{R}^\sharp f(x) = \mathcal{R} f(r, \theta),$$

where $r = x_d / \langle x' \rangle$ and $\theta = (-x', 1)/\langle x' \rangle$. This can be used to show

$$\|\mathcal{R}^\sharp f\|_{L^{d+1}(\mathbb{R}^d)} = \|\mathcal{R} f\|_{L^{d+1}(\mathfrak{G}_{d,\mu})}.$$

The connection of $\mathcal{R}$ and $\mathcal{C}$ is also simple. Define a map $\Psi : \mathbb{R}^d \to \mathbb{R}^d$ by $\Psi(x', x_d) = (x', x_d - \frac{1}{2}|x'|^2)$ and let $\Psi^* f = f \circ \Psi$. Then we have

$$\mathcal{C} = \Psi^* \circ \mathcal{R} \circ \Psi^*.$$

The proof is again a change of variables.

12.2.2 Affine invariance.

A fundamental property of the inequality (1) is its affine invariance. Namely for every $\phi \in \mathfrak{A}(d)$ we have

$$\frac{\|\mathcal{R}(f \circ \phi)\|_{L^{d+1}(\mathfrak{G}_{d,\mu})}}{\|f \circ \phi\|_{L^{(d+1)/d}(\mathbb{R}^d)}} = \frac{\|\mathcal{R} f\|_{L^{d+1}(\mathfrak{G}_{d,\mu})}}{\|f\|_{L^{(d+1)/d}(\mathbb{R}^d)}}. \tag{2}$$

In particular, if f is an extremizer for (1), then so is $f \circ \phi$ for every $\phi \in \mathfrak{A}(d)$. This identity is proven by associating $\mathcal{R}$ to a symmetric bilinear form. Define the measure λ on $\mathbb{R}^d \times \mathbb{R}^d$ by

$$d\lambda(x, y) = \lim_{\varepsilon \to 0+} \frac{1}{2\varepsilon} \mathbf{1}_{\{|x \cdot y - 1| < \varepsilon\}} dx dy$$

It is then proven that

$$\|\mathcal{R} f\|_{L^{d+1}(\mathfrak{G}_{d,\mu})} = \sup_{g \neq 0} \frac{\left| \int_{\mathbb{R}^d \times \mathbb{R}^d} f g d\lambda \right|}{\|g\|_{L^{(d+1)/d}(\mathbb{R}^d)}},$$

which implies the affine invariance (2).

12.2.3 An additional symmetry.

As alluded to above, there is an additional symmetry of the inequality (1) that is central to the analysis. Let us view $\mathbb{R}^d$ as $\mathbb{R}^{d-2}\times\mathbb{R}^1\times\mathbb{R}^1$ and consider coordinates $(u,s,t) \in \mathbb{R}^{d-2}\times\mathbb{R}^1\times\mathbb{R}^1$. Define

$$\mathcal{J}f(u,s,t) = |s|^{-d} f(s^{-1}u, s^{-1}, s^{-1}t),$$
$$\mathcal{L}f(u,s,t) = f(u,t,s).$$

Then we have $\|\mathcal{J}f\|_{(d+1)/d} = \|f\|_{(d+1)/d}$. Moreover, $\mathcal{L}\circ\mathcal{R}^\sharp = \mathcal{R}^\sharp\circ\mathcal{J}$. Combining the last two equations gives

$$\frac{\|\mathcal{R}^\sharp\mathcal{J}f\|_{d+1}}{\|\mathcal{J}f\|_{(d+1)/d}} = \frac{\|\mathcal{R}^\sharp f\|_{d+1}}{\|f\|_{(d+1)/d}}.$$

In particular, if f is an extremizer for (1), then also $\mathcal{J}f$ is.

12.2.4 Drury's identity.

We continue to use the coordinates $x=(x',x_d)\in\mathbb{R}^{d-1}\times\mathbb{R}^1$. Let $\Delta(x_1,\dots,x_d)$ be the $(d-1)$-dimensional volume of the simplex in $\mathbb{R}^d$ spanned by $x_1,\dots,x_d$ in $\mathbb{R}^d$ and $\Delta'(x_1',\dots,x_d')$ the $(d-1)$-dimensional volume of the simplex in $\mathbb{R}^{d-1}$ spanned by $x_1',\dots,x_d' \in \mathbb{R}^{d-1}$. For functions $f_0,\dots,f_d$ on $\mathbb{R}^d$ we define

$$R(f_0,\dots,f_d) = \int\limits_{(\mathbb{R}^d)^d} \Delta(x_1,\dots,x_d)^{-1}\left(\int\limits_{\pi(x_1,\dots,x_d)} f_0\,d\sigma_\pi\right)\prod_{j=1}^d f_j(x_j)\prod_{i=1}^d dx_i,$$

where $\pi=\pi(x_1,\dots,x_d)$ is the affine hyperplane in $\mathbb{R}^d$ containing the points $x_1,\dots,x_d$. This is well-defined for all up to a measure-zero set of d-tuples $(x_1,\dots,x_d)\in(\mathbb{R}^d)^d$. Also, σ_π denotes the surface measure of the hyperplane π. Drury's identity now amounts to the following:

$$\|\mathcal{R}f\|^{d+1}_{L^{d+1}(\mathfrak{G}_d,\mu)} = R(f,\dots,f). \tag{3}$$

An alternative form of this identity is required. For any $(d+1)$-tuple $x'=(x_0',\dots,x_d')$ in $(\mathbb{R}^{d-1})^{d+1}$ in general position let $v(x')\in\mathbb{R}^d$ be the unique vector such that $x_0'=\sum_{j=1}^d v_j(x')x_j'$ and $\sum_{j=1}^d v_j(x')=1$. Then we have

$$R(f_0,\dots,f_d) = \int\limits_{(\mathbb{R}^{d-1})^{d+1}} \Delta'(x_1',\dots,x_d')^{-1}\int\limits_{\mathbb{R}^d} f_0(x_0', v(x')\cdot t)\prod_{j=1}^d f_j(x_j',t_j)dt\prod_{j=0}^d dx_j'. \tag{4}$$

Therefore combining (3), (4) yields

$$\|\mathcal{R}f\|^{d+1}_{L^{d+1}(\mathfrak{G}_d,\mu)} = \int\limits_{(\mathbb{R}^{d-1})^{d+1}} \Delta'(x'_1,\ldots,x'_d)^{-1}\mathcal{T}_{v(x'_0,\ldots,x'_d)}(f(x'_0,\cdot),\cdots,f(x'_d,\cdot))\prod_{i=0}^{d} dx'_i, \tag{5}$$

where

$$\mathcal{T}_v(F_0,\ldots,F_d) = \int_{\mathbb{R}^d} F_0(t\cdot v)\prod_{j=1}^{d} F_0(t_j)dt. \tag{6}$$

12.2.5 A rearrangement inequality.

For a measurable set $E \subset \mathbb{R}^d$ let E^* be the open ball centered at the origin such that $|E| = |E^*|$. The *symmetric decreasing rearrangement* of a function f on $\mathbb{R}^d$ is defined as

$$f(x) = \int_0^\infty \mathbf{1}_{\{y:f(y)>t\}^*}(x)dt.$$

Brascamp, Lieb and Luttinger [2] proved a generalization of the Riesz-Sobolev rearrangement inequality which implies in particular

$$\mathcal{T}_v(F_0,\ldots,F_d) \le \mathcal{T}_v(F_0^*,\ldots,F_d^*). \tag{7}$$

12.2.6 Burchard's theorem.

The work of Burchard [3] implies the following characterization of the cases of equality in (7).

Theorem 2. *Let $v \in \mathbb{R}^d\backslash\{0\}$. Suppose that $F_0,\ldots,F_d$ are nonnegative, measurable functions on $\mathbb{R}^1$ and that $\mathcal{T}_v(F_0^*,\ldots,F_d^*)$. Also assume that the level sets of the f_j have measure zero and that*

$$\mathcal{T}_v(F_0,\ldots,F_d) = \mathcal{T}_v(F_0^*,\ldots,F_d^*). \tag{8}$$

Then there exist $c_j \in \mathbb{R}$ such that

$$f_j(t) = f_j^*(t-c_j)$$

for almost every $t \in \mathbb{R}$ and $c_0 = \sum_{j=1}^d c_j v_j$.

Exploiting the Drury-type identity (5) for f being an extremizer it is proven that (8) holds for $F_j = (f \circ \phi)(x'_j, \cdot)$ and $v = v(x'_0, \dots, x'_d)$ and every $\phi \in \mathfrak{A}(d)$. However, to apply the theorem it is also required that the level sets have measure zero. A priori, it is not clear whether this is true for extremizers of (1). Therefore, an alternative argument is necessary that circumvents using this theorem directly but instead exploits a more fundamental variant of Theorem 2 that is used in its proof, combined with the rearrangement inequality (7).

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13 Existence of extremizers for a family of extension operators

after L. Fanelli, L. Vega and N. Visciglia [2]
A summary written by Julien Sabin

Abstract

We present a general method to obtain the existence of optimizers for a class of variational problems coming from extension operators, for which optimizing sequences exhibit a loss of compactness.

13.1 Introduction

Let $X \subset \mathbb{R}^N$ and μ a Borel measure on X. Typically, one considers a submanifold of $\mathbb{R}^N$ together with its induced Lebesgue measure. We define the *extension operator* T associated to (X, μ) as

$$Tf(x) := \int_X f(\xi) e^{ix\cdot\xi}\, d\mu(\xi),\ \forall x \in \mathbb{R}^N,$$

for all $f \in L^1(X, \mu)$. Obviously, T is a continuous linear operator from $L^1(X, \mu)$ to $C_b^0(\mathbb{R}^N)$, the set of all bounded continuous functions on $\mathbb{R}^N$. Such operators arise naturally in the context of Strichartz estimates [9, 3], and more generally in harmonic analysis following Stein [8]. The typical result that is proved in these contexts is the boundedness of T as an operator from $L^2(X, d\mu)$ to $L^p(\mathbb{R}^N)$ for some $1 \le p \le +\infty$. This means that there exists $C > 0$ such that for all $f \in (L^1 \cap L^2)(X, \mu)$ we have the bound

$$\|Tf\|_{L^p(\mathbb{R}^N)} \le C \|f\|_{L^2(X,\mu)}\,, \tag{1}$$

and once such a bound is proved, this allows to extend T uniquely by density from $(L^1 \cap L^2)(X, \mu)$ to $L^2(X, \mu)$.

Once we know that the operator T is bounded from $L^2(X, \mu)$ to $L^p(\mathbb{R}^N)$, it is a natural question to ask about the existence of extremizers, that is functions $f \neq 0$ such that

$$\frac{\|Tf\|_{L^p(\mathbb{R}^N)}}{\|f\|_{L^2(X,\mu)}} = \sup_{h\neq 0} \frac{\|Th\|_{L^p(\mathbb{R}^N)}}{\|h\|_{L^2(X,\mu)}} = \|T\|_{L^2(X,\mu)\to L^p(\mathbb{R}^N)}\,. \tag{2}$$

One difficulty to prove the existence of extremizers is the presence of a non-compact group of symmetry leaving the quotient $\|Tf\|_{L^p} / \|f\|_{L^2}$ invariant. Indeed, for any $x_0 \in \mathbb{R}^N$ and for any $f \in L^2(X, \mu)$ we define

$$g_{x_0} f(\xi) := f(\xi) e^{ix_0 \cdot \xi}, \ \forall \xi \in X.$$

It satisfies $\|g_{x_0} f\|_{L^2(X,\mu)} = \|f\|_{L^2(X,\mu)}$, and has the following intertwinning property with the operator T:

$$T g_{x_0} = \tau_{x_0} T,$$

where $\tau_{x_0} h(x) := h(x + x_0)$. In particular, this implies that $\|T g_{x_0} f\|_{L^p} = \|Tf\|_{L^p}$ for all f. From this observation, one cannot hope for the pre-compactness of all maximizing sequences for the problem (2): indeed if it were the case, there would exist an extremizer f_* and the sequence $(g_{x_n} f_*)$ for any $|x_n| \to \infty$ would be a maximizing sequence which is not precompact.

In this note, we present a method due to Fanelli, Vega, and Visciglia [2] that allows to prove the existence of extremizers for a general family of extension operators, despite the difficulty just mentioned. Their method rely on the observation that it is enough to find a single non-trivial weak limit of a maximizing sequence, as was already emphasized by Brézis and Lieb [6, 1]. This principle is abstractly stated in Proposition 1 below. Then, this principle is applied to two situations: the finite volume case $\mu(X) < \infty$ considered in [2] and an example in the infinite volume setting when X is a paraboloid.

13.2 An abstract result on the existence of extremizers

The method of Fanelli, Vega, and Visciglia is based on the following proposition giving a sufficient condition on the existence of maximizers for a general class of linear operators.

Proposition 1. *Let $\mathfrak{H}$ a Hilbert space and $T \neq 0$ a bounded linear operator from $\mathfrak{H}$ to $L^p(\mathbb{R}^N)$ for some $2 < p < \infty$. Let $(f_n) \subset \mathfrak{H}$ a sequence satisfying*

1. $\|f_n\|_{\mathfrak{H}} = 1$ *for all* n;
2. $\|Tf_n\|_{L^p} \to \|T\|_{\mathfrak{H} \to L^p}$;
3. $f_n \rightharpoonup f_* \neq 0$ *in* $\mathfrak{H}$;

4. $Tf_n \to Tf_*$ *a.e. on* $\mathbb{R}^N$.

Then, the sequence (f_n) *converges strongly in* $\mathfrak{H}$ *to* f_*. *In particular,* $\|f_*\|_{\mathfrak{H}} = 1$ *and* $\|Tf_*\|_{L^p} = \|T\|_{\mathfrak{H}\to L^p}$.

Remark 2. *The first two conditions on the sequence* (f_n) *are the definition of a maximizing sequence. In a concrete situation, one thus only has to check the last two conditions.*

Proof. Define $r_n = f_n - f_*$. We thus have $r_n \rightharpoonup 0$ and $Tr_n \to 0$ a.e. on $\mathbb{R}^N$. This implies that

$$1 = \|f_n\|_{\mathfrak{H}}^2 = \|f_*\|_{\mathfrak{H}}^2 + \|r_n\|_{\mathfrak{H}}^2 + o(1).$$

Furthermore, by the Brézis-Lieb lemma [1], we have

$$\|T\|_{\mathfrak{H}\to L^p}^p + o(1) = \|Tf_n\|_{L^p}^p = \|Tf_*\|_{L^p}^p + \|Tr_n\|_{L^p}^p + o(1).$$

As a consequence, we deduce that

$$\|T\|_{\mathfrak{H}\to L^p}^p = \|Tf_*\|_{L^p}^p + \|Tr_n\|_{L^p}^p + o(1) \leq \|T\|_{\mathfrak{H}\to L^p}^p \left(\|f_*\|_{\mathfrak{H}}^p + \|r_n\|_{\mathfrak{H}}^p\right) + o(1).$$

Since $T \neq 0$ and $p > 2$, this leads to

$$1 \leq \|f_*\|_{\mathfrak{H}}^p + \|r_n\|_{\mathfrak{H}}^p + o(1) \leq (\|f_*\|_{\mathfrak{H}}^2 + \|r_n\|_{\mathfrak{H}}^2)^{p/2} + o(1) = 1 + o(1).$$

In the limit $n \to \infty$, the previous inequalities become equalities, and since $a^p + b^p = (a^2 + b^2)^{p/2}$ if and only if $a = 0$ or $b = 0$, we infer that $\|f_*\|_{\mathfrak{H}} = 0$ or $\lim_{n\to\infty} \|r_n\|_{\mathfrak{H}} = 0$. However, we assumed that $f_* \neq 0$, which concludes the proof. □

13.3 Application to extension operators in the finite volume setting

We now apply the abstract previous proposition to the context of extension operators. As we have already mentioned, we just have to check the last two conditions of Proposition 1 for maximizing sequences associated to extension operators. These two conditions are not satisfied for any (X, μ) and we have to add some assumptions ensuring that they hold. This is summarized in the following result.

Theorem 3. *Let (X,μ) be such that $\int_X (1+|\xi|^2)\,d\mu(\xi) < \infty$. Define*

$$p_0 = \inf\{p \geq 1,\ T \text{ satisfies } (1)\}.$$

Then, for any $p > \max(2, p_0)$ (with the convention that $p = \infty$ if $p_0 = \infty$), there exists $f \in L^2(X,\mu)$ with $\|f\|_{L^2(X)} = 1$ satisfying

$$\|Tf\|_{L^p(\mathbb{R}^N)} = \|T\|_{L^2(X)\to L^p(\mathbb{R}^N)}.$$

Remark 4. *Under the assumption that $\mu(X) < \infty$, T automatically satisfies (1) for $p = \infty$ by the Cauchy-Schwarz inequality. The fact that $p_0 < \infty$ may depend on additional properties of (X,μ) that we do not discuss here. There are several cases for which we know that $p_0 < \infty$ however [8, 11, 9].*

The proof distinguishes between the cases $p < \infty$ and $p = \infty$. In the case $p < \infty$, we apply Proposition 1, while the case $p = \infty$ requires a special treatment.

Proof in the case $p < \infty$. Since the statement of the theorem obviously holds when $\mu = 0$, we may assume that $\mu \neq 0$ and hence $T \neq 0$. Let $(f_n) \subset L^2(X,\mu)$ a sequence such that $\|f_n\|_{L^2(X)} = 1$ and

$$\|Tf_n\|_{L^p(\mathbb{R}^N)} \to \|T\|_{L^2(X)\to L^p(\mathbb{R}^N)}.$$

We first check condition (3) of Proposition 1, with the sole assumption that $\mu(X) < \infty$. Indeed, in this case T is bounded from $L^2(X)$ to $L^\infty(\mathbb{R}^N)$ as already mentioned. Let $\tilde{p}$ such that $p_0 < \tilde{p} < p < \infty$. By the Hölder inequality and the definition of p_0, there exists $0 < \theta < 1$ such that

$$\begin{aligned}\|Tf_n\|_{L^p(\mathbb{R}^N)} &\leq \|Tf_n\|^\theta_{L^{\tilde{p}}(\mathbb{R}^N)} \|Tf_n\|^{1-\theta}_{L^\infty(\mathbb{R}^N)}\\ &\leq \|T\|^\theta_{L^2(X)\to L^{\tilde{p}}(\mathbb{R}^N)} \|f_n\|^\theta_{L^2(X)} \|Tf_n\|^{1-\theta}_{L^\infty(\mathbb{R}^N)}.\end{aligned}$$

Since $\|Tf_n\|_{L^p(\mathbb{R}^N)} \to \|T\|_{L^2(X)\to L^p(\mathbb{R}^N)} \neq 0$, we deduce from the last inequality that there exists $\varepsilon_0 > 0$ such that

$$\|Tf_n\|_{L^\infty(\mathbb{R}^N)} \geq \varepsilon_0 > 0$$

for n large enough. Hence, there exists $(x_n) \subset \mathbb{R}^N$ such that

$$|Tf_n(x_n)| \geq \varepsilon_0/2 > 0$$

for n large enough (when $\mu(X) < \infty$, $L^2(X,\mu) \subset L^1(X,\mu)$ and Tf_n is a bounded continuous function on $\mathbb{R}^N$ for any n). Since

$$Tf_n(x_n) = (\tau_{x_n} Tf_n)(0) = (Tg_{x_n} f_n)(0),$$

the sequence $h_n := g_{x_n} f_n$ still satisfies $\|h_n\|_{L^2(X)} = 1$, $\|Th_n\|_{L^p(\mathbb{R}^N)} \to \|T\|_{L^2(X)\to L^p(\mathbb{R}^N)}$ with the additional property that

$$|Th_n(0)| \geq \varepsilon_0/2 > 0$$

for n large enough. Since (h_n) is bounded in $L^2(X,\mu)$, we may assume that $h_n \rightharpoonup h^*$ in $L^2(X,\mu)$ up to a subsequence. Now

$$|\langle 1, h^*\rangle_{L^2(X)}| = \lim_{n\to\infty} |\langle 1, h_n\rangle_{L^2(X)}| = \lim_{n\to\infty} \left| \int_X h_n(\xi)\, d\mu(\xi) \right| = \lim_{n\to\infty} |Th_n(0)|$$

and this implies that $h^* \neq 0$, which is exactly condition (3) of Proposition 1. To finish the proof of the theorem, we thus have to check condition (4) of Proposition 1, and this is where the assumption $\int_X |\xi|^2 d\mu(\xi)$ appears. Indeed, under this assumption, the sequence (Th_n) actually belongs to $C^1(\mathbb{R}^N)$ with the uniform bounds

$$\|Th_n\|_{L^\infty(\mathbb{R}^N)} \leq \mu(X)^{1/2} \|h_n\|_{L^2(X)} = \mu(X)^{1/2},$$

$$\|\nabla_x Th_n\|_{L^\infty(\mathbb{R}^N)} \leq \left(\int_X |\xi|^2\, d\mu(\xi) \right)^{1/2}$$

which implies by the Ascoli-Arzela theorem that the sequence (Th_n) converges uniformly on all compact sets of $\mathbb{R}^N$ towards a continuous bounded function ϕ. To prove condition (4), we thus have to check that $\phi = Th^*$. For any $\psi \in C_0^\infty(\mathbb{R}^N)$, we have by Fubini's theorem

$$\langle \phi, \psi\rangle = \lim_{n\to\infty} \langle Th_n, \psi\rangle = \lim_{n\to\infty} \langle h_n, \hat{\psi}_{|X}\rangle_{L^2(X)} = \langle h^*, \hat{\psi}_{|X}\rangle_{L^2(X)} = \langle Th^*, \psi\rangle,$$

where we have used that $\hat{\psi}_{|X} \in L^2(X,\mu)$ since μ is a finite Borel measure on X. This finishes the proof of the theorem. □

Proof in the case $p = \infty$. We define the sequence (h_n) in the fashion as in the case $p < \infty$, except that we need the additional property that

$$\begin{aligned}\|Th_n\|_{L^\infty(\mathbb{R}^N)} \geq |Th_n(0)| &= |Tf_n(x_n)| \\ &\geq (1 - 1/n)\, \|Tf_n\|_{L^\infty(\mathbb{R}^N)} = (1 - 1/n)\, \|Th_n\|_{L^\infty(\mathbb{R}^N)}.\end{aligned}$$

Again by the Ascoli-Arzela theorem, (Th_n) converges to Th^* uniformly on all compact sets of $\mathbb{R}^N$, and in particular

$$\|Th^*\|_{L^\infty(\mathbb{R}^N)} \geq |Th^*(0)| = \lim_{n\to\infty} |Th_n(0)| = \|T\|_{L^2(X)\to L^\infty(\mathbb{R}^N)} \,.$$

On the other hand we have

$$\|Th^*\|_{L^\infty(\mathbb{R}^N)} \leq \|T\|_{L^2(X)\to L^\infty(\mathbb{R}^N)} \|h^*\|_{L^2(X)} \,,$$

which implies that $\|h^*\|_{L^2(X)} \geq 1$ since $T \neq 0$. We always have $\|h^*\|_{L^2(X)} \leq 1$ by weak convergence, hence $\|h^*\|_{L^2(X)} = 1$ and $h_n \to h^*$ strongly in $L^2(X)$, and hence h^* is a maximizer. □

Remark 5. *The previous proofs show that we have actually the stronger result that any maximizing sequence is pre-compact in $L^2(X,\mu)$ up to the action of symmetry group $G = \{g_{x_0},\ x_0 \in \mathbb{R}^N\}$.*

13.4 An application in infinite volume: Strichartz estimates

As we saw in the previous proofs, the finite volume condition $\mu(X) < \infty$ is crucial, in particuliar to find a non-trivial weak limit h^* via a uniform lower bound on $\|Th_n\|_{L^\infty(\mathbb{R}^N)}$. In this section we give an example on how to obtain such a lower bound in the infinite volume situation of Strichartz estimates. In this case (and in all of this section), the set X is a paraboloid

$$X = \{(\omega,\xi) \in \mathbb{R}\times\mathbb{R}^d,\ \omega = |\xi|^2\}, \tag{3}$$

and the measure μ is given by the push-forward relation

$$\int_X f(\omega,\xi)\, d\mu(\omega,\xi) = \int_{\mathbb{R}^d} f(|\xi|^2,\xi)\, d\xi. \tag{4}$$

Strichartz estimates [9] state that there exists $C > 0$ such that

$$\|Tf\|_{L^{2+4/d}(\mathbb{R}^{d+1})} \leq C\,\|f\|_{L^2(X)}\,, \tag{5}$$

for all $f \in (L^1 \cap L^2)(X,\mu)$. Furthermore, a simple scaling argument shows that $p = 2 + 4/d$ is the only exponent such that (1) holds for this choice of (X,μ). In particular, it is hopeless to get a lower bound on $\|Tf_n\|_{L^\infty(\mathbb{R}^N)}$ as in the proof of Theorem 3 since T is not bounded from $L^2(X)$ to $L^\infty(\mathbb{R}^{d+1})$. However, there exists a more refined estimate given in [4, Prop. 4.24] based on [10]: there exists $C > 0$ and $0 < \theta < 1$ such that

$$\|Tf\|_{L^{2+4/d}(\mathbb{R}^{d+1})} \leq C \|f\|_{L^2(X)}^\theta \left(\sup_{Q \in \mathcal{D}} |Q|^{-1/2} \|Tf_Q\|_{L^\infty(\mathbb{R}^{d+1})} \right)^{1-\theta}, \qquad (6)$$

where Q denotes the set of all dyadic cubes on $\mathbb{R}^d$, that is the union over $j \in \mathbb{Z}$ of all cubes of side length 2^j and centered at $(2^j\mathbb{Z})^d$, and f_Q is defined as

$$\mathcal{F}_{\mathbb{R}^d}(f_Q) = \chi_Q \mathcal{F}_{\mathbb{R}^d}(\xi \mapsto f(|\xi|^2, \xi)),$$

where $\mathcal{F}_{\mathbb{R}^d}$ denotes the Fourier transform on $\mathbb{R}^d$ and χ_Q is the characteristic function of the set Q. From the estimate (6), one can infer the existence of a non-zero weak limit to a maximizing sequence as in the proof of Theorem 3 up to a group of symmetry that is larger than G. Indeed it contains non only the translations in $\mathbb{R}^{d+1}$ but also translations at the "source" $\mathbb{R}^d$ and dilations of the paraboloid X, due to the particular structure of X as a graph of the homogeneous function $\xi \mapsto |\xi|^2$.

The second instance in the proof of Theorem 3 where the finite volume condition $\mu(X) < \infty$ appears is when we check condition (4) of Proposition 1, namely that (Tf_n) converges almost everywhere. In the infinite-volume context of Strichartz estimates, this last fact will follow from the *local compactness* of the operator T: if $f_n \rightharpoonup 0$ in $L^2(X)$, one can prove that $Tf_n \to 0$ strongly in $L^2_{\text{loc}}(\mathbb{R}^{d+1})$. This local compactness property relies on the fundamental property of local smoothing of the operator T: there exists $C > 0$ such that

$$\left\|(1+x^2)^{-1/2}(-\Delta_x)^{1/4} Tf\right\|_{L^2(\mathbb{R}_t \times \mathbb{R}^d_x)} \leq C \|f\|_{L^2(X)} \qquad (7)$$

for any $f \in L^2(X)$. The estimate (7) means that locally in x, the function Tf has 1/2-derivatives in x more than the input f, hence the operator T is locally regularizing. Hence, if $f_n \rightharpoonup 0$ in $L^2(X)$ and in P_Λ denotes the Fourier projection in x for frequencies bigger than $\Lambda > 0$ (which commutes with T), we have

$$\begin{aligned} \|P_\Lambda T f_n\|_{L^2(\mathbb{R}_t \times K)} &\leq C_K \left\|(1+x^2)^{-1/2} T P_\Lambda f_n\right\|_{L^2(\mathbb{R}_t \times \mathbb{R}^d_x)} \\ &\leq C_K C \left\|(-\Delta_x)^{-1/4} P_\Lambda f_n\right\|_{L^2(X)} \lesssim \Lambda^{-1/2}, \end{aligned}$$

for any compact $K \subset \mathbb{R}^d$, meaning that the large frequency part of Tf_n is small in $L^2_{\mathrm{loc}}(\mathbb{R}^{d+1})$. On the other hand,

$$(1 - P_\Lambda)Tf_n(t, x) = \int_{|\xi| \le \Lambda} e^{it|\xi|^2 - ix\cdot\xi} \hat{f}_n(\xi)\, d\xi,$$

converges to 0 for any fixed $(t, x) \in \mathbb{R} \times \mathbb{R}^d$ since $\xi \mapsto \chi(|\xi| \le \Lambda)e^{it|\xi|^2 - ix\cdot\xi} \in L^2(\mathbb{R}^d_\xi)$ and (f_n) converges weakly to zero in $L^2(\mathbb{R}^d)$. This is how we prove condition (4) in the case of the paraboloid. Together with Proposition 1, this leads to the

Theorem 6. *Assume X is the paraboloid* (3) *together with the measure* (4). *Then, the Strichartz estimate* (5) *has optimizers.*

Remark 7. *This result has been proved by Kunze [5] in $d = 1$ and by Shao [7] in $d \ge 2$, using a profile decomposition. Our last proof shows that extracting a single non-trivial profile is enough to obtain the existence of extremizers.*

Remark 8. *Strichartz estimates also hold when replacing $L^{2+4/d}(\mathbb{R} \times \mathbb{R}^d)$ by $L^p_t L^q_x(\mathbb{R} \times \mathbb{R}^d)$ with $2/p + d/q = d/2$, $2 \le q \le 2d/(d-2)$ if $d \ge 3$, $2 \le q < \infty$ if $d = 2$, and $2 \le q \le \infty$ if $d = 1$. A similar proof shows the existence of optimizers when q is not equal to the endpoint $q = 2d/(d-2)$ for $d \ge 3$ and $q = \infty$ for $d = 1, 2$. The existence of optimizers at the endpoint is an open problem.*

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14 The Sharp Hausdorff-Young Inequality

after W. Beckner [1]
A summary written by Mateus Sousa

Abstract

We study multiplier inequalities in the Hermite semigroup and use the classical central limit theorem to obtain the sharp constant for the Hausdorff-Young inequality through a limiting argument.

14.1 Introduction

For $f \in L^1(\mathbb{R}^d)$ we consider the Fourier transform given by

$$\mathcal{F}(f)(\xi) = \hat{f}(\xi) := \int_{\mathbb{R}^d} f(x)e^{2\pi i x\cdot\xi}\mathrm{dx},$$

which extends to a bounded operator from $L^p(\mathbb{R}^d)$ to its dual space $L^{p'}(\mathbb{R}^d)$, $1 \leq p \leq 2$, satisfying the classical Hausdorff-Young inequality

$$\|\hat{f}\|_{p'} \leq \|f\|_p. \tag{1}$$

Inequality (1) is obtained from interpolation between the L^1 and L^2 cases, which are both sharp from the facts that for $p = 1$ any nonnegative function attains equality in (1) and $\mathcal{F}$ extends as a unitary operator in L^2.

Our main goal is to prove the sharp version of (1) for the remaining exponents, which is the following:

Theorem 1. *For $1 < p < 2$, and $f \in L^p(\mathbb{R}^d)$*

$$\|\hat{f}\|_{p'} \leq A_p^d \|f\|_p, \tag{2}$$

where $A_p = (p^{1/p}/p'^{1/p'})^{1/2}$.

The first observation to make is that the product structure of the Fourier transform will make Theorem 1 follow directly from the case $d = 1$ and Lemma 3 below, so we only need to consider the 1-dimensional case.

Another important observation is that the constant A_p^d is sharp because it is attained for the function $f(x) = e^{-\pi|x|^2}$. In fact, gaussian functions are the only maximizers of (2), but we do not address this question here and the interested reader can check reference [2].

To prove the 1-dimensional case, we consider for $\omega \in \mathbb{C}$ the operator T_ω which acts in Hermite polynomials by

$$T_\omega H_m(x) = \omega^m H_m(x),$$

where

$$H_m(x) = \int (x+iy)^m \mathrm{d}\mu(y)$$

and

$$\mathrm{d}\mu(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \mathrm{dx}.$$

T_ω can be expressed as an integral operator on $L^2(\mathrm{d}\mu)$ by

$$\begin{aligned} T_\omega(x,y) &= (1-\omega^2)^{-1/2} \exp\left\{ -\frac{\omega^2(x^2+y^2)}{2(1-\omega^2)} + \frac{\omega xy}{1-\omega^2} \right\}, \\ (T_\omega g)(x) &= \int T_\omega(x,y) g(y) \mathrm{d}\mu(y). \end{aligned}$$

The main property we need from T_ω is the following result:

Theorem 2. *For $\omega = i\sqrt{p-1}$ and $1 < p < 2$, T_ω extends to a bounded operator from $L^p(\mathrm{d}\mu)$ to $L^{p'}(\mathrm{d}\mu)$ and*

$$\|T_\omega g\|_{L^{p'}(\mathrm{d}\mu)} \leq \|g\|_{L^p(\mathrm{d}\mu)}. \tag{3}$$

Now a simple change of variables shows that (3) holding for polynomials $g(x)$ is equivalent to the 1-dimensional case of (2) holding for functions of the form $f(x) = g(\sqrt{2\pi p}x)e^{-\pi x^2}$, which yields both theorems since they would hold for a dense set of functions.

14.2 Discrete analogue of T_ω

To prove Theorem 2, we use the classical central limit (CLT) theorem to get the normal distribution $\mathrm{d}\mu$ as a weak-$\star$ limit of Bernoulli trials.

Let $\mathrm{d}\nu(x)$ be a probability measure in $\mathbb{R}$ such that

$$\int x \mathrm{d}\nu(x) = 0, \quad \int x^2 \mathrm{d}\nu(x) = 1.$$

If we define nu_n as the n-fold convolution of the measure $\mathrm{d}\nu(\sqrt{n}x)$ with itself, the classical central limit gives us that for every bounded continuous function h

$$\begin{aligned}\lim_{n\to\infty}\int h(x)\mathrm{d}\nu_n(x) &= \lim_{n\to\infty}\int h(\tfrac{x_1+\cdots+x_n}{\sqrt{n}})\mathrm{d}\nu(x_1)\cdots\mathrm{d}\nu(x_n)\\ &= \int h(x)\mathrm{d}\mu(x)\end{aligned} \tag{4}$$

For the case of a compactly supported measure $\mathrm{d}\nu$, the convergence in (4) is also holds for functions h with polynomial growth. In particular, if we consider a Bernoulli trial, i.e, a measure $\mathrm{d}\nu$ that gives weight $1/2$ to $x = \pm 1$, for every polynomial h and $1 \le p < \infty$ we have

$$\lim_{n\to\infty}\|h\|_{L^p(\mathrm{d}\nu_n)} = \|h\|_{L^p(\mathrm{d}\mu)}. \tag{5}$$

In order to establish (3) we define an analogue of the operator T_ω for the product measure $\mathrm{d}\nu^n(x_1,\cdots,x_n) = \mathrm{d}\nu(\sqrt{n}x_1)\times\cdots\times\mathrm{d}\nu(\sqrt{n}x_n)$. Since every function in the space $L^p(\mathrm{d}\nu^n)$ can be seen as a polynomial of degree 1 (in an almost everywhere sense) in each variable x_k, we can define for every $1 \le k \le n$ operators

$$C_{n,k} : a + bx_k \mapsto a + \omega b x_k,$$

where a and b are functions of the remaining $n-1$ variables, and

$$K = K_n = C_{n,n}\circ\cdots\circ C_{n,1}.$$

For the case $k = n = 1$ the operator K is simply the operator T_ω as seen in $L^p(\mathrm{d}\nu)$, because $H_0(x) = 1$ and $H_1(x) = x$ and so

$$T_w(aH_0 + bH_1) = aH_0 + b\omega H_1 = K(a + bx).$$

The operator K works as a discrete analogue of T_ω for the product measure space in $L^p(\mathrm{d}\nu^n)$. Now in order to get (3) we prove bounds for K and compare what happens to $T_\omega g$ and $K_n g_n$, where g is a polynomial and g_n are functions associated to g given in terms of basic symmetric functions.

14.3 Bounds for the discrete analogue

To get bounds for K_n we just need to look at the case $n = 1$, because of the following result about product of operators.

Lemma 3. *Let T_1 and T_2 be two integral operators defined by kernels such that for every $(f_1, f_2) \in L^p(\mathrm{d}\rho_1) \times L^p(\mathrm{d}\rho_2)$*

$$\|T_1 f_1\|_{L^p(\mathrm{d}\lambda_1)} \le \|f_1\|_{L^q(\mathrm{d}\rho_1)},$$
$$\|T_2 f_2\|_{L^p(\mathrm{d}\lambda_2)} \le \|f_2\|_{L^q(\mathrm{d}\rho_2)}$$

for σ-finite measures $\lambda_1, \lambda_2, \rho_1, \rho_2$ and $1 \le p \le q$. Then the integral operator associated to the product T_1T_2 maps $L^p(\mathrm{d}\rho_1 \times \mathrm{d}\rho_2)$ to $L^q(\mathrm{d}\lambda_1 \times \mathrm{d}\lambda_2)$ with norm 1.

Lemma 3 is simple to get just applying Fubini's theorem and Minkowski's integral inequality (which is the reason behind the $p \le q$), and with this in hand we just need to bound $C_{n,1}$ for each n. For that the case $C = C_{1,1}$ that is equivalent to the following result

Lemma 4. *Let $a, b \in \mathbb{C}$ and $1 < p < 2$. If we set $\omega = i\sqrt{p-1}$ then*

$$\left\{ \frac{|a + \omega b|^{p'} + |a - \omega b|^{p'}}{2} \right\}^{1/p'} \le \left\{ \frac{|a + b|^p + |a - b|^p}{2} \right\}^{1/p}.$$

Now to bound general $C_{n,1}$, and therefore K_n, one just needs to notice that it follows straight from Lemma 4 that for any $\alpha > 0$

$$\|Cg\|_{L^{p'}(\mathrm{d}\nu_\alpha)} \le \|g\|_{L^p(\mathrm{d}\nu_\alpha)},$$

where $\mathrm{d}\nu_\alpha(x) = \mathrm{d}\nu(\alpha x)$. Therefore, setting $\alpha = \sqrt{n}$ will give the desired norm 1 of K_n. To finish things we just need to look at the interplay between K acting on symmetric functions and T_ω acting on polynomials.

14.4 Hermite polynomials, symmetric functions and the result

Let $\sigma_{n,j}(x_1, \cdots, x_n)$ be the basic symmetric functions, that is

$$\sigma_{n,j}(x_1, \cdots, x_n) = \sum_{m_1 < \cdots < m_j} x_{m_1} \cdots x_{m_j}.$$

One can check that the functions $\varphi_{n,j} = j!\sigma_{n,j}$ form an orthonormal basis for the space of all the symmetric functions with respect to the $L^2(\mathrm{d}\nu^n)$ norm, and it follows from the definition that they are also eigenfunctions of the operator K, that is

$$K\varphi_{n,j} = \omega^j \varphi_{n,j}. \tag{6}$$

The generating function for the $\varphi_{n,j}$ is given by

$$\begin{aligned} \mathcal{T}(x_1 \cdots, x_n, t) &= \prod_{k=1}^{n}(1 + x_k t) \\ &= \sum_{j=0}^{n} \frac{\varphi_{n,j}(x_1, \cdots, x_n)}{j!} t^j. \end{aligned}$$

For the Hermite polynomials the generating function is given by

$$\mathfrak{T}(x,t) = \sum_{m=0}^{\infty} \frac{H_m(x)}{m!} t^m.$$

If we consider the case $x = x_1 + \cdots + x_n$ where each x_k is either $-1/\sqrt{n}$ or $1/\sqrt{n}$, which is the set to consider when dealing with the measures $\mathrm{d}\nu_n$ and $\mathrm{d}\nu^n$, then one can look at the relations between $\mathcal{T}$ and $\mathfrak{T}$ and get formulas relating each each $\varphi_{n,l}$ and the symmetrized functions $H_m(x_1 + \cdots + x_n)$ given by Hermite polynomials, which is the following

$$\varphi_{n,j}(x_1, \cdots, x_n) = H_j(x_1 + \cdots + x_n) + \frac{1}{n} \sum_{l=1}^{\lfloor j/2 \rfloor} a_{j,l} H_{j-2l}(x_1 + \cdots + x_n), \tag{7}$$

where the coefficients $a_{j,l}$ are bounded with respect to n for fixed j (see Appendix 1 of [1]).

Relation (7) means that for large values of n the main term of $\varphi_{n,j}$ comes from $H_j(x_1 + \cdots + x_n)$. In fact it follows from (6) and (7) that

$$T_\omega H_j(x_1 + \cdots + x_n) - K\varphi_{n,j}(x_1, \cdots, x_n) = \frac{\omega^j}{n} \sum_{l=1}^{\lfloor j/2 \rfloor} a_{j,l} H_{j-2l}(x_1 + \cdots + x_n),$$

and any $L^p(\mathrm{d}\nu^n)$-norm of the right-hand side goes to zero as n tends to infinity because of (5).

Now we consider a polynomial g. Since the Hermite polynomials generate the space of all polynomials (they are linearly independent and each H_m has degree m), we can write g as a linear combination of Hermite polynomials

$$g(x) = \sum_{m=0}^{M} b_m H_m(x).$$

We define for each $n \in \mathbb{N}$ the symmetric function

$$g_n(x_1, \cdots, x_n) = \sum_{m=0}^{M} b_m \varphi_{n,m}(x_1, \cdots, x_n).$$

From (7) we get that

$$T_\omega g(x_1 + \cdots + x_n) - K g_n(x_1, \cdots, x_n) = \sum_{m=0}^{M} b_m \frac{\omega^m}{n} \sum_{l=1}^{\lfloor m/2 \rfloor} a_{m,l} H_{m-2l},$$

hence from the triangle inequality and the convergence to zero of the $L^p(\mathrm{d}\nu^n)$ norms mentioned above, we get for every polynomial g that

$$\lim_{n\to\infty} |\|T_\omega g\|_{L^p(\mathrm{d}\nu_n)} - \|K g_n\|_{L^p(\mathrm{d}\nu^n)}| = 0. \tag{8}$$

Now if we set $\omega = i\sqrt{p-1}$, from Lemmas 3 and 4 we have

$$\|K g_n\|_{L^{p'}(\mathrm{d}\nu^n)} \leq \|g_n\|_{L^p(\mathrm{d}\nu^n)},$$

and from this last inequality, combined with (5) and (8), we get for every polynomial g that

$$\|T_\omega g\|_{L^{p'}(\mathrm{d}\mu)} \leq \|g\|_{L^p(\mathrm{d}\mu)},$$

which yields the desired result.

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15 A mass-transportation approach to sharp Sobolev and Gagliardo-Nirenberg inequalities

*after D. Cordero-Erausquin, B. Nazaret, C. Villani [**?**].*
A summary written by Gennady Uraltsev.

Abstract

We show an elementary proof of Sobolev's inequality on $\mathbb{R}^n$ with sharp constants for arbitrary norms on $\mathbb{R}^n$. Mass transportation techniques allow us to reduce the problem to a concavity argument together with a Hölder inequality and this facilitates studying the cases of equality and thus finding all the minimizers for the problem.

15.1 Introduction and generalities

The paper [**?**] is aimed at developing an approach to studying functional inequalities such as the Sobolev and the Gagliardo-Nirenberg inequalities using tools from mass transportation. This approach is particularly adapted to studying the sharp constants in the above inequalities since it relies on a concavity argument coming from mass transport theory and a Hölder inequality. The constants in both of these steps can be explicitly tracked to formulate the sharp version of the inequality and, furthermore, tracking the well-understood cases of equality in these steps gives an explicit characterization of the extremizers of the sharp inequality that turn out to be unique up to the dilation and translation symmetries of the problem.

Let us recall Sobolev's inequality on $\mathbb{R}^n$ with $n > 1$ and $p \in [1, n)$. Let $f \in W^{1,p}(\mathbb{R}^n)$ where $W^{1,p}$ is the set of functions in $L^p(\mathbb{R}^n)$ whose distributional derivatives of first order $\partial_i f$, $i \in \{1, \dots, n\}$ are also in $L^p(\mathbb{R}^n)$. Then setting $p^* = \frac{np}{n-p}$ to be the critical Sobolev exponent we have that

$$\|f\|_{L^{p^*}(\mathbb{R}^n)} \lesssim_{n,p} \|\nabla f\|_{L^p(\mathbb{R}^n)} \qquad \frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}. \tag{1}$$

The critical exponent p^* is determined by scaling: setting $f_\lambda(\cdot) = f(\lambda^{-1}\cdot)$ we have that $\nabla f_\lambda = \lambda^{-1} (\nabla f)_\lambda$ so that $\|\nabla f\|_{L^p(\mathbb{R}^n)} = \lambda^{n/p-1}$ while $\|f_\lambda\|_{L^{p^*}} = \lambda^{n/p*}$. We thus have that (**??**) is scaling invariant in addition to being translation invariant.

15.1.1 Norms and Geometry of $\mathbb{R}^n$

It is noteworthy that the expression $\|\nabla f\|_{L^p}$ cannot see constant functions i.e. it is zero if $f(x) = c$ while $\|f\|_{L^{p^*}} = +\infty$ in that case. However this doesn't contradict (1) since constants are not in $W^{1,p}$; in other words (1) can be seen as an a priori estimate on $f \in C_c^\infty(\mathbb{R}^n)$ that extends by density to $W^{1,p}$ so morally f is 0 at ∞. This underlines that Sobolev's inequality is actually a geometrical fact that depends on the structure of $\mathbb{R}^n$ and this suggests us to generalize Sobolev's inequality in the main result to arbitrary norms $\|\cdot\|_E$ on $\mathbb{R}^n$.

15.1.2 Scaling and duality

A crucial step in the present proof of Sobolev's inequality is the formulation of a dual problem. Proving (1) as finding the extremum of the minimization problem

$$\inf_{\|f\|_{L^{p^*}(\mathbb{R}^n)}=1} \|\nabla f\|_{L^p(\mathbb{R}^n)}$$

and showing that it is strictly larger than 0. We can formulate a dual problem with the same scaling symmetry:

$$\sup_{\|g\|_{L^{p^*}(\mathbb{R}^n)}=1} \frac{\int_{\mathbb{R}^n} |g|^{p^*(1-1/n)}(y)dy}{\left(\int_{\mathbb{R}^n} |y|^q |g|^{p^*}(y)dy\right)^{\frac{1}{q}}} \qquad \frac{1}{p} + \frac{1}{q} = 1. \tag{2}$$

The functional that gets maximized above has as its numerator a norm with smaller exponent than L^{p^*} so it can be made arbitrarily large using unboundedness of $\mathbb{R}^n$, however spreading out g gets penalized by the weight in the denominator.

15.2 The main result

Given the above remarks let us introduce the following notation. Let E be a finite dimensional Banach space so that $E = (\mathbb{R}^n, \|\cdot\|_E)$. The dual space is $E' = (\mathbb{R}^n, \|\cdot\|_{E'})$ where the norm $\|\cdot\|_{E'}$ is defined via

$$\|Y\|_{E'} = \sup_{X \in E} \frac{Y \cdot X}{\|X\|_E}$$

where $(Y, X) \mapsto Y \cdot X$ is the duality pairing on $E' \times E$. Following the remarks on the scaling invariance of (**??**) we introduce the homogeneous norm $\|f\|_{\dot{W}^{1,p}(E)} = \|f\|_{L^{p*}(E;\mathbb{R})} + \|df\|_{L^p(E;E')}$; as a matter of fact while f is a real-valued function on E its differential df is an E' valued function. By abuse of notation we will write $\nabla f = df$ and we will implicitly omit specifying the norms on the spaces.As previously $p^* = \frac{np}{n-p}$ is the critical Sobolev exponent.

Theorem 1 (Sharp Sobolev Inequality)**.** *Let $p \in (1, n)$ and $q = \frac{p}{p-1}$ be its dual exponent.*

1. *The duality principle holds:*

$$\sup_{g \in L^{p^*}} \frac{1}{\|g\|_{L^{p^*}}} \frac{\int_{\mathbb{R}^n} |g|^{p^*(1-1/n)}(y)dy}{\left(\int_{\mathbb{R}^n} |y|^q |g|^{p^*}(y)dy\right)^{\frac{1}{q}}} \leq \frac{p(n-1)}{n(n-p)} \inf_{f \in L^{p*}} \frac{\|\nabla f\|_{L^p}}{\|f\|_{L^{p^*}}}. \quad (3)$$

   The inequality is sharp and is attained if $f(x) = g(x) = h_p(x) := \frac{1}{\left(1+\|x\|_E^q\right)^{\frac{n-p}{p}}}$.

2. *The following sharp Sobolev inequality holds for all $f \in \dot{W}^{1,p}(E)$:*

$$\|f\|_{L^{p^*}(E;\mathbb{R})} \leq C_E \|\nabla f\|_{L^p(E;E')}. \quad (4)$$

   where $C_E = \frac{\|\nabla h_p\|_{L^p}}{\|h_p\|_{L^{p^*}}}$.

Theorem 2 (Isoperimetry)**.** *If $f \in C_c^\infty(E)$ and $f \neq 0$ then*

$$\frac{\|\nabla f\|_{L^1}}{\|f\|_{L^{n/(n-1)}}} \geq n|B|^{1/n} \quad (5)$$

where B is the unit ball in E and $|B|$ is its Lebesgue measure. This equality extends to functions $f \in BV(E)$ and equality holds if $f(x) = h_1(x) = 1_B(x)$.

15.3 Some notions about mass transport and convex functions

The proof of both the above theorems are based on a displacement concavity argument. We will avoid going into details of the general definitions and restrict ourselves to the following notions. Given a measure spaces (X, μ)

and (Y, ν) with a measurable map $\Phi : X \to Y$ we say that ν is the push-forward of the measure μ via Φ ($\Phi^{\#}\mu = \nu$) or that Φ transports μ to ν if

$$\int_Y b(y) d\Phi^{\#}\mu(y) = \int_Y b(\Phi(x)) d\mu(x) \qquad \forall b : Y \to \mathbb{R}^+. \text{ Borel function.}$$

A deep result obtained by Brenier [2] and refined by McCann [**?**] is

Proposition 3 (Brenier map)**.** *If μ and ν are probability measures on $\mathbb{R}^n$ and μ is absolutely continuous with respect to the Lebesgue measure then there exists a convex function ϕ such that $\nabla\phi^{\#}\mu = \nu$ and $\nabla\phi$ is uniquely determined μ-almost everywhere.*

The Brenier map originates as the minimizer over all transport maps Φ of μ to ν of the functional $\int |\Phi(x) - x|^2 dx$. This is not at all needed for our application but is strictly related to the convexity of ϕ that is crucial for our discussion. If both μ and ν are given by respective densities $\mu = F(x)dx$ and $\nu = G(x)dx$ on $\mathbb{R}^n$ a formal change of variables would yield that ϕ satisfies the Monge-Ampére equation:

$$F(x) = G(\nabla\phi) \det d^2\phi(x). \tag{6}$$

An approximation argument yields that (6) holds in the F-almost everywhere sense.

Furthermore we also use this crucial characterization of the Hessian of a convex function due originally to Alexandrov [4].

Proposition 4 (Second Derivatives of Convex Functions.)**.** *Let ϕ be a convex function on $\mathbb{R}^n$. Then its distributional Hessian $d^2\phi$ is a measure with values in positive semidefinite symmetric real matrixes. More precisely for $i, j \in \{1, \dots, n\}$ there exists measures $\rho^{i,j}$ so that*

$$\int \psi d\rho^{i,j}(x) = \int \partial_i \partial_j \psi(x) \phi(x) dx \qquad \forall \psi \in C_c^2(\mathbb{R}^n) \tag{7}$$

and for any $\xi \in \mathbb{R}^n$ the measures $\sum_{i,j} \xi_i \xi_j \rho^{i,j}$ are positive. It is natural to subdivide $d^2\phi = d^2\phi_a + d^2\phi_s$ into the absolutely continuous and singular part respectively.

15.4 Sketch of the proof

Here we illustrate the main points of the proof of Theorem 1. Notice that the duality statement of the Theorem implies the Sobolev inequality. The sharpness follows from the fact that the equality in the duality statement is attained.

We the first ingredient is a simple formulation of the fact that the functional $F \mapsto \int F^{1-1/n} dx$ is displacement concave according to the definition of McCann [3]. Practically let $F(x)dx$ and $G(x)dx$ be non-negative probability measures with smooth compactly supported densities on $\mathbb{R}^n$ and let ϕ be the corresponding Brenier map. Then

$$\int G^{1-1/n}(x)dx \leq \frac{1}{n} \int F^{1-1/n}(x) \Delta\phi_{ac}(x)dx. \tag{8}$$

This is proven by applying the geometric mean - arithmetic mean inequality to Equation (6).

After renormalization we apply this result to $F = |f|^{P^*}$ and $G = |g|^{p^*}$. Integration by parts and Hölder's inequality yield

$$\begin{aligned} \int |g|^{p^*} &\leq -\frac{p(n-1)}{n(n-p)} \int \overbrace{f^{\frac{p^*}{q}} \nabla\phi(x)} \, \overbrace{\nabla f(x)} \, dx \\ \underbrace{\leq}_{\text{Hölder}} & \frac{p(n-1)}{n(n-p)} \|\nabla f\|_{L^p} \left(\int |\nabla\phi|^q(x) F(x) dx \right)^{1/q}. \end{aligned} \tag{9}$$

Applying the definition of transport yields the duality statement

The extremizers can be found by imposing that equality holds for the geometric - arithmetic mean inequality on the transport map (for example setting it to be the Id operator). Imposing equality in Hölder's inequality gives a condition that results in the explicit form of h_p.

15.5 Remarks and further topics

Some remarks on the procedure we illustrated above are in order. First of all a very similar procedure allows us to prove Theorem 2. While the procedure is similar unfortunately we lose the duality principle. Furthermore also the Gagliardo-Nirenberg inequality can be proven with a similar approach. The difference is that one needs to use the concavity of the map

$$M \mapsto (\det M)^{1-\gamma} \qquad \gamma \in [1 - 1/n; 1]$$

for positive semi-definite symmetric matrixes. Unfortunately the argument breaks down outside a range of exponents that are a strict subset of the exponents for which the Gagliardo-Nirenberg inequality is known to hold. The authors of [1] believe that expanding the range of exponents could be obtained by better understanding the geometric duality.

Finally, as mentioned above, the simplicity of the argument together with the use of sharp inequalities (Hölder and concavity) allows us to track equality and to prove a stronger result:

Theorem 5. *Equality in the duality relation* (2) *holds for* $\dot{W}^{1,p}$ *functions if and only if* f *and* g *are of the form* $ch_p(\lambda^{-1}(\cdot - x_0))$ *for some* $x_0 \in \mathbb{R}^n$, $c \in \mathbb{R}$ *and* $\lambda \in \mathbb{R}^+$.

The proof of this result is complicated only by the necessity of checking that formal steps like integration by parts work for given the absence regularity of the transport map and of the extremizers.

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16 Brunn-Minkowski inequalities and log concave functions

after B. Simon [1]
A summary written by Michał Warchalski

Abstract

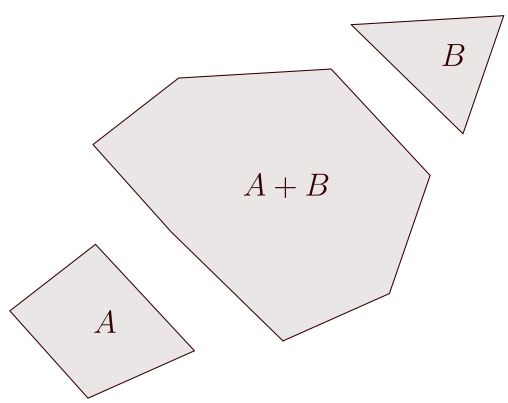
We give two proofs of the Brunn-Minkowski inequality, one rather elementary and the other being an application of the theory of log concave functions, namely Prékopa's theorem. Moreover we make a use of it proving the isoperimetric inequality. We also note some relations with mass transportation and discuss certain applications of Prékopa's theorem, among which there is a Gaussian version of the Brunn-Minkowski inequality.

16.1 Introduction

The Brunn-Minkowski inequality in its simplest form says that for two nonempty Borel sets $A, B \subset \mathbb{R}^n$ we have the following

$$|A+B|^{1/n} \geq |A|^{1/n} + |B|^{1/n}, \tag{1}$$

where $|\cdot|$ denotes the Lebesgue measure.



We will give two different proofs of this inequality and apply it to the isoperimetric inequality.

The first proof will be rather elementary, initially showing the inequality for disjoint unions of rectangles via an induction argument and passing to general sets by approximation. We will also note an implicit connection of this proof with mass transportation. The second proof, on the other hand, follows from an application of a more general fact based on the theory of log concave functions, namely Prékopa's theorem. Hence, beforehand we will introduce log concave functions and briefly look at their properties. Moreover, we will discuss a Gaussian version of the Brunn-Minkowski inequality as well as a kind of a counterpart of Prékopa's theorem for measures given by Gaussian densities.

First of all we note that we have another two equivalent formulations of (1). For two nonempty Borel sets $A_0, A_1 \subset \mathbb{R}^n$ and $\theta \in [0,1]$ let

$$A_\theta = \theta A_0 + (1-\theta) A_1.$$

Proposition 1. *The following statements are equivalent.*

(i) For any two Borel sets $A, B \subset \mathbb{R}^n$

$$|A+B|^{1/n} \geq |A|^{1/n} + |B|^{1/n}.$$

(ii) For any two Borel sets $A_0, A_1 \subset \mathbb{R}^n$

$$|A_\theta|^{1/n} \geq \theta |A_0|^{1/n} + (1-\theta)|A_1|^{1/n}. \tag{2}$$

(iii) For any two Borel sets $A_0, A_1 \subset \mathbb{R}^n$

$$|A_\theta| \geq |A_0|^\theta |A_1|^{1-\theta}. \tag{3}$$

16.2 Application - the isoperimetric inequality

In this subsection we assume the Brunn-Minkowski inequality and apply it to the isoperimetric inequality. In particular, we will make a use of the formulation (3).

Let

$$A^{(\varepsilon)} = \{x \colon d(x,A) < \varepsilon\},$$

where $d(x,A)$ the denotes the distance of the point x from the set A. Given an arbitrary measurable set A, we define its surface area by

$$s(A) := \liminf_{\varepsilon \to 0} \frac{|A^{(\varepsilon)}| - |A|}{\varepsilon}.$$

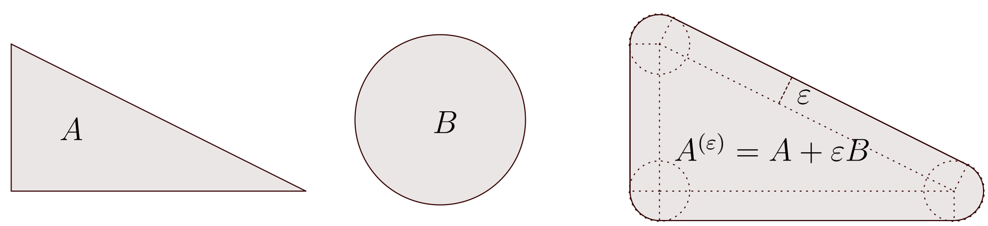
In many cases, for example when A has smooth boundary, the above limit exists. We consider $\liminf$ instead of $\limsup$, only because then the isoperimetric inequality is stronger.

Theorem 2 (Isoperimetric inequality)**.** *Let A be a measurable set in $\mathbb{R}^n$ and let B the open ball with the same volume as A. Then $s(A) \geq s(B)$.*

Proof. By scaling we may assume that A has the same volume as the unit ball B. Then we have

$$A^{(\varepsilon)} = A + \varepsilon B = (1+\varepsilon)\left(\frac{1}{1+\varepsilon}A + \frac{\varepsilon}{1+\varepsilon}B\right) = (1+\varepsilon)A_{\theta(\varepsilon)},$$

with $\theta(\varepsilon) = \varepsilon/(1+\varepsilon)$ and $A_0 = A$, $A_1 = B$.



Applying the Brunn-Minkowski inequality (3) in this setting we obtain

$$|A^{(\varepsilon)}| \geq (1+\varepsilon)^n |B|.$$

substracting $|A| = |B|$, dividing by ε and passing to the $\liminf$ on both sides we obtain

$$s(A) \geq n|B|.$$

This is exactly what we need, since

$$n|B| = \int_0^1 \int_{S^{n-1}} d\Omega \; nr^{n-1} dr = s(B).$$

□

16.3 The first proof

Here we will shortly sketch the first, more elementary, proof of the Brunn-Minkowski inequality (1).

Proof. As announced before, we proceed by induction, initially proving the result for disjoint unions of rectangles with sides parallel to the coordinate axes. If A is rectangle spanned by the sides $a_1, a_2, ..., a_n$ and B is spanned by the $b_1, b_2, ..., b_n$, then $A+B$ is spanned by the sides $a_1+b_1, a_2+b_2, ..., a_n+b_n$. Then, by the geometric-arithemtic mean inequality we get

$$\prod_{j=1}^{n} c_j^{1/n} + \prod_{j=1}^{n} d_j^{1/n} \leq 1,$$

with $c_j = a_j/(a_j + b_j)$ and $d_j = b_j/(a_j + b_j)$, what is equivalent to the Brunn-Minkowski inequality in this case.

Next, we assume that A is a disjoint union of l rectangles and B is a disjoint union of m rectangles, with $l + m \geq 3$. By disjointness, there exists a coordinate x_i and a number α, such that the hyperplane $x_i = \alpha$ separates at least one pair of rectangles of A. This means that

$$A_< = A \cap \{x \colon x_i < \alpha\},$$

$$A_> = A \cap \{x \colon x_i > \alpha\},$$

are disjoint unions of at most $l - 1$ rectangles each.

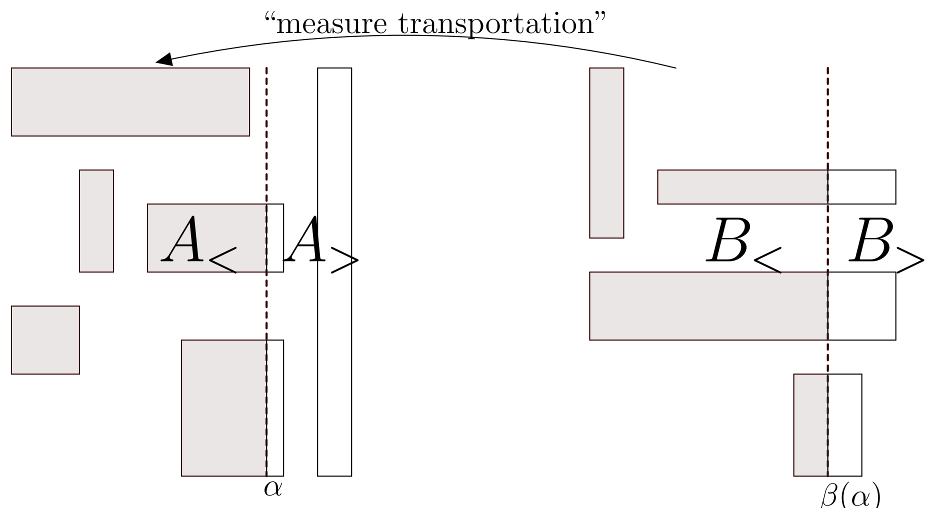
Now, by continuity of Lebesgue measure we can find a number $\beta(\alpha)$, such that $x_i = \beta(\alpha)$ separates the set B proportionally the same, so for

$$B_< = B \cap \{x \colon x_i < \beta(\alpha)\},$$

$$B_> = B \cap \{x \colon x_i > \beta(\alpha)\},$$

we have

$$\frac{|B_<|}{|A_<|} = \frac{|B_>|}{|A_>|} = \frac{|B|}{|A|}.$$



This means that the function $\beta(\cdot)$ actually *transports* the probability measure ν on the real line given by $\nu(-\infty, t) = |B \cap \{x \colon x_i < t\}|/|B|$ to the probability measure μ defined as $\mu(-\infty, t) = |A \cap \{x \colon x_i < t\}|/|A|$. Using the last display and that $A_<, B_<$ and $A_>, B_>$ are unions of at most $l - 1 + m$ rectangles, the induction gives

$$
\begin{aligned}
|A + B| &\overset{\text{disjointness}}{\geq} |A_< + B_<| + |A_> + B_>| \\
&\overset{\text{induction}}{\geq} (|A_<|^{1/n} + |B_<|^{1/n})^n + (|A_>|^{1/n} + |B_>|^{1/n})^n \\
&\overset{\text{the choice of } \beta}{=} \frac{|A_<|}{|A|}(|A|^{1/n} + |B|^{1/n})^n + \frac{|A_>|}{|A|}(|A|^{1/n} + |B|^{1/n})^n \\
&= (|A|^{1/n} + |B|^{1/n})^n.
\end{aligned}
$$

□

16.4 The second proof

The second proof of (3) follows from a more general fact, however the argument works for *convex* A_0 and A_1. It is based on the observation that if we define

$$C = \{(x, t) \colon 0 \leq t \leq 1,\ x \in tA_0 + (1 - t)A_1 = A_t\} \subset \mathbb{R}^{n+1},$$

then the characteristic function of C for $0 \leq \theta \leq 1$ obeys the condition

$$\chi_C(\theta(x, t) + (1 - \theta)(y, s)) \geq \chi_C(x, t)^\theta \chi_C(y, s)^{1-\theta} \tag{4}$$

(if we do this computation carefully, we use $\theta A_0 + (1-\theta)A_0 \subset A_0$ and $\theta A_1 + (1-\theta)A_1 \subset A_1$ for $0 \le \theta \le 1$ - that is why we need the convexity of A_0 and A_1). The Brunn-Minkowski inequality (3) says that an inequality similar to the above holds also after integrating out χ_C in $\mathbb{R}^n$

$$\int \chi_C(x,\theta)dx = |A_\theta| \\ \ge |A_0|^\theta |A_1|^{1-\theta} = \left(\int \chi_C(x,0)dx\right)^\theta \left(\int \chi_C(x,1)dx\right)^{1-\theta}$$

Note that (4) roughly means, that $\log \chi_C$ is concave. Functions of this kind we call *log concave.*

Definition 3. *A function $f\colon \mathbb{R}^n \to [0,\infty]$ is log concave if and only if it is lower semicontinuous and for any $x, y \in \mathbb{R}^n$, and $0 \le \theta \le 1$,*

$$f(\theta x + (1-\theta)y) \ge f(x)^\theta f(y)^{1-\theta}.$$

The above considerations show that $\chi_C\colon \mathbb{R}^{n+1} \to [0,\infty]$ is log concave and the Brunn-Minkowski inequality follows from *Prékopa's theorem.*

Theorem 4 (Prékopa's theorem)**.** *Let f be a log concave function on $\mathbb{R}^{n+m}$, we write $f(x,y)$ for $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$. Then the function g, defined as*

$$g(y) := \int_{\mathbb{R}^n} f(x,y)dx$$

is a log concave function on $\mathbb{R}^m$.

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17 Sharp fractional Hardy inequalities

after Rupert L. Frank and Robert Seiringer [?]
A summary written by An Zhang

Abstract

The authors found a sharp fractional version of the Hardy inequality, together with an explicit remainder term, which in particular also recovers the sharp Sobolev embedding into the Lorentz space. The proof developed a nonlinear and nonlocal analogue of the classical ground state representation in a general setting.

17.1 Introduction

In this paper, we will study several Hardy inequalities, a generalization of the classical inequality on $\mathbb{R}^N$:

$$|\nabla u|_p \geq p^{-1}|N-p||x^{-1}u|_p, \tag{1}$$

which is an important object for study in analysis and mathematical physics. Extremizer (extremal function that make equality hold) for (1) doesn't exist in the corresponding homogeneous Sobolev space $\dot{W}^{1,p}$ for $p > 1$, while for $p = 1$, any symmetric decreasing function makes the inequality an equality. It's then natural for people to consider sharp estimates for some fractional inequalities of Hardy-type, both for (non)-local and (non)-linear cases, see [?, ?, ?] and reference there. Especially, Mazya-Shaposhnikova [?] got a nice bound (but not sharp) for the fractional Hardy (FH) inequality, which was there used to improve a Sobolev embedding estimate of Bourgain-Brezis-Mironescu [?]. Then a possible question is what's the sharp constant for the FH inequality. Recently, Frank-Lieb [?] and then [?] got the sharp constant for the FH inequality, respectively for linear-nonlocal and general nonlinear-nonlocal cases, with a ground state representation formula. The main new result of [?] is the following theorem giving a sharp fractional Hardy inequality for general exponent $p \geq 1$ and fraction $0 < s < 1$ with a remainder term for $p \geq 2$. The method is basically from nonlinear analysis, and some rearrangement will be needed to solve equality.

Theorem 1. *Let $N \geq 1, 0 < s < 1, N/s \neq p \geq 1$, then for all $u \in \dot{W}^{s,p}$,*[5]

$$\iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} dxdy \geq C \int_{\mathbb{R}^N} \frac{|u(x)|^p}{|x|^{ps}} dx \tag{2}$$

with the sharp constant

$$C = 2 \int_0^1 r^{ps-1} |1 - r^{(N-ps)/p}|^p \phi(r) dr,$$

where

$$\phi(r) = \begin{cases} |\mathbb{S}^{N-2}| \int_{-1}^1 \frac{(1-t^2)^{(N-3)/2} dt}{(1-2rt+r^2)^{(N+ps)/2}}, & N \geq 2 \\ (1+r)^{-(1+ps)} + (1-r)^{-(1+ps)}, & N = 1. \end{cases}$$

Extremizers for (2) *don't exist for $p > 1$, while are all symmetric decreasing functions for $p = 1$. Moreover, if $p \geq 2$,* (2) *has a remainder term*

$$c \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|v(x) - v(y)|^p dxdy}{|x - y|^{N+ps} (|x||y|)^{(N-ps)/2}}, \tag{3}$$

where

$$v = |x|^{(N-ps)/p} u, \;\; c = \min_{\tau \in (0,1/2)} \{(1 - \tau)^p - \tau^p + p\tau^{p-1}\}$$

Remark 2. *1. For simplicity, we fix C and c in this note. 2. The FH inequality* (2) *with remainder term* (3) *is an equality when $p = 2$ and $c = 1$, and the ground state representation for this special linear case has been given by* [**?**]. *The Fourier transform is a powerful tool in this case. 3. For nonlinear case, the core observation is that function $\omega = |x|^{(N-ps)/p}$ is a critical point of the FH functional* (2) *but fails to lie in the handling space $\dot{W}^{s,p}$. It would be easier to see this from a local case in next subsection. 4. A direct corollary is a sharp Sobolev embedding into Lorentz space $W^{s,p} \hookrightarrow L^{p^*,p}, p < N/s, p^* = Np/(N - ps)$, and the fractional Sobolev estimate of Bourgain-Brezis-Mironescu and Mazya-Shaposhnikova. The first sharp estimate for Lorentz space is got from a rearrangement of Almgren-Lieb-type and the relation between the Lorentz norm and weighted L^p-norm. The second estimate comes from trivial Lorentz embedding.*

[5] In default, we use the Sobolev space on $\mathbb{R}^N$ for $p < N/s$, and $\mathbb{R}^N \setminus \{0\}$ for $p < N/s$.

17.2 A general setting and examples

17.2.1 General setting

To generalize, we take a nonnegative symmetric measurable function $k(x, y)$, consider the functional $E[u] = \iint |u(x) - u(y)|^p k(x, y)$, and if assume that a ground state condition holds for a positive function ω and a real-valued potential V,

$$2 \int_{\mathbb{R}^N} (\omega(x) - \omega(y))|\omega(x) - \omega(y)|^{p-2} k(x, y) dy = V(x)\omega(x)^{p-1},$$

then we anticipate analogues of the sharp FH inequality, after setting $u = \omega v$,

$$E[u] - \int V|u|^p \geq c \iint |v(x) - v(y)|^p (\omega(x)\omega(y))^{p/2} k(x, y) dx dy \ \chi_{p \geq 2} \qquad (4)$$

To repair the possible diagonal singularity of the kernel, we give the following precise "virtual ground state condition".

Condition 3. *There exist positive measurable* ω *and a family of nonnegative symmetric measurable functions* $(k_\varepsilon)_{\varepsilon>0} (\leq k) \to k, \varepsilon \to 0$*, satisfying that*

$$V_\varepsilon = 2\omega(x)^{-p+1} \int_{\mathbb{R}^N} (\omega(x) - \omega(y))|\omega(x) - \omega(y)|^{p-2} k_\varepsilon(x, y) dy$$

are absolutely convergent a.e., and have a weak limit V *in* L^1_{loc}*.*

Proposition 4. *Under above condition, the general sharp FH inequality with remainder term, the representation formula* (4) *holds.*

17.2.2 Examples

Linear case ($p = 2$).

1. pseudo-relativistic Schrodinger operators $\sqrt{-\Delta + m^2} + V_0$.

2. Schrodinger-type operator $t(-i\nabla) + V_0$, where locally bounded function $0 \leq t \lesssim |x|^{2s}$ for large x and some $s \in (0, 1)$, and $V_0 \in L^{N/(2s)} + L^\infty$.

Intuition from a local case. Take $E[u] = \int g|\nabla u|^p$ with a positive weight, then if ω is the ground state of the energy functional $I = E[u] - \int V|u|^p$, that is a positive weak solution of the corresponding weighted p-Laplace equation

$$-div(g|\nabla\omega|^{p-2}\nabla\omega) = V\omega^{p-1}$$

then we have sharp FH inequality $I \geq 0$.
Proof. If we write $u = wv$, then from convexity inequality

$$|a+b|^p \geq |a|^p + p|a|^{p-2}Re\bar{a}\cdot b, \quad for\ any\ a, b \in \mathbb{C}^N, p \geq 1,$$

and the p-Laplacian equation for ω, we get

$$\begin{aligned}
E[u] &= \int g|v\nabla\omega + \omega\nabla v|^p \\
&\geq \int g|v|^p|\nabla\omega|^p + p\int g|\nabla\omega|^{p-2}\omega Re\bar{v}|v|^{p-2}\nabla v\cdot\nabla\omega \\
&= \int g|v|^p|\nabla\omega|^p + \int g|\nabla\omega|^{p-2}\omega\nabla(|v|^p)\cdot\nabla\omega \\
&= \int g|v|^p|\nabla\omega|^p - \int div(g|\nabla\omega|^{p-2}\omega\nabla\omega)|v|^p \\
&= \int g|v|^p|\nabla\omega|^p - \int div(g|\nabla\omega|^{p-2}\nabla\omega)\omega|v|^p - \int g|\nabla\omega|^{p-2}\Delta\omega|v|^p \\
&= \int V|u|^p.
\end{aligned}$$

From an enhanced convexity with remainder term $c|b|^p$, we can get further a remainder term $I \geq c\int g\omega^p|\nabla v|^p$ similarly. This is the well-known ground state representation. In the special case $g = 1, w = |x|^{(N-p)/p}$, we recover that for the classical inequality (1) uniformly.

17.3 Proof of the theorem

Using the idea from the proof for the local case in above subsection, proposition 4 is trivial from the following lemma, using the ground state condition.

Lemma 5. *Let $p \geq 1$. For all $t \in [0,1], a \in \mathbb{C}$,*

$$|a-t|^p \geq (1-t)^{p-1}(|a|^p - t).$$

For $p > 1$, the inequality is strict unless $a = 1$ or $t = 0$. Moreover, if $p \geq 2$, then the inequality has a remainder term $ct^{p/2}|a-1|^p$, which reaches equality for $p = 2$, while is strict unless $a = 1$ or $t = 0$ for $p > 2$.

Theorem 1 comes from proposition 4 and a lemma which checks the ground state condition. From this condition, we can give two other formulas for the sharp constant C after setting $\alpha = (N - ps)/p$,

$$\begin{aligned} C &= 2\lim_{\varepsilon\to 0}\int_{|\rho-1|>\varepsilon} \frac{d\rho sgn(\rho^\alpha - 1)}{|\rho-1|^{2-p(1-s)}}\left|\frac{1-\rho^{-\alpha}}{1-\rho}\right|^{p-1} \begin{cases} \rho^{N-1}(1-\rho)^{1+ps}\phi(\rho), & \rho < 1 \\ (1-\rho^{-1})^{1+ps}\phi(\rho^{-1}), & \rho > 1, \end{cases} \\ &= \frac{2|N-ps|}{p|\mathbb{S}^{N-1}|}\iint_{|x|<1<|y|} \frac{||x|^{-\alpha} - |y|^{-\alpha}|^{p-1} dxdy}{|x-y|^{N+ps}}. \end{aligned}$$

Sharpness of C. For $p = 1$, it's trivial by inserting symmetric decreasing functions. For $p > 1$, extremizer doesn't exist and to prove the sharpness of C, we need to find a family of functions s.t. the quotient functional goes to C. Here we can choose, e.g. for $N > ps$, $u_n = 1 - n^{-\alpha}, |x| < 1; = |x|^{-\alpha} - n^{-\alpha}, 1 \leq |x| < n; = 0$ otherwise.

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18 Optimal Young inequality: a symmetric proof

after F. Barthe [2]
A summary written by Pavel Zorin-Kranich

Abstract

We give a symmetric mass transport proof of the fact that Gaussians are extremizers of the Young convolution inequality in dimension 1.

This summary answers a question that came up during the summer school: is there a version of the mass transport argument from Barthe's article [1] (presented by Johanna Richter) that is symmetric in the three functions in the dualized form of Young's convolution inequality? This is indeed the case, and in fact even more general multilinear forms (appearing in Brascamp–Lieb inequalities) have been treated by this type of arguments, see [2]. Here we present a particularly simple instance in which the form is of convolution type and all functions are one-dimensional.

Theorem 1. *Let $n \geq 2$ and consider exponents $1 \leq p_i < \infty$, $1 \leq i \leq n$, with $\sum_i \frac{1}{p_i} = n-1$. Let also $\alpha \in \mathbb{R}^n$ be given by $\alpha_i = (p_i')^{-1/2}$, where p_i' is the Hölder conjugate defined by $\frac{1}{p_i} + \frac{1}{p_i'} = 1$. Then the form*

$$\Lambda(\vec{f}) := \int_{\alpha^\perp \subset \mathbb{R}^n} \otimes_{i=1}^n f_i^{1/p_i}$$

is maximized among the functions with $\|f_1\|_1 = \cdots = \|f_n\|_1 = 1$ by the Gaussians

$$F_i(x) = \sqrt{\frac{p_i}{\pi}} e^{-p_i x^2}.$$

The sharp Young convolution inequality can be recovered by a change of variables from the case $n = 3$. Unlike the proof in [1], our proof does not use Hölder's inequality, and the above result in fact includes it as the special case $n = 2$ (again after a change of variables).

Proof. By standard truncation and approximation arguments we may restrict attention to strictly positive, smooth functions f_i. In this case for each

$i = 1, \dots, n$ there exists a unique monotonically increasing smooth map $u_i : \mathbb{R} \to \mathbb{R}$ such that the pushforward measure $(u_i)_\sharp f_i(x)dx$ coincides with $F_i(x)dx$. This can be equivalently formulated as

$$f_i \cdot (u_i')^{-1} = F_i \circ u_i. \tag{1}$$

Let now $U := \otimes_{i=1}^n u_i$ and let $\pi : \mathbb{R} \to \alpha^\perp$ denote the orthogonal projection onto the hyperplane $\alpha^\perp$. Consider the change of variables $\pi \circ U : \alpha^\perp \to \alpha^\perp$. Strict monotonicity of the u_i's and positivity of α_i's imply that this change of variables is bijective.

The Jacobian $J : \alpha^\perp \to \alpha^\perp$ of this change of variables equals $\pi U' \pi$. Writing $U' : \mathbb{R}^n \to \mathbb{R}^n$ in an orthogonal basis $(\alpha, *)$ we see that it has the form

$$\begin{pmatrix} * & * \\ * & J \end{pmatrix}.$$

Hence its $(1,1)$-minor equals $\det J$. On the other hand, by Cramer's rule it equals $\det U$ times the $(1,1)$-entry of $(U')^{-1}$, which in turn equals $\langle \alpha, (U')^{-1}\alpha \rangle$, since α is a unit vector by our scaling condition. Hence

$$\det(J) = \frac{\sum_{i=1}^n \alpha_i (u_i')^{-1}}{\prod_{i=1}^n (u_i')^{-1}}.$$

The change of variables formula now gives

$$\begin{aligned}
\Lambda(\vec{f}) &= \int_{\alpha^\perp} \frac{\otimes_{i=1}^n f_i^{1/p_i}}{\det(J)} \circ U^{-1} \circ \pi^{-1} \\
&= \int_{\alpha^\perp} \frac{\otimes_{i=1}^n (f_i^{1/p_i} (u_i')^{-1})}{\sum_{i=1}^n \alpha_i (u_i')^{-1}} \circ U^{-1} \circ \pi^{-1} \\
&= \int_{\alpha^\perp} \Big(\otimes_{i=1}^n F_i^{1/p_i} \circ \pi^{-1} \Big) \Big(\frac{\otimes_{i=1}^n (u_i')^{-1/p_i'}}{\sum_{i=1}^n \alpha_i (u_i')^{-1}} \circ U^{-1} \circ \pi^{-1} \Big),
\end{aligned}$$

where we have used (1) in the last line. In the second tensor product we use once again $\sum_{i=1}^n \frac{1}{p_i'} = 1$ and the arithmetic-geometric mean inequality to estimate

$$\prod_{i=1}^n (u_i')^{-1/p_i'} \leq \sum_{i=1}^n \alpha_i (u_i')^{-1}.$$

It follows that

$$\Lambda(\vec{f}) \leq \int_{\alpha^\perp} \otimes_{i=1}^n F_i^{1/p_i} \circ \pi^{-1}.$$

Now note that the function $\otimes_{i=1}^n F_i^{1/p_i}$ equals a constant times a Gaussian, so that

$$\otimes_{i=1}^n F_i^{1/p_i} \circ \pi^{-1} \leq \otimes_{i=1}^n F_i^{1/p_i}.$$

It follows that

$$\Lambda(\vec{f}) \leq \int_{\alpha^\perp} \otimes_{i=1}^n F_i^{1/p_i} = \Lambda(\vec{F})$$

as claimed. □

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