Nonlinear Fourier analysis and applications

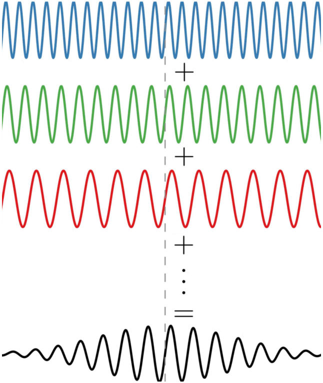
Brouwer Medal Lecture

Christoph Thiele

Antwerpen, April 8th 2026

Fourier series

$$v(\theta) = \sum_{n=-N}^{N} f_n e^{in\theta}$$



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Fourier series

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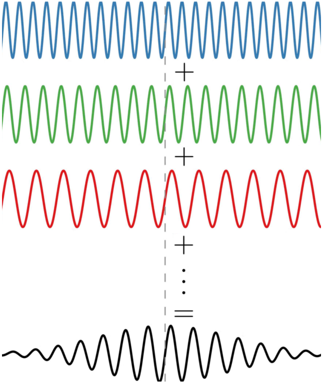
$$= \sum_{n=-N}^{N} f_n \cos(n\theta) + i f_n \sin(n\theta)$$



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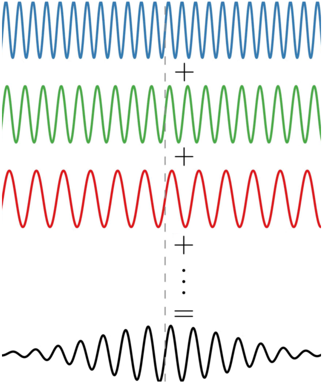
$$= \sum_{n=-N}^{N} f_n z^n$$



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Fourier's motivation: solving PDE, e.g. Airy equation

$$u(\theta, 0) = v(\theta)$$

$$\partial_t u(\theta, t) = \partial_\theta^3 u(\theta, t)$$

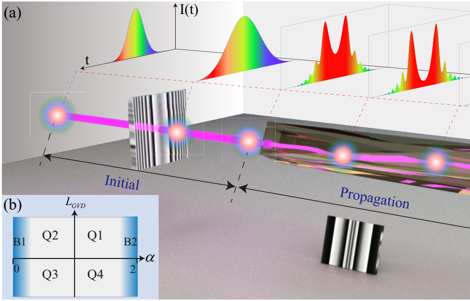

Image: S. Liu, Y. Zhang, B. A. Malomed, E. Karimi

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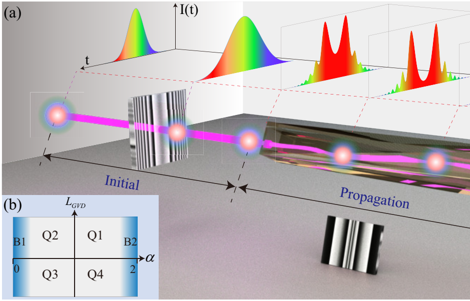

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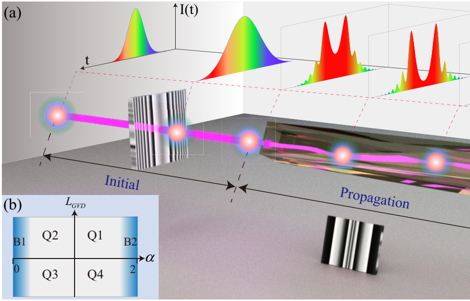
$$v \to f_n \to f_n e^{in^3 t} \to u(\theta,t)$$



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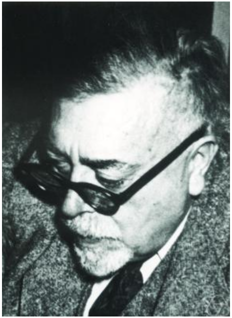
Representing general functions

For a sequence f_n with $\|f_n\|_1 \leq 1$,

$$v_N(\theta) := \sum_{n=-N}^{N} f_n e^{in\theta}$$

has a limit $v(\theta)$ for every θ as $N \to \infty$



Norbert Wiener, Wikipedia

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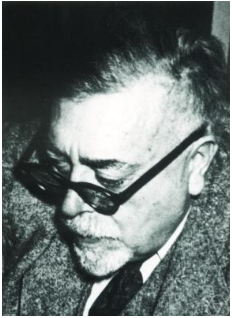

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Representing general functions

$\|f_n\|_p \leq 1$, if for all N have $\sum_{n=-N}^{N} |f_n|^p \leq 1$.

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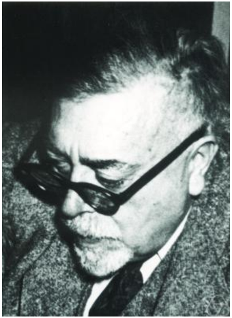

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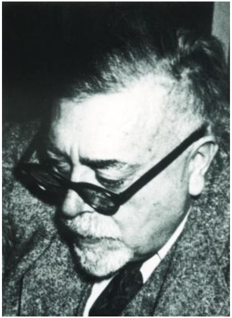


Norbert Wiener, Wikipedia

$\|f_n\|_p \le 1$ for $p > 1$ does not imply convergence at every θ.

Even more general functions

Plancherel:

$$\sum_{n=-N}^{N} |f_n|^2 = \frac{1}{2\pi}\int_0^{2\pi} |v_n(\theta)|^2 \, d\theta$$

David Hilbert, Wikipedia

Even more general functions

Plancherel:

$$\sum_{n=-N}^{N} |f_n|^2 = \frac{1}{2\pi} \int_0^{2\pi} |v_n(\theta)|^2 \, d\theta$$

If $\|f_n\|_2 \le 1$, there is a unique square integrable function v such that

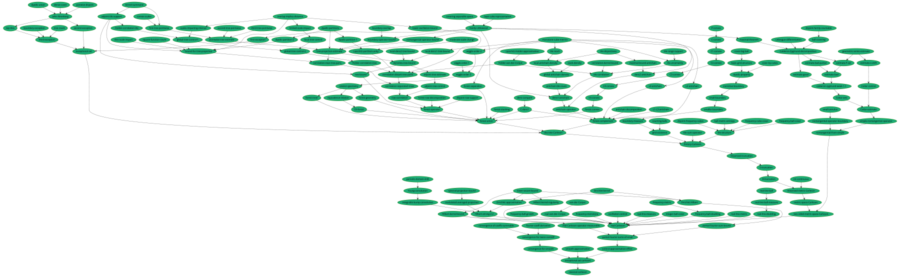
$$\int_0^{2\pi} |v_N(\theta) - v(\theta)|^2 d\theta$$

tends to 0 as $N \to \infty$.

David Hilbert, Wikipedia

Carleson's 1966 theorem, formalized 2025

If $\|f_n\|_2 \leq 1$, then for almost every θ the values $v_N(\theta)$ tend to $v(\theta)$.



Formalization '25 [F. van Doorn, CT, et al (+15)]

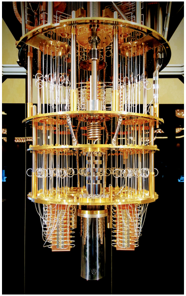
Lean Carleson project

Further Dutch influence: N.G.(Dick) de Bruijn, L.E.J.(Bertus) Brouwer

Quantum computer

Polynomials as alternation of multiplication by z and addition of f_n

$$\sum_{n=-N}^{N} f_n z^n = [\dots(((f_N z + f_{N-1})z + f_{N-2})z + f_{N-3})z + \dots + f_{-N}]\, z^{-N}$$



IBM Kryostat, Wikipedia

Quantum computer

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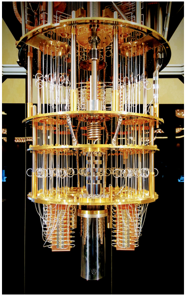


IBM Kryostat, Wikipedia

Alternate multiplication by two types of $SU(2)$ matrices:

$$Z = \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}$$

$$\frac{1}{\sqrt{1+f_n^2}} \begin{pmatrix} 1 & f_n \\ -f_n & 1 \end{pmatrix}$$

Nonlinear Fourier series suited for quantum computers.

Quantum computer

Polynomials as alternation of multiplication by z and addition of f_n

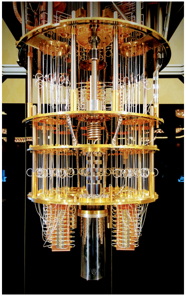
$$\sum_{n=-N}^{N} f_n z^n = [\ldots(((f_N z + f_{N-1})z + f_{N-2})z + f_{N-3})z + \cdots + f_{-N}]\, z^{-N}$$

Alternate multiplication by two types of $SU(2)$ matrices:

$$Z = \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} = \exp\left(i\theta \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right)$$

$$\frac{1}{\sqrt{1+f_n^2}} \begin{pmatrix} 1 & f_n \\ -f_n & 1 \end{pmatrix} = \exp\left(-i \arctan(f_n) \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \right)$$

Nonlinear Fourier series suited for quantum computers.



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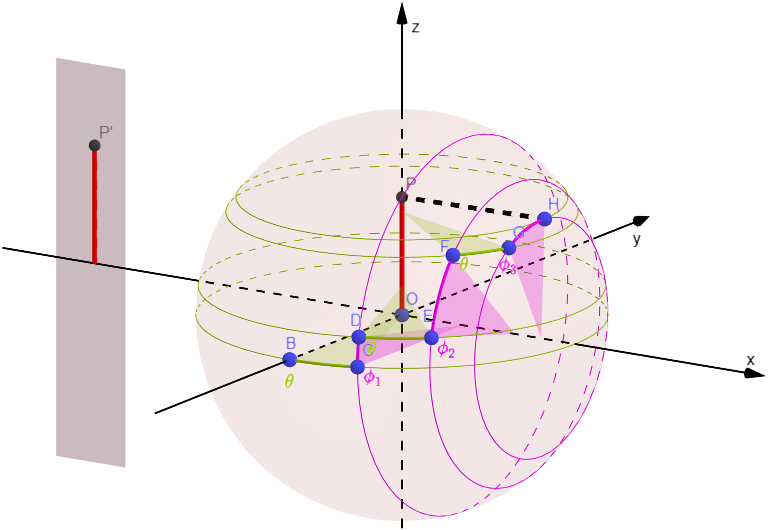
Visualization of nonlinear Fourier series

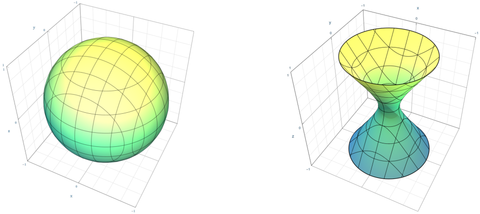


Two models: SU(2) and $SU(1,1)$

$$z^{\frac{-2-3N}{2}}\left(\prod_{n=-N}^{\nearrow N}\begin{pmatrix} z & 0 \\ 0 & z^{-1}\end{pmatrix}\frac{1}{\sqrt{1+|f_n|^2}}\begin{pmatrix} 1 & f_n \\ -\overline{f_n} & 1\end{pmatrix}\right)z^{\frac{-N}{2}}$$

$$=\prod_{n=-N}^{\nearrow N}\frac{1}{\sqrt{1+|f_n|^2}}\begin{pmatrix} 1 & f_n z^n \\ -\overline{f_n}z^{-n} & 1\end{pmatrix}=\begin{pmatrix} a_N(z) & b_N(z) \\ -\overline{b_N(z)} & \overline{a_N(z)}\end{pmatrix}$$

In linear approximation, b_N becomes the linear Fourier series.
Does limit (a, b) of (a_N, b_N) exists as $N \to \infty$?



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$$z^{\frac{-2-3N}{2}} \left(\prod_{n=-N}^{\nearrow N} \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} \frac{1}{\sqrt{1 \pm |f_n|^2}} \begin{pmatrix} 1 & f_n \\ \mp \overline{f_n} & 1 \end{pmatrix} \right) z^{\frac{-N}{2}}$$

$$= \prod_{n=-N}^{\nearrow N} \frac{1}{\sqrt{1 \pm |f_n|^2}} \begin{pmatrix} 1 & f_n z^n \\ \mp \overline{f_n} z^{-n} & 1 \end{pmatrix} = \begin{pmatrix} a_N(z) & b_N(z) \\ \mp \overline{b_N(z)} & \overline{a_N(z)} \end{pmatrix}$$

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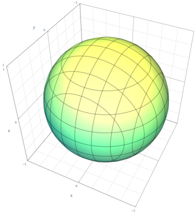
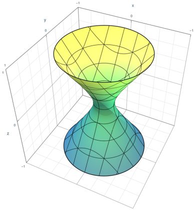
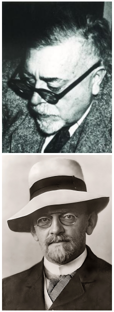




Quantum signal processing results

If $\|f_n\|_1 < \infty$, then limit $b(z)$ of $b_N(z)$ as $N \to \infty$ exists for all $|z| = 1$. The function b is "Wiener".



Plancherel: for "outer" a,

$$\int -\log(1 - |b(e^{i\theta}|^2)\, d\theta = \pi \sum_{n=-N}^{N} |f_n|^2$$

Thm '24 [M Alexis, L Lin, G.Mnatsakanyan, C.T., J. Wang]
$SU(2)$: For b with finite log-integral, there exists unique square summable sequence f_n such that b_N converge to b in mean and a is "outer".
Thm '25 [H. Ni, R. Sarkar, L. Ying, L. Lin]
The sequence f_n for $|n| \leq N$ can be computed in $CN \log(N)^2$ steps.

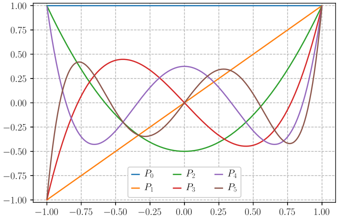
Orthogonal polynomials

In $SU(1,1)$ model, f_n supported on $n \geq 0$,

$$\phi_N(z) := z^N a_N(z) + \overline{b_N(\overline{z})}$$

are polynomials. For $N \neq M$, orthogonality

$$\langle \phi_N, \phi_M \rangle = 0\,,$$



where $\langle \phi, \psi \rangle$ is defined as

$$\langle \phi, \psi \rangle := \lim_{N \to \infty} \int_{\mathbb{T}} \frac{\phi(z)\overline{\psi(z)}}{(a_N(z) + \overline{b_N(\overline{z})})(\overline{a_N(\overline{z})} + b_N(z))}\,.$$

Convergence in measure.

Left and right orthogonal polynomials

In $SU(2)$ model, consider polynomials

$$\phi_N(z) := z^N(a_N(z) + \overline{b_N(\overline{z})}), \quad \tilde{\phi}_N(z) := z^N(a_N(z) - \overline{b_N(\overline{z})})$$

One observes for $N \neq M$,

$$\left\langle \phi_N, \tilde{\phi}_M \right\rangle = 0\,,$$

where $\langle \phi, \psi \rangle$ is defined as

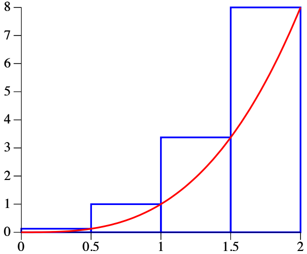
$$\langle \phi, \psi \rangle := \lim_{N \to \infty} \int_{\mathbb{T}} \frac{\phi(z)\overline{\psi(z)}}{(a_N(z) + \overline{b_N(\overline{z})})(\overline{a_N(\overline{z})} - b_N(z))}\,.$$

Thm '25 [M. Alexis, G. Mnatsakanyan, C.T.] If $|b(z)| < \frac{1}{2}$ for all z and a is "outer", then for almost all z the following converges for $k \to \infty$

$$\left[(a_{2^k}(z) + \overline{b_{2^k}(\overline{z})})(\overline{a_{2^k}(\overline{z})} - b_{2^k}(z))\right]^2$$

Fourier integral instead of sum

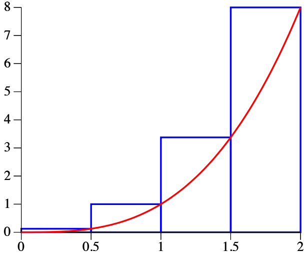
$$\widehat{f}(\xi) = \int_{-\infty}^{\infty} f(t)e^{-it\xi}dt$$



Fourier integral instead of sum

$$\widehat{f}(\xi) = \int_{-\infty}^{\infty} f(t)e^{-it\xi}dt = u(\xi,\infty)$$

$$u(\xi,x) := \int_{-\infty}^{x} f(t)e^{-it\xi}dt$$



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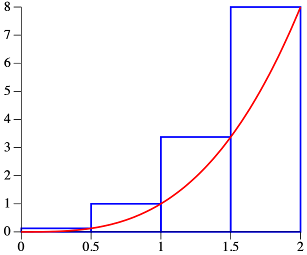


Differential equation

$$\partial_x u(\xi,x) = f(x)e^{-ix\xi},$$

Initial condition

$$u(\xi,-\infty) = 0,$$

Result

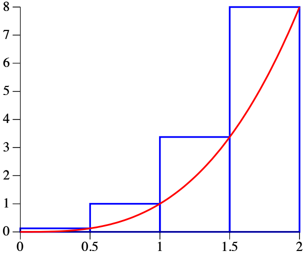
$$u(\xi,+\infty) = \widehat{f}(\xi)$$

Fourier integral instead of sum

$$\widehat{f}(\xi) = \int_{-\infty}^{\infty} f(t)e^{-it\xi}dt = u(\xi, \infty)$$

$$u(\xi, x) := \int_{-\infty}^{x} f(t)e^{-it\xi}dt$$

$$g(\xi, x) := e^{u(\xi,x)}$$



Differential equation	Initial condition	Result
$\partial_x u(\xi, x) = f(x)e^{-ix\xi}$,	$u(\xi, -\infty) = 0$,	$u(\xi, +\infty) = \widehat{f}(\xi)$
$\partial_x g(\xi, x) = g(\xi, x)f(x)e^{-ix\xi}$,	$g(\xi, -\infty) = 1$,	$g(\xi, \infty) = e^{\widehat{f}(\xi)}$

Two matrix models ($SU(2)$,$SU(1,1)$)

Differential equation

$$\partial_x G(\xi,x) = G(\xi,x)\begin{pmatrix} 0 & f(x)e^{-ix\xi} \\ \mp\overline{f}(x)e^{ix\xi} & 0 \end{pmatrix}$$

Initial condition

$$G(\xi,-\infty) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Result

$$G(\xi,\infty) = \begin{pmatrix} a(\xi) & b(\xi) \\ \mp\overline{b}(\xi) & \overline{a}(\xi) \end{pmatrix}$$

modified Korteweg de Vries equation

$$u(x,0) = v(x)$$

$$\partial_t u(x,t) = \partial_x^3 u(x,t) \pm 6u^2(x,t)\partial_x u(x,t)$$

$$v \to \widehat{v} \to \widehat{v} e^{i\xi^3 t} \to u(x,t)$$

$$v \to (a,b) \to (a, be^{i\xi^3 t}) \to u(x,t)$$

Solution method works if $|b(\xi)| < 1/2$ for all ξ and a "outer".

▸ solitary wave

Thank you

Stadspark Lier, 2018