ORTHOGONAL SPECTRA AND STABLE HOMOTOPY THEORY

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Abstract. These are course notes for the class *Algebraic Topology II* taught by the author at Bonn University in the summer terms 2022 and 2026.

Contents

Introduction

These notes aim to develop into an introduction to the foundations of stable homotopy theory and 'algebra' over structured ring spectra, based on orthogonal spectra. One possible road towards 'spectral algebra' is an associative and commutative smash product on a good point-set level category of spectra, which lifts the well-known smash product pairing on the *homotopy category*. Between the mid 1990's until around 2010, this was the preferred approach, and it is the path we will take in these notes. In recent years, the higher categorical approach pioneered by Lurie [24, 25] has become increasingly popular and versatile. And as more and more foundations are being worked out in this language, the theory grew more and more powerful; in the long run, I expect the higher categorical approach to prevail. Still, higher categorical theory demands a substantial intellectual investment for learning the basic formalism, and a model based approach should continue to have its merits.

We begin with a quick historical review and attempt at a motivation. A much more comprehensive and detailed history and of the early days of stable homotopy theory with many more references can be found in May's [32]. The first construction of what is now called 'the stable homotopy category', including its symmetric monoidal smash product, is due to Boardman [4, 5] (unpublished); accounts of Boardman's construction appear in [51], [48] and [1, Part III] (Adams devotes more than 30 pages to the construction and formal properties of the smash product).

To illustrate the drastic simplification that occurred in the foundations in the mid-90s, let us draw an analogy with the algebraic context. Let R be a commutative ring and imagine for a moment that the notion of a chain complex (of R-modules) has not been discovered, but nevertheless various complicated constructions of the unbounded derived category $\mathcal{D}(R)$ of the ring R exist. Moreover, constructions of

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the *derived* tensor product on the *derived* category exist, but they are complicated and the proof that the derived tensor product is associative and commutative occupies 30 pages. In this situation, you could talk about objects A in the derived category together with morphisms $A \otimes_R^L A \longrightarrow A$, in the derived category, which are associative and unital, and possibly commutative, again in the derived category. This notion may be useful for some purposes, but it suffers from many defects – as one example, the category of modules (under derived tensor product in the derived category), does not in general form a triangulated category.

Now imagine that someone proposes the definition of a chain complex of R-modules and shows that by formally inverting the quasi-isomorphisms, one can construct the derived category. She also defines the tensor product of chain complexes and proves that tensoring with suitably nice (i.e., *homotopically projective*) complexes preserves quasi-isomorphisms. It immediately follows that the tensor product descends to an associative and commutative product on the derived category. What is even better, now one can suddenly consider differential graded algebras, a 'rigidified' version of the crude multiplication 'up-to-chain homotopy'. We would quickly discover that this notion is much more powerful and that differential graded algebras arise all over the place (while chain complexes with a multiplication which is merely associative up to chain homotopy seldom come up in nature).

Fortunately, this is not the historical course of development in homological algebra, but the development in stable homotopy theory was, in several aspects, as indicated above. In the stable homotopy category people could consider ring spectra 'up to homotopy', which are closely related to multiplicative cohomology theories. However, the need and usefulness of ring spectra with rigidified multiplications soon became apparent, and topologists developed different ways of dealing with them. One line of approach uses operads for the bookkeeping of the homotopies which encode all higher forms of associativity and commutativity, and this led to the notions of A_∞- respectively E_∞-ring spectra. Various notions of point-set level ring spectra had been used (which were only later recognized as the monoids in a symmetric monoidal model category). For example, the orthogonal ring spectra had appeared as *$\mathscr{I}_*$-prefunctors* in [31], the *functors with smash product* were introduced in [6] and symmetric ring spectra appeared as *FSPs defined on spheres* in [17, 2.7].

At this point it had become clear that many technicalities could be avoided if one had a smash product on a good point-set category of spectra which was associative and unital *before* passage to the homotopy category. For a long time no such category was known, and there was even evidence that it might not exist [23]. In retrospect, the modern spectra categories could maybe have been found earlier if Quillen's formalism of *model categories* [36] had been taken more seriously; from the model category perspective, one should not expect an intrinsically 'left adjoint' construction like a smash product to have a good homotopical behavior in general, and along with the search for a smash product, one should look for a compatible notion of cofibrations.

In the mid-90s, several categories of spectra with nice smash products were discovered, and simultaneously, model categories experienced a major renaissance. Around 1993, Elmendorf, Kriz, Mandell and May introduced the *S-modules* [14] and Jeff Smith gave the first talks about *symmetric spectra*; the details of the model structure were later worked out and written up by Hovey, Shipley and Smith [19]. In 1995, Lydakis [26] independently discovered and studied the smash product for $\mathbf{\Gamma}$-spaces (in the sense of Segal [43]), and a little later he developed model structures and smash product for *simplicial functors* [27]. Except for the *S-modules* of Elmendorf, Kriz, Mandell and May, all other known models for spectra with nice smash product have a very similar flavor; they all arise as categories of continuous (or simplicial), space-valued functors from a symmetric monoidal indexing category, and the smash product is a convolution product (defined as a left Kan extension), which had much earlier been studied by the category theorist Day [10]. This unifying context was made explicit by Mandell, May, Schwede and Shipley in [30], where another example, the *orthogonal spectra* were first worked out in detail. The different approaches to spectra categories with smash product have been generalized and adapted to equivariant homotopy theory [12, 28, 29] and motivic homotopy theory [13, 20, 21].

There are already several good sources available which explain the stable homotopy category, starting with Adams' classic [1], and including [2, 38, 46]; these references do not focus on structured ring and module spectra, though. The monograph [14] by Elmendorf, Kriz, Mandell and May develops this theory in one of the competing frameworks, the S-modules, in detail. It has had a big impact and is widely used, for example because many standard tools can simply be quoted from that book. The theory of orthogonal spectra is by now also highly developed, but the results are spread over many research papers. The aim of these notes is to collect some basic facts about orthogonal spectra in one place, and use them to introduce the stable homotopy category as a tensor triangulated category. Needless to say that the tensor triangulated category stable homotopy category is only a shadow of the 'true' structure, i.e., the symmetric monoidal stable ∞-category of spectra...

Prerequisites. As a general principle, I assume knowledge of basic algebraic topology and unstable homotopy theory. On the other hand, no prior knowledge of *stable* homotopy theory is assumed. In particular, the eventual plan is to define the stable homotopy category using orthogonal spectra and develop its basic properties from scratch.

Conventions. Throughout this book, a *space* is a *compactly generated space* in the sense of [33], i.e., a k-space (also called *Kelley space*) that satisfies the weak Hausdorff condition. Two extensive resources with background material about compactly generated spaces are Section 7.9 of tom Dieck's textbook [49] and Appendix A of the author's book [40]. Two other influential – but unpublished – sources about compactly generated spaces are the Appendix A of Gaunce Lewis's thesis [22] and Neil Strickland's preprint [45]. We denote the category of compactly generated spaces and continuous maps by $\mathbf{T}$.

It will be convenient to define the n-sphere S^n as the one-point compactification of n-dimensional euclidean space $\mathbb{R}^n$, with the point at infinity as the basepoint. We will sometimes need to identify the 1-sphere with the space $[0,1]/\{0,1\}$, the quotient space of the unit interval with identified endpoints. The precise identifications do not matter, but for definiteness we fix a homeomorphism now. Our preferred homeomorphism is

$$\mathbf{t} \ : \ [0,1]/\{0,1\} \ \xrightarrow{\cong} \ S^1 \ , \quad x \longmapsto \frac{2x-1}{x(1-x)} \ .$$

Here the understanding is that the formula describes the function on the open interval $(0,1)$ (which is mapped homeomorphically to $\mathbb{R}$), and that the map extends continuously to the quotient space by sending the identified endpoints to the point at infinity in S^1.

The topological spaces we consider are usually pointed, and we use the notation $\pi_n(X)$ for the n-th homotopy group with respect to the distinguished basepoint, which we do not record in the notation.

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1. Sequential spectra and stable homotopy groups

Definition 1.1. A *sequential spectrum* consists of a sequence of pointed spaces X_n and continuous based maps $\sigma_n \colon S^1 \wedge X_n \longrightarrow X_{1+n}$ for $n \geq 0$. A *morphism* $f \colon X \longrightarrow Y$ of sequential spectra consists of based maps $f_n \colon X_n \longrightarrow Y_n$ for $n \geq 0$, which are compatible with the structure maps in the sense that $f_{1+n} \circ \sigma_n = \sigma_n \circ (S^1 \wedge f_n)$ for all $n \geq 0$. We denote the category of sequential spectra by $\mathcal{S}p^{\mathbb{N}}$.

We refer to the space X_n as the *n-th level* of the sequential spectrum X.

Construction 1.2 (Stable homotopy groups)**.** Primary invariants of spectra are their homotopy groups: for $k \in \mathbb{Z}$, the *k-th homotopy group* of a sequential spectrum X is defined as the colimit

$$\pi_k(X) \ = \ \mathrm{colim}_n \ \pi_{n+k}(X_n)$$

taken over the *stabilization maps* defined as the composite

$$\pi_{n+k}(X_n) \ \xrightarrow{S^1 \wedge -} \ \pi_{1+n+k}\left(S^1 \wedge X_n\right) \ \xrightarrow{(\sigma_n)_*} \ \pi_{1+n+k}(X_{1+n}) \ .$$

If k is negative, then the colimit system is only defined for $n \geq -k$. For large enough n, the set $\pi_{n+k}(X_n)$ has a natural abelian group structure and the stabilization maps are homomorphisms, so the colimit $\pi_k(X)$ inherits a natural abelian group structure.

Example 1.3 (Sphere spectrum and suspension spectra)**.** The *sphere spectrum* $\mathbb{S}$ is given by $\mathbb{S}_n = S^n$, the n-sphere. Then structure maps

$$\sigma_n \ : \ S^1 \wedge S^n \ \longrightarrow \ S^{1+n}$$

are the canonical homeomorphisms.

Every pointed space K gives rise to a *suspension spectrum* $\Sigma^\infty K$ with values

$$(\Sigma^\infty K)_n \ = \ S^n \wedge K \ .$$

The structure map $\sigma_n \colon S^1 \wedge S^n \wedge K \longrightarrow S^{1+n} \wedge K$ is the smash product of the canonical homeomorphism with K. For example, the sphere spectrum $\mathbb{S}$ is isomorphic to the suspension spectrum $\Sigma^\infty S^0$. A sequential spectrum X is isomorphic to a suspension spectrum (necessarily that of its zeroth space X_0) if and only if every structure map $\sigma_n \colon S^1 \wedge X_n \longrightarrow X_{1+n}$ is a homeomorphism. The homotopy group

$$\pi_k^{\mathrm{s}}(K) \ = \ \pi_k\left(\Sigma^\infty K\right) \ = \ \operatorname{colim}_n \ \pi_{n+k}(S^n \wedge K)$$

is called the k-th *stable homotopy group* of K. If K is a *well-pointed* based space, (i.e., the inclusion of the basepoint $\{k_0\} \longrightarrow K$ has the homotopy extension property in the category $\mathbf{T}$ of unbased spaces), then $S^n \wedge K$ is $(n-1)$-connected, see for example [49, Corollary 6.7.10]. So the groups $\pi_k(\Sigma^\infty K)$ vanish in negative dimensions, i.e., the suspension spectrum $\Sigma^\infty K$ is *connective.* For example, every space that admits a CW-structure is well-pointed for every choice of basepoint.

The homotopy group $\pi_k(\mathbb{S}) = \operatorname{colim}_n \pi_{n+k}(S^n)$ is called the *k-th stable homotopy group of spheres*, or the *k-th stable stem*, and will be denoted π_k^{s}. The group π_k^{s} is trivial for negative values of k. The degree of a self-map of a sphere provides an isomorphism $\pi_0^{\mathrm{s}} \cong \mathbb{Z}$. For $k \geq 1$, the homotopy group π_k^{s} is finite. This is a direct consequence of the Freudenthal's suspension theorem and Serre's calculation of the homotopy groups of spheres modulo torsion, which we recall without giving a proof.

Theorem 1.4 (Serre)**.** *Let $m > n \geq 1$. Then*

$$\pi_m(S^n) \cong \begin{cases} \text{(finite group)} \oplus \mathbb{Z} & \textit{if } n \textit{ is even and } m = 2n-1 \\ \text{(finite group)} & \textit{else.} \end{cases}$$

Thus for $k \geq 1$, the stable stem $\pi_k^{\mathrm{s}} = \pi_k(\mathbb{S})$ is finite.

As a concrete example, we inspect the colimit system defining π_1^{s}, the first stable stem. Since the universal cover of S^1 is the real line, which is contractible, the theory of covering spaces shows that the groups $\pi_n S^1$ are trivial for $n \geq 2$. The Hopf map

$$\eta \ : \ S^3 \subseteq \mathbb{C}^2 \backslash \{0\} \ \xrightarrow{\text{proj}} \ \mathbb{C}\mathrm{P}^1 \cong S^2$$

is a locally trivial fiber bundle with fiber S^1, so it gives rise to a long exact sequence of homotopy groups. Since the fiber S^1 has no homotopy above dimension one, the group $\pi_3 S^2$ is free abelian of rank one, generated by the class of η. Here, and throughout the book, we identify the complex projective space $\mathbb{C}P^1$ with the 2-sphere S^2 via the homeomorphism from S^2 to $\mathbb{C}P^1$ that sends $(x, y) \in \mathbb{R}^2$ to $[x + iy, 1] \in \mathbb{C}P^1$ and the point at infinity in S^2 to the line $[1, 0]$.

By Freudenthal's suspension theorem the suspension homomorphism $- \wedge S^1 \colon \pi_3(S^2) \longrightarrow \pi_4(S^3)$ is surjective and from $\pi_4(S^3)$ on the suspension homomorphism is an isomorphism. So the first stable stem π_1^{s} is cyclic, generated by the image of η, and its order equals the order of the suspension of η. On the one hand, η itself is stably essential, since the Steenrod operation Sq^2 acts non-trivially on the mod-2 cohomology of the mapping cone of η, which is homeomorphic to $\mathbb{C}\mathrm{P}^2$.

On the other hand, twice the suspension of η is null-homotopic. To see this we consider the commutative square

$$\begin{tikzcd} (x,y) \arrow[d, mapsto] & S^3 \arrow[r, "\eta"] \arrow[d] & \mathbb{C}\mathrm{P}^1 \arrow[d] & {[x:y]} \arrow[d, mapsto] \\ (\bar{x},\bar{y}) & S^3 \arrow[r, "\eta"'] & \mathbb{C}\mathrm{P}^1 & {[\bar{x}:\bar{y}]} \end{tikzcd}$$

in which the vertical maps are induced by complex conjugation in both coordinates of $\mathbb{C}^2$. The left vertical map has degree 1, so it is homotopic to the identity, whereas complex conjugation on $\mathbb{C}\mathrm{P}^1 \cong S^2$ has degree -1. So $(-1) \circ \eta$ is homotopic to η. Thus the suspension of η is homotopic to the suspension of $(-1) \circ \eta$, which by the following lemma is homotopic to the negative of $\eta \wedge S^1$.

Lemma 1.5. *Let Y be a based space, $m \geq 0$ and $f \colon S^m \longrightarrow S^m$ a continuous based map of degree k. Then for every homotopy class $x \in \pi_n(S^m \wedge Y)$ the classes $(f \wedge Y)_*(x)$ and $k \cdot x$ become equal in $\pi_{1+n}(S^{1+m} \wedge Y)$ after one suspension.*

Proof. Let $d_k \colon S^1 \longrightarrow S^1$ be any pointed map of degree k. Then the maps $S^1 \wedge f, d_k \wedge S^m \colon S^{1+m} \longrightarrow S^{1+m}$ have the same degree k, hence they are based homotopic. Suppose x is represented by $\varphi \colon S^n \longrightarrow S^m \wedge Y$. Then the suspension of $(f \wedge Y)_*(x)$ is represented by $(S^1 \wedge f \wedge Y) \circ (S^1 \wedge \varphi)$ which is homotopic to $(d_k \wedge S^m \wedge Y) \circ (S^1 \wedge \varphi) = (S^1 \wedge \varphi) \circ (d_k \wedge S^n)$. Precomposition with the degree k map $d_k \wedge S^n$ of S^{1+n} induces multiplication by k, so the last map represents the suspension of $k \cdot x$. □

The conclusion of Lemma 1.5 does not in general hold without the extra suspension, i.e., $(f \wedge Y)_*(x)$ need not equal $k \cdot x$ in $\pi_n(S^m \wedge Y)$: as we showed above, $(-1) \circ \eta$ is homotopic to η, which is *not* homotopic to $-\eta$ since η generates the infinite cyclic group $\pi_3(S^2)$.

As far as we know, the stable homotopy groups of spheres don't follow any simple pattern. Much machinery of algebraic topology has been developed to calculate homotopy groups of spheres, both unstable and stable, but no one expects to ever get explicit formulae for all stable homotopy groups of spheres. The Adams spectral sequence based on mod-p cohomology and the Adams-Novikov spectral sequence based on MU (complex cobordism) or BP (the Brown–Peterson spectrum at a fixed prime p) are the most effective tools we have for explicit calculations as well as for discovering systematic phenomena.

Example 1.6 (Multiplication in the stable stems)**.** The stable stems $\pi_*^{\mathrm{s}} = \pi_*(\mathbb{S})$ form a graded commutative ring which acts on homotopy groups of every other spectrum X. We denote the action simply by a 'dot'

$$\cdot \;:\; \pi_k^{\mathrm{s}} \times \pi_l(X) \;\longrightarrow\; \pi_{k+l}(X) \ .$$

The definition is essentially straightforward, but there is one subtlety in showing that the product is well-defined. We let $f \colon S^{m+k} \longrightarrow S^m$ and $g \colon S^{n+l} \longrightarrow X_n$ represent classes in π_k^{s} and $\pi_l(X)$, respectively. We denote by $f \cdot g$ the composite

$$S^{m+k+n+l} \xrightarrow{f \wedge g} S^m \wedge X_n \xrightarrow{\sigma^m} X_{m+n}$$

and then define

$$[f] \cdot [g] \;=\; (-1)^{kn} \cdot [f \cdot g] \tag{1.7}$$

in the group $\pi_{k+l}(X)$.

We check that the multiplication is well-defined. If we replace $f \colon S^{m+k} \longrightarrow S^m$ by its suspension $S^1 \wedge f \colon S^{1+m+k} \longrightarrow S^{1+m}$, then

$$(S^1 \wedge f) \cdot g \;=\; \sigma^{1+m} \circ (S^1 \wedge f \wedge g) \;=\; \sigma_{m+n} \circ (S^1 \wedge \sigma^m) \circ (S^1 \wedge f \wedge g) \;=\; \sigma_{m+n} \circ (S^1 \wedge (f \cdot g)) \ .$$

Since the sign in the formula (1.7) does not change, the resulting stable class is independent of the representative f of the stable class in π_k^{s}. Independence of the representative for $\pi_l(X)$ is slightly more subtle. If we replace $g \colon S^{n+l} \longrightarrow X_n$ by the representative $\sigma_n \circ (S^1 \wedge g) \colon S^{1+n+l} \longrightarrow X_{1+n}$, then $f \cdot g$ gets replaced by the lower horizontal composite in the commutative diagram

$$\begin{tikzcd}
S^{1+m+k+n+l} \arrow[r, "S^1 \wedge f \wedge g"] \arrow[d, "\chi_{1,m+k} \wedge S^{n+l}"'] & S^{1+m} \wedge X_n \arrow[d, "\chi_{1,m} \wedge X_n"] & \\
S^{m+k+1+n+l} \arrow[r, "f \wedge S^1 \wedge g"] \arrow[rr, bend right, "f \cdot (\sigma_n \circ (S^1 \wedge g))"'] & S^{m+1} \wedge X_n \arrow[r, "\sigma^{m+1}"] & X_{m+1+n}
\end{tikzcd}$$

By Lemma 1.5 the map $\chi_{1,m}\wedge X_n$ induces multiplication by $(-1)^m$ on homotopy groups *after one suspension.* This cancels part of the sign $(-1)^{m+k}$ that is the effect of precomposition with the shuffle permutation $\chi_{1,m+k}$ on the left. So in the colimit $\pi_{k+l}(X)$ we have

$$[f\cdot(\sigma_n\circ(S^1\wedge g))] \;=\; (-1)^k\cdot[\sigma^{m+1}(S^1\wedge f\wedge g)] \;=\; (-1)^k\cdot[f\cdot g]\ .$$

Since the dimension of $S^1\wedge g$ is one more than the dimension of g, the extra factor $(-1)^k$ makes sure that product $[f]\cdot[g]$ as defined in (1.7) is independent of the representative of the stable class $[g]$.

Now we verify that the dot product is biadditive. We only show the relation $x\cdot(y+y')=x\cdot y+x\cdot y'$, and additivity in x is similar. Suppose as before that $f\colon S^{m+k}\longrightarrow S^m$ and $g,g'\colon S^{n+l}\longrightarrow X_n$ represent classes in π_k^{s} and $\pi_l(X)$, respectively. Then the sum of g and g' in $\pi_{n+l}(X_n)$ is represented by the composite

$$S^{n+l}\xrightarrow{\text{pinch}} S^{n+l}\vee S^{n+l}\xrightarrow{g\vee g'} X_n\ .$$

In the square

$$\begin{tikzcd}
S^{m+n+k+l} \arrow[rr, "S^m\wedge\chi_{n,k}\wedge S^l"] \arrow[dd, "\text{pinch}"'] & & S^{m+k+n+l} \arrow[d, "\text{pinch}\wedge\mathrm{Id}"'] \arrow[dr, bend left, "f\wedge(g+g')"] & \\
& & S^{m+k}\wedge(S^{n+l}\vee S^{n+l}) \arrow[r, "f\wedge(g\vee g')"] & S^m\wedge X_n \\
S^{m+n+k+l}\vee S^{m+n+k+l} \arrow[rr, "(S^m\wedge\chi_{n,k}\wedge S^l)\vee(S^m\wedge\chi_{n,k}\wedge S^l)"'] & & S^{m+k+n+l}\vee S^{m+k+n+l} \arrow[u, "\cong"'] \arrow[ur, bend right, "(f\wedge g)\vee(f\wedge g')"'] &
\end{tikzcd}$$

the right part commutes on the nose and the left square commutes up to homotopy. After composing with the iterated structure map $\sigma^m\colon S^m\wedge X_n\longrightarrow X_{m+n}$, the composite around the top of the diagram becomes $f\cdot(g+g')$, whereas the composite around the bottom represents $[f]\cdot[g]+[f]\cdot[g']$. This proves additivity of the dot product in the right variable.

If we specialize to $X=\mathbb{S}$ then the product provides a biadditive graded pairing $\cdot\colon\pi_k^{\mathrm{s}}\times\pi_l^{\mathrm{s}}\longrightarrow\pi_{k+l}^{\mathrm{s}}$ of the stable homotopy groups of spheres. We claim that for every sequential spectrum X the diagram

$$\begin{tikzcd}
\pi_j^{\mathrm{s}}\times\pi_k^{\mathrm{s}}\times\pi_l(X) \arrow[r, "\pi_j^{\mathrm{s}}\times\cdot"] \arrow[d, "\cdot\times\pi_l(X)"'] & \pi_j^{\mathrm{s}}\times\pi_{k+l}(X) \arrow[d, "\cdot"] \\
\pi_{j+k}^{\mathrm{s}}\times\pi_l(X) \arrow[r, "\cdot"'] & \pi_{j+k+l}(X)
\end{tikzcd}$$

commutes, so the product on the stable stems and the action on the homotopy groups of a spectrum are associative. After unraveling all the definitions, this associativity ultimately boils down to the equality

$$(-1)^{jm}\cdot(-1)^{(j+k)n} \;=\; (-1)^{kn}\cdot(-1)^{j(m+n)}$$

and commutativity of the square

$$\begin{tikzcd}
S^q\wedge S^m\wedge X_n \arrow[r, "S^q\wedge\sigma^m"] \arrow[d, "\cong\wedge X_n"'] & S^q\wedge X_{m+n} \arrow[d, "\sigma^q"] \\
S^{q+m}\wedge X_n \arrow[r, "\sigma^{q+m}"'] & X_{q+m+n}
\end{tikzcd}$$

Finally, the multiplication in the homotopy groups of spheres is commutative in the graded sense. Indeed, for representing maps $f\colon S^{m+k} \longrightarrow S^m$ and $g\colon S^{n+l} \longrightarrow S^n$ the square

$$\begin{tikzcd} S^{m+k+n+l} \arrow[r, "f\wedge g"] \arrow[d, "\chi_{m+k,n+l}"'] & S^{m+n} \arrow[d, "\chi_{m,n}"] \\ S^{n+l+m+k} \arrow[r, "g\wedge f"'] & S^{n+m} \end{tikzcd}$$

commutes. After one suspension, the two vertical coordinate permutations induce the signs $(-1)^{(m+k)(n+l)}$ and $(-1)^{mn}$, respectively, on homotopy groups. So in the stable group we have

$$[f]\cdot[g] \ = \ (-1)^{kn}\cdot[f\cdot g] \ = \ (-1)^{kl+lm}\cdot[g\cdot f] \ = \ (-1)^{kl}\cdot[g]\cdot[f] \ .$$

The following table gives the stable homotopy groups of spheres through dimension 8:

n	0	1	2	3	4	5	6	7	8
π_n^{s}	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	0	0	$\mathbb{Z}/2$	$\mathbb{Z}/240$	$(\mathbb{Z}/2)^2$
generator	ι	η	η^2	ν			ν^2	σ	$\eta\sigma,\ \varepsilon$

Here ν and σ are the Hopf maps which arises unstably as fiber bundles $S^7 \longrightarrow S^4$ respectively $S^{15} \longrightarrow S^8$. The element ε in the 8-stem can be defined using Toda brackets as $\varepsilon = \eta\sigma + \langle \nu, \eta, \nu\rangle$. The table contains or determines all multiplicative relations in this range except for $\eta^3 = 12\nu$. A theorem of Nishida's [35] says that every homotopy element of positive dimension is nilpotent.

Example 1.8 (Eilenberg–Mac Lane spectra, sequential version)**.** For an abelian group A and $n \geq 0$, we let $K(A,n)$ be an Eilenberg–MacLane space of type (A,n), i.e., a pair $(K(A,n),\varphi_n)$ consisting of a based space admitting a CW-structure such that the homotopy group $\pi_k(K(A,n),*)$ is trivial for $k \neq n$, and an isomorphism $\varphi_n\colon \pi_n(K(A,n),*) \cong A$. Since $\Omega K(A,1+n)$ is also an Eilenberg–MacLane space of type (A,n), there is a continuous based map $\rho_n\colon K(A,n) \longrightarrow \Omega K(A,1+n)$ making the following diagram commute:

$$\begin{tikzcd} \pi_n(K(A,n),*) \arrow[r, "(\rho_n)_*"] \arrow[dr, "\cong"', "\varphi_n"] & \pi_n(\Omega K(A,1+n),*) \arrow[r, "\cong"] & \pi_{1+n}(K(A,1+n),*) \arrow[dl, "\cong", "\varphi_{1+n}"'] \\ & A & \end{tikzcd} \tag{1.9}$$

The unnamed isomorphism is the adjunction isomorphism. Moreover, such a ρ_n is unique up to based homotopy, and it is a weak homotopy equivalence. We write $\sigma_n\colon S^1\wedge K(A,n) \longrightarrow K(A,1+n)$ for the adjoint of the map ρ_n under the adjunction $(S^1\wedge -,\Omega)$. Then the data

$$HA \ = \ \{K(A,n),\ \sigma_n\}_{n\geq 0}$$

form a sequential spectrum, an *Eilenberg–Mac Lane spectrum* for the group A. Because the group $\pi_{k+n}(K(A,n),*)$ is trivial for all $k\neq 0$ with $k+n\geq 0$, the stable homotopy group $\pi_k(HA)$ is trivial for $k\neq 0$. And the commutativity of the diagrams (1.9) guarantees that the isomorphisms $\varphi_n\colon \pi_n(K(A,n),*) \cong A$ assemble into an isomorphism

$$\varphi \ : \ \pi_0(HA) \ \xrightarrow{\ \cong\ } \ A \ .$$

In Example 3.3 we discuss a more refined version of the Eilenberg–MacLane spectra, one that does not depend on choices of Eilenberg–MacLane spaces, and that assembles into a lax symmetric monoidal functor from abelian groups (under tensor product) to orthogonal spectra (under smash product).

Example 1.10 (Smash products with and functions from spaces)**.** For a based space K, smashing with K and taking based mapping space from K are an adjoint functor pair

$$-\wedge K \ : \ \rightleftarrows \ : \ \mathrm{map}_*(K,-) \ = \ (-)^K$$

We can lift these functors to sequential spectra by applying them levelwise. More precisely, for a sequential spectrum X we define new sequential spectra $X \wedge K$ and $\mathrm{map}_*(K, X)$ by

$$(X \wedge K)_n \;=\; X_n \wedge K \qquad \text{and} \qquad \mathrm{map}_*(K,X)_n \;=\; \mathrm{map}_*(K, X_n)\ .$$

The structure maps do not interact with K: the n-structure map for $X \wedge K$ is

$$S^1 \wedge (X \wedge K)_n \;=\; S^1 \wedge X_n \wedge K \xrightarrow{\ \sigma_n \wedge K\ } X_{1+n} \wedge K \;=\; (X \wedge K)_{1+n}\ .$$

The n-structure map for $\mathrm{map}_*(K, X)$ is the composite

$$S^1 \wedge \mathrm{map}_*(K, X_n) \longrightarrow \mathrm{map}_*(K, S^1 \wedge X_n) \xrightarrow{\ \mathrm{map}_*(K,\sigma_n)\ } \mathrm{map}_*(K, X_{1+n})$$

where the first is an assembly map that sends $x \wedge f$ to the map sending $k \in K$ to $x \wedge f(k)$.

Just as the functors $- \wedge K$ and $\mathrm{map}_*(K, -)$ are adjoint on the level of based spaces, the two functors just introduced are an adjoint pair for sequential spectra. The adjunction unit $\eta\colon X \longrightarrow \mathrm{map}_*(K, X \wedge K)$ and counit $\epsilon\colon \mathrm{map}_*(K, X) \wedge K \longrightarrow X$ are defined levelwise as coevaluation and evaluation maps:

$$\begin{aligned} \eta_n \;&:\; X_n \longrightarrow \mathrm{map}_*(K, X_n \wedge K)\ , \quad && \eta_n(x)(k) \;=\; x \wedge k \\ \epsilon_n \;&:\; \mathrm{map}_*(K, X_n) \wedge K \longrightarrow X\ , \quad && \epsilon_n(f)(k) \;=\; f(k)\ . \end{aligned}$$

An important special case of this construction is when $K = S^1$ is a 1-sphere, i.e., the one-point compactification of $\mathbb{R}$. In this case we call $X \wedge S^1$ the *suspension* of X, and we call $\Omega X = \mathrm{map}_*(S^1, X)$ the *loop spectrum* of X. We obtain an adjunction between $- \wedge S^1$ and Ω as the special case $K = S^1$ of the previous adjunction.

Definition 1.11. A morphism $f\colon X \longrightarrow Y$ of sequential spectra is a *stable equivalence* if the induced map $\pi_k(f)\colon \pi_k(X) \longrightarrow \pi_k(Y)$ is an isomorphism for all integers k.

In Proposition 1.27 we prove that stable equivalences are closed under various constructions such as suspensions, loop, shift adjoint, wedges, and finite products.

We will develop some of the basic properties of homotopy groups for sequential spectra. We being by showing that looping and suspending a spectrum shifts the homotopy groups. The loop homomorphism starts from the isomorphism

$$\alpha \;:\; \pi_{n+k}(\Omega X_n) \;\cong\; \pi_{n+k+1}(X_n)$$

that is defined by the same adjunction as above, i.e., the class represented by a continuous based map $f\colon S^{n+k} \longrightarrow \Omega X_n$ is sent to the class of the map $\hat{f}\colon S^{n+k+1} \longrightarrow X_n$ given by $\hat{f}(s \wedge t) = f(s)(t)$, where $s \in S^{n+k}$ and $t \in S^1$. As n varies, these particular isomorphisms are compatible with stabilization maps, so they induce an isomorphism

$$\alpha \;:\; \pi_k(\Omega X) \xrightarrow{\ \cong\ } \pi_{k+1}(X)$$

on colimits.

The maps $- \wedge S^1\colon \pi_{n+k}(X_n) \longrightarrow \pi_{n+k+1}(X_n \wedge S^1)$ given by smashing from the right with the identity of the circle are compatible with the stabilization process for the homotopy groups of X and $X \wedge S^1$, respectively, so upon passage to colimits they induce a natural map of homotopy groups

$$- \wedge S^1 \;:\; \pi_k(X) \longrightarrow \pi_{k+1}(X \wedge S^1)\ ,$$

which we call the *suspension homomorphism*.

As before we let $\eta\colon X \longrightarrow \Omega(X \wedge S^1)$ and $\epsilon\colon (\Omega X) \wedge S^1 \longrightarrow X$ denote the unit respectively counit of the adjunction. Then for every map $f\colon S^{n+k} \longrightarrow \Omega X_n$ we have $\hat{f} = \epsilon_n \circ (f \wedge S^1)$ and for every map $g\colon S^{n+k} \longrightarrow X_n$ we have $g \wedge S^1 = \widehat{\eta_n \circ g}$. This means that the two triangles

$$(1.12) \qquad \begin{tikzcd} \pi_k(\Omega X) \arrow[rr, "\alpha"] \arrow[dr, "- \wedge S^1"'] & & \pi_{k+1}(X) \\ & \pi_{k+1}((\Omega X) \wedge S^1) \arrow[ur, "\pi_{k+1}(\epsilon)"'] & \end{tikzcd} \qquad \begin{tikzcd} \pi_k(X) \arrow[rr, "- \wedge S^1"] \arrow[dr, "\pi_k(\eta)"'] & & \pi_{k+1}(X \wedge S^1) \\ & \pi_k(\Omega(X \wedge S^1)) \arrow[ur, "\alpha"'] & \end{tikzcd}$$

commute.

Proposition 1.13. *Let X be a sequential spectrum.*

(i) *The loop and suspension homomorphisms*

$$\alpha \ : \ \pi_k(\Omega X) \ \longrightarrow \ \pi_{k+1}(X) \qquad \textit{and} \qquad - \wedge S^1 \ : \ \pi_k(X) \ \longrightarrow \ \pi_{k+1}(X \wedge S^1)$$

are isomorphisms of homotopy groups.

(ii) *The unit $\eta\colon X \longrightarrow \Omega(X \wedge S^1)$ and counit $\epsilon\colon (\Omega X) \wedge S^1 \longrightarrow X$ of the adjunction are stable equivalences.*

(iii) *For every continuous based map $h\colon S^m \longrightarrow S^m$, the morphism of sequential spectra $X \wedge h\colon X \wedge S^m \longrightarrow X \wedge S^m$ induces multiplication by the degree of h on all homotopy groups.*

Proof. (i) We already argued that the loop homomorphism α on homotopy groups is bijective since it is the colimit of compatible bijections. The case of the suspension homomorphism $- \wedge S^1$ is slightly more involved. We show injectivity first. Let $f\colon S^{n+k} \longrightarrow X_n$ represent an element in the kernel of the suspension homomorphism. By stabilizing, if necessary, we can assume that the suspension $f \wedge S^1\colon S^{n+k+1} \longrightarrow X_n \wedge S^1$ is nullhomotopic. Then $\sigma_n \circ \tau \circ (f \wedge S^1)\colon S^{n+k+1} \longrightarrow X_{n+1}$ is also nullhomotopic, where $\tau\colon X_n \wedge S^1 \cong S^1 \wedge X_n$ is the twist homeomorphism. The maps $\sigma_n \circ \tau \circ (f \wedge S^1)$ and $\sigma_n \circ (S^1 \wedge f)$, the stabilization of f, only differ by a coordinate permutation of the source sphere, hence the stabilization of f is nullhomotopic. So f represents the trivial element in $\pi_k(X)$, which shows that the suspension homomorphism is injective.

It remains to show that the suspension homomorphism is surjective. Let $g\colon S^{n+k+1} \longrightarrow X_n \wedge S^1$ be a map which represents a class in $\pi_{k+1}(X \wedge S^1)$. We consider the map $f = \sigma_n \circ \tau \circ g\colon S^{n+k+1} \longrightarrow X_{1+n}$ where τ is again the twist homeomorphism. We claim that $(-1)^{n+k} \cdot (f \wedge S^1)\colon S^{n+k+1+1} \longrightarrow X_{1+n} \wedge S^1$ represents the same class as g in $\pi_{k+1}(X \wedge S^1)$. To see this, we contemplate the diagram:

$$\begin{tikzcd}
S^{1+n+k+1} \arrow[d, "\chi_{1,n+k}\wedge S^1"'] \arrow[drr, bend left, "S^1\wedge g"] & & \\
S^{n+k+1+1} \arrow[r, "g\wedge S^1"] \arrow[drr, bend right, "f\wedge S^1"'] & X_n \wedge S^1 \wedge S^1 \arrow[r, "\tau\wedge S^1"] & S^1 \wedge X_n \wedge S^1 \arrow[d, "\sigma_n\wedge S^1"] \\
& & X_{1+n} \wedge S^1
\end{tikzcd}$$

The composite through the upper right is the stabilization of g, and the composite through the lower left represents $(-1)^{n+k} \cdot (f \wedge S^1)$. However, the upper triangle does *not* commute! The failure to commutativity are the involutions of $S^{1+n+k+1}$ and $S^1 \wedge X_n \wedge S^1$ which interchange the outer two sphere coordinates in each case. This coordinate change in the source induces multiplication by -1; the coordinate change in the target is a map of degree -1, so after a single suspension it also induces multiplication by -1 on homotopy groups (see Lemma 1.5). Altogether this shows that the upper triangle commutes up to homotopy *after one suspension*, and so the suspension map on homotopy groups is also surjective.

(ii) Since loops and suspension homomorphism are bijective and the triangles (1.12) commute, the unit and counit of the adjunction are stable equivalences.

(iii) Because the iterated suspension homomorphism

$$- \wedge S^m \ : \ \pi_k(X) \ \longrightarrow \ \pi_{k+m}(X \wedge S^m)$$

is bijective by part (i), every class in $\pi_{k+m}(X \wedge S^m)$ has a representative of the form $f \wedge S^m\colon S^{n+k+m} \longrightarrow X_n \wedge S^m$ for some continuous based map $f\colon S^{k+n} \longrightarrow X_n$. So

$$(X \wedge h)_*[f \wedge S^m] \ = \ [(X_n \wedge h) \circ (f \wedge S^m)] \ = \ [(f \wedge S^m) \circ (S^{n+k} \wedge h)] \ = \ [f \wedge S^m] \cdot \deg(h) \ . \qquad \square$$

Corollary 1.14. *For every morphism $f\colon X \longrightarrow Y$ of sequential spectra, the following conditions are equivalent.*

(a) *The morphism $f\colon X \longrightarrow Y$ is a stable equivalence.*

(b) *The morphism $\Omega f\colon \Omega X \longrightarrow \Omega Y$ is a stable equivalence.*

(c) *The morphism $f\colon X \wedge S^1 \longrightarrow Y \wedge S^1$ is a stable equivalence.*

A morphism $g\colon A\wedge S^1 \longrightarrow X$ *of sequential spectra is a stable equivalence if and only if its adjoint* $\hat g\colon A \longrightarrow \Omega X$ *is a stable equivalence.*

Proof. We only need to prove the last statement. The morphism g and its adjoint are related by $g = \epsilon\circ(\hat g\wedge S^1)$ where $\epsilon\colon (\Omega X)\wedge S^1 \longrightarrow X$ is the counit of the adjunction. The counit is a stable equivalence by Proposition 1.13. We conclude that the morphism g is a stable equivalence if and only if the morphism $\hat g\wedge S^1\colon A\wedge S^1 \longrightarrow (\Omega X)\wedge S^1$ is. Since suspension shifts homotopy groups, this happens if and only if g is a stable equivalence. □

As far as I can see, the suspension functor $-\wedge S^1$ does not in general preserve weak equivalences of spaces if these are not well-pointed. Hence $-\wedge S^1$ does not in general preserve level equivalences of sequential spectra. However, $-\wedge S^1$ preserves stable equivalences, ultimately because of the suspension isomorphism.

1.1. **Mapping cone and homotopy fiber.** Now we review the mapping cone and the homotopy fiber of a map of based spaces in some detail, along with their relationships to one another and to suspension and loop space. The *(reduced) mapping cone* Cf of a morphism of based spaces $f\colon A \longrightarrow B$ is defined by

$$Cf \;=\; (A\wedge[0,1])\cup_f B\ .$$

Here the unit interval $[0,1]$ is pointed by $0\in[0,1]$, so that $A\wedge[0,1]$ is the reduced cone of A. The mapping cone comes with an inclusion $i\colon B\longrightarrow Cf$ and a projection $p\colon Cf\longrightarrow A\wedge S^1$; the projection sends B to the basepoint and is given on $A\wedge[0,1]$ by $p(a\wedge x) = a\wedge \mathbf{t}(x)$ where

$$\mathbf{t}\colon [0,1]\longrightarrow S^1 \qquad \text{is defined as} \qquad \mathbf{t}(x) = \frac{2x-1}{x(1-x)}\ .$$

What is relevant about the map $\mathbf{t}$ is not the precise formula, but that it passes to a homeomorphism between the quotient space $[0,1]/\{0,1\}$ and the circle S^1, and that it satisfies $\mathbf{t}(1-x) = -\mathbf{t}(x)$.

We observe that an iteration of the mapping cone construction yields the suspension of A, up to homotopy.

Lemma 1.15. *Let* $f\colon A\longrightarrow B$ *be any continuous based map.*

(i) *The collapse map*
$$*\cup p\ :\ Ci = (B\wedge[0,1])\cup_i Cf \ \longrightarrow\ A\wedge S^1$$
is a based homotopy equivalence.

(ii) *The square*
$$\begin{tikzcd} Ci \arrow[r, "p\cup *"] \arrow[d, "*\cup p"'] & B\wedge S^1 \arrow[d, "B\wedge\tau"] \\ A\wedge S^1 \arrow[r, "f\wedge S^1"'] & B\wedge S^1 \end{tikzcd}$$
commutes up to natural, based homotopy, where τ *is the involution of* S^1 *given by* $\tau(x) = -x$.

(iii) *Let* $\beta\colon Z\longrightarrow B$ *be a continuous based map such that the composite* $i\beta\colon Z\longrightarrow Cf$ *is null-homotopic. Then there exists a based map* $h\colon Z\wedge S^1\longrightarrow A\wedge S^1$ *such that* $(f\wedge S^1)\circ h\colon Z\wedge S^1\longrightarrow B\wedge S^1$ *is homotopic to* $\beta\wedge S^1$.

Proof. (i) A homotopy inverse $r\colon A\wedge S^1\longrightarrow (B\wedge[0,1])\cup_i Cf$ of $*\cup p$ is defined by the formula

$$r(a\wedge x) \;=\; \begin{cases} a\wedge 2x & \text{in } Cf \text{ for } 0\le x\le 1/2, \text{ and} \\ f(a)\wedge(2-2x) & \text{in } B\wedge[0,1] \text{ for } 1/2\le x\le 1. \end{cases}$$

We give explicit based homotopies between the two composites r and $*\cup p$ and the respective identity maps. The space $Ci = (B\wedge[0,1])\cup_i Cf$ is homeomorphic to the quotient of the disjoint union of $B\wedge[0,1]$ and $A\wedge[0,1]$ by the equivalence relation that identifies $f(a)\wedge 1$ in $B\wedge[0,1]$ with $a\wedge 1$ in $A\wedge[0,1]$ for all $a\in A$. So we can define a homotopy on the space Ci by gluing two compatible homotopies. The homotopy

$$[0,1]\times(B\wedge[0,1]) \ \longrightarrow\ Ci\ , \quad (t, b\wedge x)\longmapsto b\wedge(1-t)x \quad \text{in } B\wedge[0,1]\ .$$

and the homotopy

$$[0,1]\times(A\wedge[0,1]) \longrightarrow Ci\ ,\ (t,a\wedge x)\mapsto \begin{cases} a\wedge(1+t)x & \text{in } Cf \text{ for } 0\le x\le 1/(1+t), \text{ and} \\ f(a)\wedge(2-x(1+t)) & \text{in } B\wedge[0,1] \text{ for } 1/(1+t)\le x\le 1, \end{cases}$$

are compatible, and the combined homotopy starts at $t=0$ with the identity and ends at $t=1$ with the map $r\circ(*\cup p)$.

A homotopy from the identity of $A\wedge S^1$ to $(*\cup p)\circ r$ is given by

$$[0,1]\times(A\wedge S^1) \longrightarrow A\wedge S^1\ ,\quad (t,\ a\wedge x) \longmapsto a\wedge(1+t)$$

which is to be interpreted as the basepoint if $(1+t)x\ge 1$.

(ii) Again we glue the desired homotopy from two pieces, namely

$$[0,1]\times(B\wedge[0,1]) \longrightarrow B\wedge S^1\ ,\quad (t,\ b\wedge x) \longmapsto b\wedge(1+t-x)\ ,$$

which has to be interpreted as the basepoint if $x\le t$ and

$$[0,1]\times(A\wedge[0,1]) \longrightarrow B\wedge S^1\ ,\quad (t,\ a\wedge x) \longmapsto f(a)\wedge(t+x-1)$$

which has to be interpreted as the basepoint if $t+x\le 1$. The two homotopies are compatible and the combined homotopy starts with the map $(B\wedge\tau)\circ(p\cup *)$ for $t=0$ and it ends with the map $(f\wedge S^1)\circ(*\cup p)$ for $t=1$.

(iii) Let $H\colon Z\wedge[0,1]\longrightarrow Cf$ be a based null-homotopy of the composite $i\beta\colon Z\longrightarrow Cf$, i.e., $H(z\wedge 1)=i(\beta(z))$ for all $z\in Z$. The composite $p_AH\colon Z\wedge[0,1]\longrightarrow A\wedge S^1$ then factors as $p_AH=hp_Z$ for a unique map $h\colon Z\wedge S^1\longrightarrow A\wedge S^1$.

To analyze $(f\wedge S^1)\circ h$ we compose it with the map $*\cup p_Z\colon (Z\wedge[0,1])\cup_{Z\times 1}(Z\wedge[0,1])\longrightarrow Z\wedge S^1$ which collapses the second cone and which is a homotopy equivalence by (i). We obtain a sequence of equalities and homotopies

$$\begin{aligned}(f\wedge S^1)\circ h\circ(*\cup p_Z) &= (f\wedge S^1)\circ(*\cup p_A)\circ((\beta\wedge[0,1])\cup H)\\ &\simeq (B\wedge\tau)\circ(p_B\cup *)\circ((\beta\wedge[0,1])\cup H)\\ &= (B\wedge\tau)\circ(\beta\wedge S^1)\circ(p_Z\cup *)\\ &= (\beta\wedge S^1)\circ(Z\wedge\tau)\circ(p_Z\cup *)\ \simeq\ (\beta\wedge S^1)\circ(*\cup p_Z)\end{aligned}$$

Here $(\beta\wedge[0,1])\cup H\colon CZ\cup_{Z\times 1}CZ\longrightarrow CB\cup_i Cf=C(i)$. The two homotopies result from part (ii) applied to f respectively the identity of Z. Since the map $*\cup p_Z$ is a homotopy equivalence, this proves that $(f\wedge S^1)\circ h$ is homotopic to $\beta\wedge S^1$. □

Now we can introduce mapping cones for sequential spectra. The *mapping cone* Cf of a morphism of sequential spectra $f\colon X\longrightarrow Y$ is define levelwise:

$$(Cf)_n\ =\ C(f_n)\ =\ (X_n\wedge[0,1])\cup_{f_n} Y_n\ , \tag{1.16}$$

the reduced mapping cone of $f_n\colon X_n\longrightarrow Y_n$. The structure maps are induced by the structure maps of X and Y, and they do not interact with the cone coordinate. The inclusions $i_n\colon Y_n\longrightarrow C(f_n)$ and projections $p_n\colon C(f_n)\longrightarrow X_n\wedge S^1$ assemble into morphisms of sequential spectra $i\colon Y\longrightarrow Cf$ and $p\colon Cf\longrightarrow X\wedge S^1$.

We define a *connecting homomorphism* $\delta\colon\pi_{k+1}(Cf)\longrightarrow\pi_k(X)$ as the composite

$$\pi_{k+1}(Cf)\ \xrightarrow{p_*}\ \pi_{k+1}(X\wedge S^1)\ \xrightarrow{-\wedge S^{-1}}\ \pi_k(X)\ , \tag{1.17}$$

where the second map is the inverse of the suspension isomorphism $-\wedge S^1\colon\pi_k(X)\longrightarrow\pi_{k+1}(X\wedge S^1)$. If we unravel all the definitions, we see that δ sends the class represented by a based map $g\colon S^{n+k+1}\longrightarrow Cf_n$ to $(-1)^{n+k}$ times the class of the composite

$$S^{n+k+1}\ \xrightarrow{g}\ Cf_n\ \xrightarrow{p_n}\ X_n\wedge S^1\ \xrightarrow{\text{twist}}\ S^1\wedge X_n\ \xrightarrow{\sigma_n}\ X_{1+n}\ .$$

Proposition 1.18. *For every morphism $f\colon X \longrightarrow Y$ of sequential spectra the long sequence of abelian groups*

$$\cdots \longrightarrow \pi_k(X) \xrightarrow{f_*} \pi_k(Y) \xrightarrow{i_*} \pi_k(Cf) \xrightarrow{\delta} \pi_{k-1}(X) \longrightarrow \cdots$$

is exact.

Proof. We start with exactness at $\pi_k(Y)$. The composite of $f\colon X \longrightarrow Y$ and the inclusion $Y \longrightarrow Cf$ is levelwise the constant map at the basepoint, so it induces the trivial map on π_k. It remains to show that every element in the kernel of i_* is in the image of f_*. Let $\beta\colon S^{n+k} \longrightarrow Y_n$ represent an element in the kernel. By increasing n, if necessary, we can assume that $i\beta\colon S^{n+k} \longrightarrow C(f_n)$ is null-homotopic. By Lemma 1.15 (iii) there is a based map $h\colon S^{n+k+1} \longrightarrow X_n \wedge S^1$ such that $(f_n \wedge S^1) \circ h$ is homotopic to $\beta \wedge S^1$. The composite

$$\tilde{h} \; : \; S^{1+n+k} \xrightarrow{\chi_{1,n+k}} S^{n+k+1} \xrightarrow{h} X_n \wedge S^1 \xrightarrow{\tau_{X_n,S^1}} S^1 \wedge X_n$$

then has the property that $(S^1 \wedge f_n) \circ \tilde{h}$ is homotopic to $S^1 \wedge \beta$. The map $\sigma_n \circ \tilde{h}\colon S^{1+n+k} \longrightarrow X_{1+n}$ represents a homotopy class in $\pi_k(X)$ and we have

$$f_*[\sigma_n \circ \tilde{h}] \; = \; [f_{1+n} \circ \sigma_n \circ \tilde{h}] \; = \; [\sigma_n \circ (S^1 \wedge f_n) \circ \tilde{h}] \; = \; [\sigma_n \circ (S^1 \wedge \beta)] \; = \; [\beta] \; .$$

So the class represented by β is in the image of $f_*\colon \pi_k(X) \longrightarrow \pi_k(Y)$.

We now deduce the exactness at $\pi_k(Cf)$ and $\pi_{k-1}(X)$ by comparing the mapping cone sequence for $f\colon X \longrightarrow Y$ to the mapping cone sequence for the morphism $i\colon Y \longrightarrow Cf$ (shifted to the left). The collapse map

$$* \cup p \; : \; Ci \; = \; CY \cup_i Cf \; \longrightarrow \; X \wedge S^1$$

is levelwise a homotopy equivalence by Lemma 1.15 (i), and thus induces an isomorphism of homotopy groups. Now we consider the diagram

$$\begin{tikzcd}
Cf \arrow[r, "i_i"] \arrow[dr, "p"'] & Ci \arrow[r, "p\cup *"] \arrow[d, "*\cup p"] & Y \wedge S^1 \arrow[d, "Y\wedge\tau"] \\
 & X \wedge S^1 \arrow[r, "f\wedge S^1"'] & Y \wedge S^1
\end{tikzcd}$$

whose upper row is part of the mapping cone sequence for the morphism $i\colon Y \longrightarrow Cf$. The left triangle commutes on the nose and the right triangle commutes up to based homotopy by Lemma 1.15 (ii). The involution $\tau\colon S^1 \longrightarrow S^1$ has degree -1, so the automorphism $Y \wedge \tau$ of $Y \wedge S^1$ induces multiplication by -1 on homotopy groups. We get a commutative diagram

$$\begin{tikzcd}
\pi_k(Y) \arrow[r, "i_*"] \arrow[d, equal] & \pi_k(Cf) \arrow[r, "(i_i)_*"] \arrow[d, equal] & \pi_k(Ci) \arrow[r, "\delta"] \arrow[d, "(-\wedge S^{-1})\circ(*\cup p)_*"', "\cong"] & \pi_{k-1}(Y) \arrow[d, "(-1)\cdot"] \\
\pi_k(Y) \arrow[r, "i_*"'] & \pi_k(Cf) \arrow[r, "\delta"'] & \pi_{k-1}(X) \arrow[r, "f_*"'] & \pi_{k-1}(Y)
\end{tikzcd}$$

(using for the right square the naturality of the suspension isomorphism). By the previous paragraph, applied to $i\colon Y \longrightarrow Cf$ instead of f, the upper row is exact at $\pi_k(Cf)$. Since all vertical maps are isomorphisms, the original lower row is exact at $\pi_k(Cf)$. But the morphism f was arbitrary, so when applied to $i\colon Y \longrightarrow Cf$ instead of f, we obtain that the upper row is exact at $\pi_k(Ci)$. Since all vertical maps are isomorphisms, the original lower row is exact at $\pi_{k-1}(X)$. This finishes the proof. $\square$

A continuous map $f\colon A \longrightarrow B$ of spaces is an *h-cofibration* if it has the homotopy extension property, i.e., given a continuous map $\varphi\colon B \longrightarrow X$ and a homotopy $H\colon A \times [0,1] \longrightarrow X$ such that $H(-,0) = \varphi f$, there is a homotopy $\bar{H}\colon B \times [0,1] \longrightarrow X$ such that $\bar{H} \circ (f \times [0,1]) = H$ and $\bar{H}(-,0) = \varphi$. An equivalent condition is that the map $A \times [0,1] \cup_{f\times 0} B \longrightarrow B \times [0,1]$ has a retraction. For every h-cofibration the map

$Cf \longrightarrow B/A$ which collapses the cone of A to a point is a based homotopy equivalence, see for example [16, Proposition 0.17] or [49, Proposition 5.1.10] with $B = *$.

Let $f\colon X \longrightarrow Y$ be a morphism of sequential spectra that is levelwise an h-cofibration. Then by the above, the morphism $c\colon Cf \longrightarrow Y/X$ that collapses the cone of X is a level equivalence, and so it induces an isomorphism of homotopy groups. We can thus define another connecting homomorphism

$$\delta \ :\ \pi_k(Y/X) \ \longrightarrow\ \pi_{k-1}(X) \tag{1.19}$$

as the composite of the inverse of the isomorphism $c_*\colon \pi_k(Cf) \longrightarrow \pi_k(Y/X)$ and the connecting homomorphism $\pi_k(Cf) \longrightarrow \pi_{k-1}(X)$ defined in (1.17).

Corollary 1.20. *Let $f\colon X \longrightarrow Y$ be a morphism of sequential spectra that is levelwise an h-cofibration and denote by $q\colon Y \longrightarrow Y/X$ the quotient map. Then the long sequence of homotopy groups*

$$\cdots \ \longrightarrow\ \pi_k(X) \ \xrightarrow{f_*}\ \pi_k(Y) \ \xrightarrow{q_*}\ \pi_k(Y/X) \ \xrightarrow{\delta}\ \pi_{k-1}(X) \ \longrightarrow\ \cdots$$

is exact.

Now we discuss the *homotopy fiber*, a construction 'dual' to the mapping cone. The homotopy fiber of a morphism $f\colon A \longrightarrow B$ of based spaces is the fiber product

$$Ff \ =\ * \times_B B^{[0,1]} \times_B A \ =\ \{(\lambda, a) \in B^{[0,1]} \times A \mid \lambda(0) = *,\ \lambda(1) = f(a)\}\ ,$$

i.e., the space of paths in B starting at the basepoint and equipped with a lift of the endpoint to A. As basepoint of the homotopy fiber we take the pair consisting of the constant path at the basepoint of B and the basepoint of A. The homotopy fiber comes with maps

$$\Omega B \ \xrightarrow{i}\ Ff \ \xrightarrow{q}\ A\ ;$$

the map q is the projection to the second factor and the value of the map i on a loop $\omega\colon S^1 \longrightarrow B$ is $i(\omega) = (\omega \circ \mathbf{t}, *)$.

We can apply the homotopy fiber levelwise to a morphism $f\colon X \longrightarrow Y$ of sequential spectra. The homotopy fiber Ff is defined by

$$(Ff)_n \ =\ F(f_n)\ ,$$

the homotopy fiber of $f_n\colon X_n \longrightarrow Y_n$. The inclusions $i_n\colon \Omega(Y_n) \longrightarrow (Ff)_n$ and projections $q_n\colon (Ff)_n \longrightarrow X_n$ assemble into morphisms of sequential spectra $i\colon \Omega Y \longrightarrow Ff$ and $p\colon Ff \longrightarrow X$.

We define a *connecting homomorphism* $\delta\colon \pi_{k+1}(Y) \longrightarrow \pi_k(Ff)$ as the composite

$$\pi_{k+1}(Y) \ \xrightarrow{\alpha^{-1}}\ \pi_k(\Omega Y) \ \xrightarrow{i_*}\ \pi_k(Ff)\ , \tag{1.21}$$

where $\alpha\colon \pi_k(\Omega Y) \longrightarrow \pi_{1+k}(Y)$ is the loop isomorphism.

We can compare the mapping cone and homotopy fiber as follows. For a map $f\colon A \longrightarrow B$ of based spaces we define a map $\bar{h}\colon F(f) \times [0,1] \longrightarrow (A \wedge [0,1]) \cup_f B = Cf$ by

$$(\lambda, a, t) \ \longmapsto\ \begin{cases} a \wedge 2t & \text{for } 0 \leq t \leq 1/2, \text{ and} \\ \lambda(2-2t) & \text{for } 1/2 \leq t \leq 1. \end{cases}$$

We note that the two formulas match at $t = 1/2$ because $\lambda(1) = f(a) = a \wedge 1$ in Cf. Since $\bar{h}(\lambda, a, 0)$ and $\bar{h}(\lambda, a, 1)$ are the basepoint of the mapping cone for all (λ, a) in Ff, the map $\bar{h}$ factors over a based map

$$h \ :\ (Ff) \wedge S^1 \ \longrightarrow\ Cf\ ,$$

which satisfies $h \circ q = \bar{h}$ and is natural in f. So for a morphism $f\colon X \longrightarrow Y$ of sequential spectra, the maps h for the various levels together form a natural morphism

$$h \ :\ (Ff) \wedge S^1 \ \longrightarrow\ Cf\ .$$

Proposition 1.22. *For every morphism $f\colon X \longrightarrow Y$ of sequential spectra the long sequence of abelian groups*
$$\cdots \longrightarrow \pi_k(Ff) \xrightarrow{q_*} \pi_k(X) \xrightarrow{f_*} \pi_k(Y) \xrightarrow{\delta} \pi_{k-1}(Ff) \longrightarrow \cdots$$
is exact and the morphism $h\colon (Ff)\wedge S^1 \longrightarrow Cf$ is a stable equivalence.

Proof. The long sequence is exact because it is obtained from the unstable long exact sequences for the homotopy fiber sequences $(Ff)_n \longrightarrow X_n \longrightarrow Y_n$ by passage to the colimit (which is exact).

For showing that h is a stable equivalence it suffices to show that the composite $h_*\circ(-\wedge S^1)\colon \pi_k(Ff) \longrightarrow \pi_{k+1}(Cf)$ is an isomorphism. We claim that the diagram

$$\begin{tikzcd}
\pi_{k+1}(Y) \arrow[r, "\delta"] \arrow[d, "(-1)\cdot"'] & \pi_k(Ff) \arrow[r, "q_*"] \arrow[d, "h_*\circ(-\wedge S^1)"'] & \pi_k(X) \arrow[d, equal] \\
\pi_{k+1}(Y) \arrow[r, "i_*"'] & \pi_{k+1}(Cf) \arrow[r, "\delta"'] & \pi_k(X)
\end{tikzcd}$$

commutes. The morphism $h_*\circ(-\wedge S^1)\colon \pi_k(Ff) \longrightarrow \pi_{k+1}(Cf)$ and the identity maps of the homotopy groups of X and Y thus give a natural map from the long exact sequence of the homotopy fiber to the long exact sequence of the mapping cone, with an extra sign. A sign does not affect exactness of a sequence, and so the five lemma shows that $h_*\circ(-\wedge S^1)$ is an isomorphism. Hence h is a stable equivalence.

We still have to justify the commutativity of the previous diagram. For the right square this is the definition of the connecting homomorphism, naturality of the suspension isomorphism and the fact that the composite
$$(Ff)\wedge S^1 \xrightarrow{h} Cf \xrightarrow{p} X\wedge S^1$$
is homotopic to $q\wedge S^1$ via the homotopy
$$[0,1]\times((Ff)\wedge S^1) \longrightarrow X\wedge S^1\ , \quad (t,\ (\lambda,a)\wedge s) \longmapsto \begin{cases} a\wedge 2s/(2-t) & \text{for } 0\le s\le 1-t/2, \text{ and} \\ * & \text{for } 1-t/2\le s\le 1, \end{cases}$$
(to be interpreted levelwise). Indeed, these facts together supply the relation
$$\begin{aligned}
\delta\circ h_*\circ(-\wedge S^1) &= (-\wedge S^{-1})\circ p_*\circ h_*\circ(-\wedge S^1) \\
&= (-\wedge S^{-1})\circ(q\wedge S^1)_*\circ(-\wedge S^1) \\
&= (-\wedge S^{-1})\circ(-\wedge S^1)\circ q_* \ = \ q_*\ .
\end{aligned}$$
For the left square we need that the diagram

$$\begin{tikzcd}
(\Omega Y)\wedge S^1 \arrow[r, "i\wedge\tau"] \arrow[d, "\epsilon"'] & (Ff)\wedge S^1 \arrow[d, "h"] \\
Y \arrow[r, "i"'] & Cf
\end{tikzcd}$$

commutes up to based homotopy, where ϵ is the adjunction counit. One possible such homotopy is
$$\begin{aligned}
[0,1]\times((\Omega Y)\wedge S^1) &\longrightarrow \quad Cf \\
(t,\ \omega\wedge x) &\longmapsto \begin{cases} * & \text{for } 0\le x\le t/2, \text{ and} \\ \omega(2(1-t)/(2-x)) & \text{for } t/2\le x\le 1. \end{cases}
\end{aligned}$$
Given this, we have
$$\begin{aligned}
h_*(\delta(y)\wedge S^1) &= h_*(i_*(\alpha^{-1}(y))\wedge S^1) \ = \ (h\circ(i\wedge S^1))_*(\alpha^{-1}(y)\wedge S^1) \\
&= -(i\circ\epsilon)_*(\alpha^{-1}(y)\wedge S^1) \ =_{(1.12)} \ -i_*(y)
\end{aligned}$$
and this finishes the proof. □

For every Serre fibration $\varphi\colon E \longrightarrow B$ of topological spaces the map $c\colon F \longrightarrow F(\varphi)$ from the strict fiber to the homotopy fiber that sends $e \in F$ to (const_*, e). is a weak equivalence. We let $f\colon X \longrightarrow Y$ be a morphism of sequential spectra that is levelwise a Serre fibration; then by the above the morphism $c\colon F \longrightarrow F(f)$ from the strict fiber to the homotopy fiber of f is a level equivalence. So we can define another connecting morphism

$$\delta \ : \ \pi_k(Y) \ \longrightarrow \ \pi_{k-1}(F)$$

as the composite of the connecting homomorphism $\pi_k(Y) \longrightarrow \pi_{k-1}(Ff)$ defined in (1.21) and the inverse of the isomorphism $c_*\colon \pi_{k-1}(Ff) \longrightarrow \pi_{k-1}(F)$.

Corollary 1.23. *Let $f\colon X \longrightarrow Y$ be a morphism of sequential spectra that is levelwise a Serre fibration; let $\iota\colon F \longrightarrow X$ denote the inclusion of the fiber over the basepoint. Then the long sequence of homotopy groups*

$$\cdots \ \longrightarrow \ \pi_k(F) \ \xrightarrow{\iota_*} \pi_k(X) \ \xrightarrow{f_*} \ \pi_k(Y) \ \xrightarrow{\delta} \pi_{k-1}(F) \ \longrightarrow \ \cdots$$

is exact.

We draw some consequences of our previous results.

Proposition 1.24. (i) *For every family of sequential spectra $\{X^i\}_{i\in I}$ and every integer k the canonical map*

$$\bigoplus_{i\in I} \pi_k(X^i) \ \longrightarrow \ \pi_k\left(\bigvee_{i\in I} X^i\right)$$

is an isomorphism of abelian groups.

(ii) *For every finite indexing set I, every family $\{X^i\}_{i\in I}$ of sequential spectra and every integer k the canonical map*

$$\pi_k\left(\prod_{i\in I} X^i\right) \ \longrightarrow \ \prod_{i\in I} \pi_k(X^i)$$

is an isomorphism of abelian groups.

(iii) *For every finite family of sequential spectra the canonical morphism from the wedge to the product is a stable equivalence.*

Proof. (i) We first show the special case of two summands. If A and B are two sequential spectra, then the wedge inclusion $i_A\colon A \longrightarrow A \vee B$ has a retraction. So the associated long exact homotopy group sequence of Proposition 1.18 splits into short exact sequences

$$0 \ \longrightarrow \ \pi_k(A) \ \xrightarrow{(i_A)_*} \ \pi_k(A \vee B) \ \xrightarrow{i_*} \ \pi_k(C(i_A)) \ \longrightarrow \ 0\ .$$

The mapping cone $C(i_A)$ is isomorphic to $(CA) \vee B$ and thus homotopy equivalent to B. So we can replace $\pi_k(C(i_A))$ by $\pi_k(B)$ and conclude that $\pi_k(A \vee B)$ splits as the sum of $\pi_k(A)$ and $\pi_k(B)$, via the canonical map. The case of a finite indexing set I now follows by induction.

In the general case we consider the composite

$$\bigoplus\nolimits_{i\in I} \pi_k(X^i) \ \longrightarrow \ \pi_k\left(\bigvee\nolimits_{i\in I} X^i\right) \ \longrightarrow \ \prod\nolimits_{i\in I} \pi_k(X^i)\ ,$$

where the second map is induced by the projections to the wedge summands. This composite is the canonical map from a direct sum to a product of abelian groups, hence injective. So the first map is injective as well. For surjectivity we consider a continuous based map $f\colon S^{n+k} \longrightarrow \bigvee_{i\in I} X^i_n$ that represents an element in the k-th homotopy group of $\bigvee_{i\in I} X^i$. Since the source of f is compact, there is a finite subset J of I such that f has image in $\bigvee_{j\in J} X^j_n$, see for example [40, Proposition A.18]. Then the given class is in the image of $\pi_k\left(\bigvee_{j\in J} X^j\right)$; since J is finite, the class is in the image of the canonical map, by the previous paragraph.

(ii) Unstable homotopy groups commute with products, which for finite indexing sets are also sums, which commute with filtered colimits.

(iii) This is a direct consequence of (i) and (ii). More precisely, for finite indexing set I and every integer k the composite map

$$\bigoplus_{i\in I}\pi_k(X^i)\longrightarrow\pi_k(\bigvee_{i\in I}X^i)\ \longrightarrow\ \pi_k(\prod_{i\in I}X^i)\longrightarrow\prod_{i\in I}\pi_k(X^i)$$

is an isomorphism, where the first and last maps are the canonical ones. These canonical maps are isomorphisms by parts (i) respectively (ii), hence so is the middle map. □

Remark 1.25. The restriction to *finite* indexing sets in parts (ii) of the previous corollary is essential, and it ultimately comes from the fact that infinite products do not in general commute with sequential colimits. Here is an explicit example: we consider the spectra $\mathbb{S}^{\leq i}$ obtained by truncating the sphere spectrum above level i, i.e.,

$$(\mathbb{S}^{\leq i})_n\ =\ \begin{cases} S^n & \text{for } n\leq i,\\ * & \text{for } n\geq i+1\end{cases}$$

with structure maps as a quotient spectrum of $\mathbb{S}$. Then $\mathbb{S}^{\leq i}$ has trivial homotopy groups for all i. The 0th homotopy group of the product $\prod_{i\geq 1}\mathbb{S}^{\leq i}$ is the colimit of the sequence of maps

$$\prod_{i\geq n}\pi_n(S^n)\ \longrightarrow\ \prod_{i\geq n+1}\pi_{n+1}(S^{n+1})$$

which first projects away from the factor indexed by $i=n$ and then takes a product of the suspensions homomorphisms $-\wedge S^1\colon\pi_n(S^n)\longrightarrow\pi_{n+1}(S^{n+1})$. The colimit is thus isomorphic to the quotient of an infinite product of copies of the group $\mathbb{Z}$ by the direct sum of the same number of copies of $\mathbb{Z}$. Hence the right hand side of the canonical map

$$\pi_0\left(\prod\nolimits_{i\geq 1}\mathbb{S}^{\leq i}\right)\ \longrightarrow\ \prod\nolimits_{i\geq 1}\pi_0(\mathbb{S}^{\leq i})$$

is trivial, while the left hand side is not.

Proposition 1.26. (i) *Let $e^m\colon X^m\longrightarrow X^{m+1}$ be morphisms of sequential spectra that are levelwise closed embeddings, for $m\geq 0$. Let X^∞ be a colimit of the sequence $\{e^m\}_{m\geq 0}$. Then for every integer k the canonical map*

$$\operatorname{colim}_{m\geq 0}\,\pi_k(X^m)\ \longrightarrow\ \pi_k(X^\infty)$$

is an isomorphism.

(ii) *Let $e^m\colon X^m\longrightarrow X^{m+1}$ and $f^m\colon Y^m\longrightarrow Y^{m+1}$ be morphisms of sequential spectra that are levelwise closed embeddings, for $m\geq 0$. Let $\psi^m\colon X^m\longrightarrow Y^m$ be stable equivalences that satisfy $\psi^{m+1}\circ e^m = f^m\circ\psi^m$ for all $m\geq 0$. Then the induced morphism $\psi^\infty\colon X^\infty\longrightarrow Y^\infty$ between the colimits of the sequences is a stable equivalence.*

(iii) *Let $f^m\colon Y^m\longrightarrow Y^{m+1}$ be stable equivalences of sequential spectra that are levelwise closed embeddings, for $m\geq 0$. Then the canonical morphism $f^\infty\colon Y^0\longrightarrow Y^\infty$ to a colimit of the sequence $\{f^m\}_{m\geq 0}$ is a stable equivalence.*

Proof. (i) We let $f\colon S^{n+k}\longrightarrow X^\infty_n$ be a based continuous map that represents a class in $\pi_k(X^\infty)$. Since the sphere S^{n+k} is compact and X^∞_n is a colimit of the sequence of closed embeddings $X^m_n\longrightarrow X^{m+1}_n$, the map f factors through a continuous map

$$\bar f\ :\ S^{n+k}\ \longrightarrow\ X^m_n$$

for some $m\geq 0$, see for example [18, Proposition 2.4.2] or [40, Proposition A.15]. The same reasoning applies to homotopies, so the canonical map

$$\operatorname{colim}_{m\geq 0}\,\pi_{n+k}(X^m_n)\ \longrightarrow\ \pi_{n+k}(X^\infty_n)$$

is bijective. Passing to colimits over n proves the claim.

Parts (ii) and (iii) are direct consequences of (i). □

Proposition 1.27. (i) *A wedge of stable equivalences is a stable equivalence.*

(ii) *A finite product of stable equivalences is a stable equivalence.*

(iii) *Consider a commutative square of orthogonal spectra*

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "f"'] & B \arrow[d, "g"] \\ C \arrow[r, "j"'] & D \end{tikzcd} \tag{1.28}$$

and let $h = (Cf) \cup g \colon Ci \longrightarrow Cj$ *be the map induced by* f *and* g *on mapping cones. Then if two of the three morphisms* f, g *and* h *are stable equivalences, so is the third.*

(iv) *Consider a commutative square* (1.28) *of sequential spectra. Let* $e \colon Fi \longrightarrow Fj$ *be the map induced by* f *and* g *on homotopy fibers. Then if two of the three morphisms* e, f *and* g *are stable equivalences, so is the third.*

(v) *Consider a commutative square* (1.28) *of sequential spectra for which one of the following conditions holds:*
(a) *the square is a pushout and* i *or* f *is levelwise an h-cofibration.*
(b) *the square is a pullback and* j *or* g *is a levelwise a Serre fibration.*
Then f *is a stable equivalence if and only if* g *is.*

(vi) *Let* K *be a based space that admits a CW-structure. Then the functor* $- \wedge K$ *preserves stable equivalences of sequential spectra.*

(vii) *Let* K *be a based space that admits a finite CW-structure. Then the functor* $\mathrm{map}_*(K, -)$ *preserves stable equivalences of sequential spectra.*

Proof. Part (i) holds because the homotopy groups of a wedge are the direct sum of the homotopy groups of the wedge summands (Proposition 1.24 (i)). Part (ii) holds because the homotopy groups of a finite product are the product of the homotopy groups of the factors (Proposition 1.24 (ii)). Part (iii) follows by applying the 5-lemma to the long exact sequences of the mapping cones of i and j (Proposition 1.18). Part (iv) follows by applying the 5-lemma to the long exact sequences of the homotopy fibers of i and j (Proposition 1.22).

(v) We start with case (a) of a pushout square. Since the square is a pushout, the morphism $j \colon C \longrightarrow D$ descends to an isomorphism $j/i \colon C/A \cong D/B$ between the two vertical quotient spectra; and the morphism $g \colon B \longrightarrow D$ descends to an isomorphism $g/f \colon B/A \cong D/C$ between the two horizontal quotient spectra.

If f is levelwise an h-cofibration, then the long exact homotopy group sequence for the quotient spectrum C/A (Corollary 1.20) shows that f is a stable equivalence if and only of C/A has trivial homotopy groups. Since h-cofibrations are stable under cobase change, the same is true for g: the morphism g is a stable equivalence if and only of D/B has trivial homotopy groups. So f is a stable equivalence if and only if g is.

If i is an h-cofibration, the argument is similar, but slightly different. In this case we compare the two long exact homotopy group sequences for the horizontal quotient spectra (Corollary 1.20). Since $g/f \colon B/A \cong D/C$ is an isomorphism, the 5-lemma shows that f is stable equivalence if and only if g is. The comparison map for the homotopy groups of the

The case (b) of a pullback square is strictly dual, using strict fibers instead of quotient spectra, and Corollary 1.23 in place of Corollary 1.20.

(vi) The functor $- \wedge K$ preserves mapping cones, so by the long exact homotopy group sequence of Proposition 1.18 it suffices to show the following special case: let X be a sequential spectrum all of whose homotopy groups vanish; then all homotopy groups of the spectrum $X \wedge K$ vanish, too.

We let K_n denote the n-skeleton in a CW-structure on K. We show first, by induction on n, that the spectrum $X \wedge K_n$ has trivial homotopy groups. The induction starts with $n = -1$, where there is nothing to show. For $n \geq 0$ the quotient K_n/K_{n-1} is homeomorphic to a wedge of n-spheres. Since homotopy groups take wedges to sums, the suspension isomorphism allows us to rewrite the homotopy groups of $X \wedge (K_n/K_{n-1})$ as

$$\pi_k(X \wedge (K_n/K_{n-1})) \;\cong\; \pi_k(X \wedge (\bigvee\nolimits_I S^n)) \;\cong\; \pi_k(\bigvee\nolimits_I (X \wedge S^n)) \;\cong\; \bigoplus\nolimits_I \pi_{k-n}(X) \; .$$

This group is trivial by the hypothesis on X.

The inclusion $K_{n-1} \longrightarrow K_n$ is an h-cofibration of based spaces, so the induced morphism $X \wedge K_{n-1} \longrightarrow X \wedge K_n$ is an h-cofibration of sequential spectra; these morphisms are then in particular levelwise closed embeddings, giving rise to a long exact sequence of homotopy groups (Corollary 1.20). By the previous paragraph and the inductive hypothesis, the spectrum $X \wedge K_n$ has vanishing homotopy groups. This completes the inductive step.

Since K is the sequential colimit, along h-cofibrations of based spaces, of the skeleta K_n, the spectrum $X \wedge K$ is the sequential colimit, along h-cofibrations sequential spectra, of the sequence with terms $X \wedge K_n$. These h-cofibrations are in particular levelwise closed embeddings, see for example [40, Proposition A.31]. So homotopy groups commute with such sequential colimits (Proposition 1.26 (i)), so also $X \wedge K$ has vanishing homotopy groups.

(vii) We start with a special case and let X be a sequential spectrum whose homotopy groups vanish. We show first that then the homotopy groups of the spectrum $\mathrm{map}_*(K, X)$ vanish, too. We argue by induction over the number of cells in a CW-structure of K. The induction starts when K consists only of the basepoint, in which case $\mathrm{map}_*(K, X)$ is a trivial spectrum and there is nothing to show. For the inductive step we assume that the homotopy groups of $\mathrm{map}_*(K, X)$ vanish and L is obtained from K by attaching an n-cell. Then the restriction map $\mathrm{map}_*(L, X) \longrightarrow \mathrm{map}_*(K, X)$ is levelwise a Serre fibration whose fiber is isomorphic to

$$\mathrm{map}_*(L/K, X) \ \cong\ \mathrm{map}_*(S^n, X) \ \cong\ \Omega^n X \ .$$

The homotopy groups of this spectrum are isomorphic to the shifted homotopy groups of X, and these vanish by assumption. The long exact sequence of Corollary 1.23 and the inductive hypothesis then show that the homotopy groups of $\mathrm{map}_*(K, X)$ vanish.

The functor $\mathrm{map}_*(K, -)$ commutes with homotopy fibers; so two applications of the long exact homotopy group sequence of a homotopy fiber (Proposition 1.22) reduce the general case to the special case. □

1.2. **Homology theories from spectra.** We write $\mathbf{T}_*$ for the category of based spaces, i.e., compactly generated spaces equipped with a distinguished basepoint. Morphisms in $\mathbf{T}_*$ are all based continuous maps.

Definition 1.29. A *generalized homology theory* $\{E_k, \partial\}$ consists of the following data:

- a functor
$$E_k \ :\ \mathbf{T}_* \ \longrightarrow\ \text{(abelian groups)}$$
for every integer k;
- a natural transformation $\partial : E_{k+1}(Cf) \longrightarrow E_k(A)$ for every integer k, where $f : A \longrightarrow B$ is a based continuous map, and $Cf = (A \wedge [0,1]) \cup_f B$ is the reduced mapping cone of f.

This data has to satisfy the following axioms:

- (Additivity) For every family $\{A_i\}_{i\in I}$ of based spaces the canonical map
$$\bigoplus_{i\in I} E_k(A_i) \ \longrightarrow\ E_k(\bigvee_{i\in I} A_i)$$
is an isomorphism.
- (Homotopy invariance) If $f, g : A \longrightarrow B$ are based homotopic, then $E_k(f) = E_k(g) \colon E_k(A) \longrightarrow E_k(B)$.
- (Exactness) For every continuous based map $f : A \longrightarrow B$, the sequence
$$\cdots \longrightarrow E_{k+1}(Cf) \xrightarrow{\ \partial\ } E_k(A) \xrightarrow{E_k(f)} E_k(B) \xrightarrow{E_k(i)} E_k(Cf) \longrightarrow \cdots$$
of E-homology groups is exact.

Example 1.30 (Generalized homology from a spectrum)**.** We let E be a sequential spectrum, and k an integer. We define the *E-homology* of a based space A as

$$E_k(A) \ =\ \pi_k(E \wedge A) \ .$$

The functors E_k form a generalized homology theory with respect to the connecting homomorphism (1.17) of the long exact homotopy group sequence

$$\partial \ : \ E_{k+1}(Cf) \ \longrightarrow \ E_k(A) \ ,$$

i.e., the composite of the map

$$E_{k+1}(Cf) \ \xrightarrow{\pi_{k+1}(E\wedge p)} \ \pi_{k+1}(E\wedge A\wedge S^1)$$

and the inverse of the suspension isomorphism $\pi_k(E\wedge A_+)\cong \pi_{k+1}(E\wedge A_+\wedge S^1)$. The additivity property is Proposition 1.24 (i), combined with the fact that $E\wedge -$ takes wedges of based spaces to coproducts of spectra. Homotopy invariance is the homotopy invariance of stable homotopy groups. And exactness is a special case of Proposition 1.18.

Remark 1.31. Every generalized homology theory has the following additional properties.

(a) The additivity property for $X=Y=\{*\}$ one-point spaces shows that the canonical map and the fact that the wedge of X and Y is another one-point spaces shows that the addition homomorphism

$$E_k(*)\oplus E_k(*) \ \longrightarrow \ E_k(*)$$

is an isomorphism. But this forces $E_k=0$ to be the trivial group.

(b) Every based homotopy equivalence $f\colon X\longrightarrow Y$ induces an isomorphism $E_k(f)\colon E_k(X)\xrightarrow{\cong} E_k(Y)$.

(c) The previous two items show that $E_k(X)=0$ whenever X is contractible to its basepoint.

(d) Because $E_k(*)=0$, the long exact sequence for the E-homology of the unique map $t_A\colon A\longrightarrow *$ specializes to isomorphisms

$$\partial \ : \ E_{k+1}(C(t_A)) \ \xrightarrow{\cong} \ E_k(A) \ .$$

The 'projection' $p\colon C(t_A)\longrightarrow A\wedge S^1$ is a homeomorphism, so we obtain a *suspension isomorphism*

$$E_{k+1}(p)\circ\partial^{-1} \ : \ E_k(A) \ \xrightarrow{\cong} \ E_{k+1}(A\wedge S^1) \ .$$

(e) Let (B,A) be a pair of spaces with the homotopy extension property. Then the quotient map $0\cup \mathrm{Id}_B\colon (A\wedge[0,1])\cup_A B = C(\mathrm{incl})\longrightarrow B/A$ is a based homotopy equivalence; so it induces an isomorphism $E_k(C(\mathrm{incl}))\cong E_k(B/A)$. The long exact sequence of the inclusion can thus be turned into an exact sequence

$$\cdots\longrightarrow E_{k+1}(B/A)\xrightarrow{\delta} E_k(A)\xrightarrow{E_k(\mathrm{incl})} E_k(B)\xrightarrow{E_k(\mathrm{proj})} E_k(B/A)\longrightarrow\cdots \ ,$$

where

$$\delta \ = \ \partial\circ E_{k+1}(0\cup \mathrm{Id}_B)^{-1} \ .$$

Remark 1.32 (Relative and unreduced theories)**.** We let $\{E_k,\partial\}$ be a generalized homology theory.

For an unbased space X, we define the *unreduced E-homology* by

$$E_k^+(X) \ = \ E_k(X_+) \ ,$$

i.e., the original theory of X with a disjoint basepoint added. Because the functor $(-)_+\colon \mathbf{T}\longrightarrow \mathbf{T}_*$ takes disjoint unions to wedges, the additivity property in the unreduced version becomes the statement that for every family $\{X_i\}_{i\in I}$ of unbased spaces the canonical map

$$\bigoplus_{i\in I} E_k^+(X_i) \ \longrightarrow \ E_k^+(\coprod_{i\in I} X_i)$$

is an isomorphism.

For a pair (X,Y) of unbased spaces, we write

$$CY\cup_Y X \ = \ Y\times[0,1]\cup_{Y\times 1} X/\sim$$

for the unreduced mapping cone of the inclusion. Here the equivalence relation identifies $(1,y)$ with $y\in X$ for all $y\in Y$, and it collapses $Y\times 0$ to a point. We take the cone point – the equivalence class of $Y\times 0$ – as the basepoint. We then define the *relative E-homology* by

$$E_k(X,Y) \ = \ E_k(CY\cup_Y X) \ .$$

The based space $CY \cup_Y X$ is homeomorphic to the reduced mapping cone of the based inclusion $\iota\colon Y_+ \longrightarrow X_+$, where we added disjoint basepoints to X and Y. So the long exact sequence for ι is an exact sequence

$$\cdots \longrightarrow E_{k+1}(X,Y) \xrightarrow{\partial} E_k^+(Y) \xrightarrow{E_k(\mathrm{incl})} E_k^+(X) \longrightarrow E_k(X,Y) \longrightarrow \cdots$$

If $Y = \{x_0\}$ is a single point, then $CY \cup_Y X$ deformation retracts onto X; for if X is pointed by x_0, then

$$E_k(X, \{x_0\}) \cong E_k(X) .$$

The original functors can be recovered from the unreduced theory as follows. The inclusion $i\colon X \longrightarrow C()$

Proposition 1.33. *Let $\{E_k, \partial\}$ be a generalized homology theory. Let $\mathrm{tel}_i X_i$ be the mapping telescope of a sequence of continuous maps*

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \ldots .$$

Then the canonical maps $j_m\colon X_m \longrightarrow \mathrm{tel}_i X_i$ provide an isomorphism

$$\mathrm{colim}_{m\geq 0}\, E_k^+(X_m) \xrightarrow{\cong} E_k^+(\mathrm{tel}_{i\geq 0} X_i) .$$

Proof. The mapping telescope participates in a pushout square

$$\begin{tikzcd} \coprod_{i\geq 0} X_i \times \{0,1\} \arrow[r] \arrow[d, "\mathrm{incl}"'] & \coprod_{i\geq 0} X_i \arrow[d, "j"] \\ \coprod_{i\geq 0} X_i \times [0,1] \arrow[r] & \mathrm{tel}_i X_i \end{tikzcd}$$

For $x \in X_i$, the upper vertical map sends $(x, 0)$ to x, and it sends $(x, 1)$ to $f_i(x)$. The two vertical maps have the homotopy extension property, so we obtain a long exact sequence involving the E-homology of $\coprod_{i\geq 0} X_i$, of $\mathrm{tel}_i X_i$, and of the cokernel of the h-cofibration j. This cokernel is homeomorphic to $\bigvee_{i\geq 0}(X_i)_+ \wedge S^1$, so the sequence becomes an exact sequence

$$\cdots \longrightarrow E_{k+1}(\bigvee_{i\geq 0}(X_i)_+ \wedge S^1) \xrightarrow{\partial} E_k^+(\amalg_{i\geq 0} X_i) \longrightarrow E_k^+(\mathrm{tel}_i X_i) \longrightarrow E_k(\bigvee_{i\geq 0}(X_i)_+ \wedge S^1) \longrightarrow \cdots$$

Additivity and the suspension isomorphism provide isomorphisms

$$\bigoplus_{i\geq 0} E_k^+(X_i) \xrightarrow{\cong} E_k^+(\amalg_{i\geq 0} X_i) \qquad \text{and}$$

$$\bigoplus_{i\geq 0} E_k^+(X_i) \xrightarrow{\cong} \bigoplus_{i\geq 0} E_{k+1}((X_i)_+ \wedge S^1) \xrightarrow{\cong} E_{k+1}(\bigvee_{i\geq 0}(X_i)_+ \wedge S^1) .$$

These isomorphism turn to exact sequence into another exact sequence

$$\cdots \longrightarrow \bigoplus_{i\geq 0} E_k^+(X_i) \xrightarrow{\Delta} \bigoplus_{i\geq 0} E_k^+(X_i) \longrightarrow E_k^+(\mathrm{tel}_i X_i) \longrightarrow \bigoplus_{i\geq 0} E_{k-1}^+(X_i) \longrightarrow \cdots$$

We omit the verification that the morphism Δ is given by

$$\Delta(x) = x - (f_i)_*(x)$$

for all $x \in E_k^+(X_i)$. Since this map is injective, the sequence provides an isomorphism

$$\mathrm{coker}\left(\mathrm{Id} - f_*\colon \bigoplus_{i\geq 0} E_k^+(X_i) \longrightarrow \bigoplus_{i\geq 0} E_k^+(X_i) \right) \cong E_k^+(\mathrm{tel}_i X_i) .$$

The cokernel is a presentation of the colimit of the sequence of abelian groups

$$E_k(X_0) \xrightarrow{E_k(f_0)} E_k(X_1) \xrightarrow{E_k(f_1)} E_k(X_2) \xrightarrow{E_k(f_2)} \ldots .$$

This proves the claim. $\square$

If X is a CW-complex with skeleta X_n, then the maps $\mathrm{tel}_{i\geq 0} X_i \longrightarrow X$ that collapses the intervals in the mapping telescope is a homotopy equivalence. So Proposition 1.33 yields:

Corollary 1.34. *Let $\{E_k, \partial\}$ be a generalized homology theory. Let X be a CW-complex with skeleta X_m. Then the inclusions $X_m \longrightarrow X$ provide an isomorphism*

$$\mathrm{colim}_{m\geq 0}\, E_k^+(X_m) \xrightarrow{\cong} E_k^+(X) \ .$$

Example 1.35. We let A be an abelian group, and we let HA be an Eilenberg–MacLane spectrum as in Example 1.8. Then for spaces X admitting a CW-structure, the generalized homology theory $(HA)_k(X)$ is naturally isomorphic to singular homology $H_k(X;A)$. We will construct a natural isomorphism later, once we have discussed the orthogonal spectrum model for HA in Example 3.3.

2. Orthogonal spectra

Orthogonal spectra are used, at least implicitly, already in [31]; the term 'orthogonal spectrum' was introduced by Mandell, May, Shipley and the author in [30], where the (non-equivariant) stable model structure for orthogonal spectra was constructed. Before giving the formal definition we try to motivate it. An orthogonal spectrum X assigns a based space $X(V)$ to every inner product space, and it keeps track of an $O(V)$-action on $X(V)$ and of a way to stabilize by suspensions. When doing this in a coordinate-free way, the stabilization data assigns to a linear isometric embedding $\varphi\colon V \longrightarrow W$ a continuous based map

$$\varphi_\star \ : \ S^{W-\varphi(V)} \wedge X(V) \ \longrightarrow \ X(W)$$

where $W - \varphi(V)$ is the orthogonal complement of the image of φ. This structure map should 'vary continuously with φ', but this phrase has no literal meaning because the source of $\varphi_\star$ depends on φ. The way to make the continuous dependence rigorous is to exploit the fact that the complements $W-\varphi(V)$ vary in a locally trivial way, i.e., they are the fibers of a distinguished vector bundle, the 'orthogonal complement bundle', over the space of $\mathbf{L}(V,W)$ of linear isometric embeddings. All the structure maps $\varphi_\star$ together define a map on the smash product of $X(V)$ with the Thom space of this complement bundle, and the continuity of the dependence on φ is formalized by requiring continuity of that map. All these Thom spaces together form the morphism spaces of a based topological category, and the data of an orthogonal spectrum can conveniently be packaged as a continuous based functor on this category.

Construction 2.1. We let V and W be inner product spaces. Over the space $\mathbf{L}(V,W)$ of linear isometric embeddings sits a certain 'orthogonal complement' vector bundle with total space

$$\xi(V,W) \ = \ \{\, (w,\varphi) \in W \times \mathbf{L}(V,W) \mid w \perp \varphi(V) \,\} \ .$$

The structure map $\xi(V,W) \longrightarrow \mathbf{L}(V,W)$ is the projection to the second factor. The vector bundle structure of $\xi(V,W)$ is as a vector subbundle of the trivial vector bundle $W\times\mathbf{L}(V,W)$, and the fiber over $\varphi\colon V\longrightarrow W$ is the orthogonal complement $W-\varphi(V)$ of the image of φ.

We let $\mathbf{O}(V,W)$ be the Thom space of the bundle $\xi(V,W)$, i.e., the one-point compactification of the total space of $\xi(V,W)$. Up to non-canonical homeomorphism, we can describe the space $\mathbf{O}(V,W)$ differently as follows. If the dimension of W is smaller than the dimension of V, then the space $\mathbf{L}(V,W)$ is empty and $\mathbf{O}(V,W)$ consists of a single point at infinity. Otherwise we can choose a linear isometric embedding $\varphi\colon V\longrightarrow W$, and then the maps

$$\begin{aligned} O(W)/O(W-\varphi(V)) \quad &\longrightarrow \ \mathbf{L}(V,W) \ , \quad A\cdot O(W-\varphi(V)) \ \longmapsto \ A\varphi \quad \text{and} \\ O(W)\ltimes_{O(W-\varphi(V))} S^{W-\varphi(V)} \ &\longrightarrow \ \mathbf{O}(V,W) \ , \quad [A,w] \ \longmapsto \ (Aw, A\varphi) \end{aligned}$$

are homeomorphisms. Here, and in the following, we write

$$G\ltimes_H A \ = \ (G_+)\wedge_H A \ = \ (G_+\wedge A)/\sim$$

for a closed subgroup H of G and a based G-space A; the equivalence relation is $gh\wedge a \sim g\wedge ha$ for all $(g,h,a)\in G\times H\times A$. Put yet another way: if $\dim V = n$ and $\dim W = n+m$, then $\mathbf{L}(V,W)$ is homeomorphic to the homogeneous space $O(n+m)/O(m)$ and $\mathbf{O}(V,W)$ is homeomorphic to $O(n+m)\ltimes_{O(m)} S^m$. The vector bundle $\xi(V,W)$ becomes trivial upon product with the trivial bundle V, via the trivialization

$$\xi(V,W)\times V \;\cong\; W\times \mathbf{L}(V,W)\ , \quad ((w,\varphi),v) \;\longmapsto\; (w+\varphi(v),\varphi)\ .$$

When we pass to Thom spaces on both sides this becomes the *untwisting homeomorphism:*

$$\mathbf{O}(V,W)\wedge S^V \;\cong\; S^W\wedge \mathbf{L}(V,W)_+\ . \tag{2.2}$$

The Thom spaces $\mathbf{O}(V,W)$ are the morphism spaces of a based topological category. Given a third inner product space U, the bundle map

$$\xi(V,W)\times\xi(U,V) \;\longrightarrow\; \xi(U,W)\ , \quad ((w,\varphi),\,(v,\psi)) \;\longmapsto\; (w+\varphi(v),\,\varphi\psi)$$

covers the composition map $\mathbf{L}(V,W)\times\mathbf{L}(U,V)\longrightarrow\mathbf{L}(U,W)$. Passage to Thom spaces gives a based map

$$\circ \;:\; \mathbf{O}(V,W)\wedge\mathbf{O}(U,V) \;\longrightarrow\; \mathbf{O}(U,W)$$

which is clearly associative, and is the composition in the category $\mathbf{O}$. The identity of V is $(0,\mathrm{Id}_V)$ in $\mathbf{O}(V,V)$.

Definition 2.3. An *orthogonal spectrum* is a based continuous functor from $\mathbf{O}$ to the category $\mathbf{T}_*$ of based spaces. A *morphism* of orthogonal spectra is a natural transformation of functors. We denote the category of orthogonal spectra by $\mathcal{S}p$.

Given two inner product spaces V and W we define a continuous based map

$$i_V \;:\; S^V \;\longrightarrow\; \mathbf{O}(W,V\oplus W) \qquad \text{by} \qquad v \;\longmapsto\; ((v,0),(0,-))\ ,$$

where $(0,-)\colon W\longrightarrow V\oplus W$ is the embedding of the second summand. We define the *structure map* $\sigma_{V,W}\colon S^V\wedge X(W)\longrightarrow X(V\oplus W)$ of the orthogonal spectrum X as the composite

$$S^V\wedge X(W) \xrightarrow{\;i_V\wedge X(W)\;} \mathbf{O}(W,V\oplus W)\wedge X(W) \xrightarrow{\;X\;} X(V\oplus W)\ .$$

Often it will be convenient to use the *opposite structure map*

$$\sigma^{\mathrm{op}}_{V,W} \;:\; X(V)\wedge S^W \;\longrightarrow\; X(V\oplus W) \tag{2.4}$$

which we define as the following composite:

$$X(V)\wedge S^W \xrightarrow{\;\text{twist}\;} S^W\wedge X(V) \xrightarrow{\;\sigma_{W,V}\;} X(W\oplus V) \xrightarrow{\;X(\tau_{V,W})\;} X(V\oplus W)$$

Remark 2.5 (Coordinatized orthogonal spectra)**.** Every inner product space is isometrically isomorphic to $\mathbb{R}^n$ with standard inner product, for some $n\geq 0$. So the topological category $\mathbf{O}$ has a small skeleton, and the functor category of orthogonal spectra has 'small' morphism sets. Up to isomorphism, an orthogonal spectrum X is determined by the values $X_n = X(\mathbb{R}^n)$, the action of $O(n)$ on it, and the structure maps $\sigma_n\colon S^1\wedge X_n\longrightarrow X_{1+n}$ for $n\geq 0$. This also leads to a more explicit 'coordinatized' description of orthogonal spectra in a way that resembles a presentation by generators and relations. We refer to Exercise E.1 for more details.

2.1. **Constructions.** We discuss various constructions which produce new orthogonal spectra from old ones. Whenever possible, we describe the effect that a certain construction has on the homotopy groups.

Example 2.6 (Limits and colimits)**.** In any category of continuous based functors, limits and colimits exist, and they are defined objectwise. In particular, the category of orthogonal spectra has all limits and colimits, and they are defined levelwise. Let us be a bit more precise and consider a functor $F\colon J\longrightarrow\mathcal{S}p$ from a small category J to the category of orthogonal spectra. Then we define an orthogonal spectrum $\mathrm{colim}_J F$ at an inner product space V by

$$(\mathrm{colim}_J F)(V) \;=\; \mathrm{colim}_{j\in J} F(j)(V)\ ,$$

the colimit in the category of pointed spaces. The structure maps ares the composite

$$\mathbf{O}(V,W)\wedge(\mathrm{colim}_{j\in J}F(j)(V))\cong \mathrm{colim}_{j\in J}(\mathbf{O}(V,W)\wedge F(j)(V)) \xrightarrow{\;\mathrm{colim}_J F(j)(V,W)\;} \mathrm{colim}_{j\in J}F(j)(W)\ ;$$

here we exploit that smashing with any based space is a left adjoint, and thus the natural map

$$\mathrm{colim}_{j\in J}(\mathbf{O}(V,W)\wedge F(j)(V)) \;\longrightarrow\; \mathbf{O}(V,W)\wedge(\mathrm{colim}_{j\in J}F(V))$$

is an isomorphism, whose inverse is the first map above.

The argument for inverse limits is similar, but we have to use that structure maps

$$\mathbf{O}(V,W)\wedge X(V) \;\longrightarrow\; X(W)$$

of an orthogonal spectrum X can also be defined in the adjoint form, as maps

$$\tilde{X}(V,W) \;:\; X(V) \;\longrightarrow\; \mathrm{map}_*(\mathbf{O}(V,W),X(W)) \;.$$

We can thus construct a limit of $F\colon J\longrightarrow \mathcal{S}p$ by taking

$$(\lim_J F)(V) \;=\; \lim_{j\in J}F(j)(V) \;,$$

and the structure map is adjoint to the composite

$$\lim_{j\in J}F(W) \xrightarrow{\lim_J \hat{F}(j)(V,W)} \lim_{j\in J}\mathrm{map}_*(\mathbf{O}(V,W),F(j)(W)) \cong \mathrm{map}_*(\mathbf{O}(V,W),(\lim_{j\in J}F(j)(W)) \;;$$

we exploit that as a right adjoint, the functor $\mathrm{map}_*(\mathbf{O}(V,W)-)$ preserves limits.

Example 2.7. We let $F\colon \mathbf{T}_*\longrightarrow\mathbf{T}_*$ be any continuous endofunctor on the category of based spaces. Then for every orthogonal spectrum X, the composite functor

$$\mathbf{O} \xrightarrow{\;X\;} \mathbf{T}_* \xrightarrow{\;F\;} \mathbf{T}_*$$

is another orthogonal spectra. Similarly, we can apply F levelwise to morphisms of orthogonal spectra, so that composition with F becomes a functor

$$F\circ - \;:\; \mathcal{S}p \;\longrightarrow\; \mathcal{S}p \;.$$

Example 2.8 (Smash products with and functions from spaces)**.** For a based space A, smashing with A and taking based mapping space from A are an adjoint pair of continuous functors

$$-\wedge A \;:\; \rightleftarrows \;:\; \mathrm{map}_*(A,-) \;=\; (-)^A$$

We can thus apply these functors levelwise to orthogonal spectra as explained in Example 2.7; for every orthogonal spectrum X, this yields two new orthogonal spectra $X\wedge A$ and X^A. More explicitly, we have

$$(X\wedge A)(V) \;=\; X(V)\wedge A \qquad\text{and}\qquad \mathrm{map}_*(A,X)(V) \;=\; \mathrm{map}_*(A,X(V))$$

for an inner product space V. The structure maps and actions of the orthogonal groups do not interact with A: the group $O(V)$ acts through its action on $X(V)$, and the structure maps are given by the composite

$$S^V\wedge(X\wedge A)(W) \;=\; S^V\wedge X(W)\wedge A \xrightarrow{\sigma_{V,W}\wedge A} X(V\oplus W)\wedge A \;=\; (X\wedge A)(V\oplus W)$$

and by the composite

$$S^V\wedge \mathrm{map}_*(A,X(W)) \;\longrightarrow\; \mathrm{map}_*(A,S^V\wedge X(W)) \xrightarrow{\mathrm{map}_*(A,\sigma_{V,W})} \mathrm{map}_*(A,X(V\oplus W))$$

where the first is an assembly map that sends $v\wedge f$ to the map sending $a\in A$ to $v\wedge f(a)$.

Just as the functors $-\wedge A$ and $\mathrm{map}_*(A,-)$ are adjoint on the level of based spaces, the two functors just introduced are an adjoint pair on the level of orthogonal spectra. The adjunction

$$\mathcal{S}p(X,\mathrm{map}_*(A,Y)) \xrightarrow{\;\cong\;} \mathcal{S}p(X\wedge A,Y) \tag{2.9}$$

takes a morphism $f\colon X\longrightarrow \mathrm{map}_*(A,Y)$ to the morphism $f^\flat\colon X\wedge A\longrightarrow Y$ whose Vth level $f^\flat(V)\colon X(V)\wedge A\longrightarrow Y(V)$ is $f^\flat(V)(x\wedge a)=f(V)(x)(a)$.

An important special case of this construction is when $A=S^1$ is a 1-sphere, i.e., the one-point compactification of $\mathbb{R}$. The *suspension* $X\wedge S^1$ is defined by

$$(X\wedge S^1)(V) \;=\; X(V)\wedge S^1 \;,$$

the smash product of the Vth level of X with S^1. The *loop spectrum* $\Omega X = \mathrm{map}_*(S^1, X)$, defined by

$$(\Omega X)(V) \;=\; \Omega X(V) \;=\; \mathrm{map}_*(S^1, X(V)) \;,$$

the based mapping space from S^1 to the Vth level of X. We obtain an adjunction between $-\wedge S^1$ and Ω as the special case $A = S^1$ of (2.9).

We note that if X is an Ω-spectrum, then so is X^A, provided we also assume that A is cofibrant (for example a CW-complex). Indeed, under this hypothesis, the mapping space functor $\mathrm{map}_*(A, -)$ takes the weak equivalence $\tilde{\sigma}_n \colon X_n \longrightarrow \Omega X_{n+1}$ to a weak equivalence

$$X_n^A = \mathrm{map}_*(A, X_n) \xrightarrow{\mathrm{map}_*(A, \tilde{\sigma}_n)} \mathrm{map}_*(A, \Omega X_{n+1}) \cong \Omega(X_{n+1}^A) \;.$$

In Proposition 1.27 we prove that stable equivalences are closed under various constructions such as suspensions, loop, shift adjoint, wedges, and finite products. As we hope to show later, up to stable equivalence, every orthogonal spectrum can be replaced by an Ω-spectrum.

Clearly, every orthogonal spectrum has an underlying sequential spectrum, obtained by forgetting the actions of the orthogonal groups. The homotopy groups of an orthogonal spectra are, by definition, the homotopy groups of the underlying sequential spectrum.

Now we can introduce mapping cones for orthogonal spectra. In essence, we take mapping cones 'objectiwse' as in the case of sequential spectra (1.16), and the extra structure goes along for the ride. In more detail, the *mapping cone* Cf of a morphism of orthogonal spectra $f \colon X \longrightarrow Y$ is defined by the pushout square:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "-\wedge 1"'] & Y \arrow[d, "i"] \\ X \wedge [0,1] \arrow[r] & Cf \end{tikzcd}$$

Because colimits and smash product with spaces in the categories of sequential and orthogonal spectra are objectwise, this amounts to defining Cf by taking mapping cones levelwise/objectwise:

$$(Cf)(V) \;=\; C(f(V)) \;=\; (X(V) \wedge [0,1]) \cup_{f(V)} Y(V) \;,$$

where V is any inner product space. Moreover, the orthogonal group $O(V)$ acts on $(Cf)(V)$ through the given action on $X(V)$ and $Y(V)$ and trivially on the interval. The inclusions $i(V) \colon Y(V) \longrightarrow C(f(V))$ and projections $p(V) \colon C(f(V)) \longrightarrow X(V) \wedge S^1$ assemble into morphisms of orthogonal spectra $i \colon Y \longrightarrow Cf$ and $p \colon Cf \longrightarrow X \wedge S^1$. The homotopy fiber $F(f)$ is the fiber product

$$Ff \;=\; * \times_Y Y^{[0,1]} \times_Y X$$

i.e., the pullback in the cartesian square of orthogonal spectra:

$$\begin{tikzcd} Ff \arrow[r, "q"] \arrow[d] & X \arrow[d, "{(*,f)}"] \\ Y^{[0,1]} \arrow[r, "{(\mathrm{ev}_0, \mathrm{ev}_1)}"'] & Y \times Y \end{tikzcd}$$

Here $\mathrm{ev}_i \colon Y^{[0,1]} \longrightarrow Y$ for $i = 0, 1$ is the ith evaluation map which takes a path $\omega \in Y^{[0,1]}$ to $\omega(i)$, i.e., the start or endpoint. Limits and mapping objects in spectra are taken objectwise, so $(Ff)(V) = F(f(V))$, the orthogonal group $O(V)$ acts on $(Ff)(V)$ through the given action on $X(V)$ and $Y(V)$ and trivially on the interval. The inclusions $i(V) \colon \Omega Y(V) \longrightarrow (Ff)(V)$ and projections $q(V) \colon (Ff)(V) \longrightarrow X(V)$ assemble into morphisms of orthogonal spectra $i \colon \Omega Y \longrightarrow Ff$ and $p \colon Ff \longrightarrow X$.

Construction 2.10. We define a forgetful functor

$$u \;:\; \mathcal{S}p \;\longrightarrow\; \mathcal{S}p^{\mathbb{N}}$$

from orthogonal to sequential spectra. For an orthogonal spectrum X, we define n-level of the the sequential spectrum uX by

$$(uX)_n \;=\; X(\mathbb{R}^n)\ ,$$

the value at $\mathbb{R}^n$ endowed with the standard inner product $\langle x,y\rangle = \sum_{i=1}^n x_i y_i$. The n-th structure map is

$$\sigma_n \;=\; \sigma_{\mathbb{R},\mathbb{R}^n} \;:\; S^1 \wedge X_n \;=\; S^{\mathbb{R}} \wedge X(\mathbb{R}^n) \;\longrightarrow\; X(\mathbb{R}^{1+n}) \;=\; X_{1+n}\ .$$

On morphisms, the forgetful functor evaluates at $\mathbb{R}^n$. We will refer to uX as the *underlying sequential spectrum* of X.

Definition 2.11. The *homotopy groups* of an orthogonal spectrum are the homotopy groups of the underlying sequential spectrum. A morphism $f\colon X \longrightarrow Y$ of orthogonal spectra is a *stable equivalence* if the induced map $\pi_k(f)\colon \pi_k(X) \longrightarrow \pi_k(Y)$ is an isomorphism for all integers k.

By design, the forgetful functor $u\colon \mathcal{S}p \longrightarrow \mathcal{S}p^{\mathbb{N}}$ creates homotopy groups and preserves and detects stable equivalences. It also commutes with all categorical construction that we considered so far, namely limits, colimits, and smash product with $-\wedge K$, and mapping spectra from $\mathrm{map}_*(K,-)$ any based space K. This means that all structure results about homotopy groups and stable equivalence that we proved for sequential spectra in Section 1 also apply to orthogonal spectra. In particular, the following features and results are also apply to orthogonal spectra:

- the loop and suspension isomorphism (Proposition 1.13 (i));
- the unit $\eta\colon X \longrightarrow \Omega(X\wedge S^1)$ and counit $\epsilon\colon (\Omega X)\wedge S^1 \longrightarrow X$ of the loop-suspension adjunction are stable equivalences for every orthogonal spectrum X (Proposition 1.13 (ii));
- the long exact homotopy group sequence for the mapping cone of a morphism of orthogonal spectra (Proposition 1.18);
- the long exact homotopy group sequence for the strict cofiber of a morphism of orthogonal spectra that is levelwise an h-cofibration (Corollary 1.20);
- the long exact homotopy group sequence for the homotopy fiber of a morphism of orthogonal spectra (Proposition 1.22);
- the long exact homotopy group sequence for the strict fiber of a morphism of orthogonal spectra that is levelwise a Serre fibration (Corollary 1.23);
- homotopy groups takes wedges of orthogonal spectra to directs sums, and they take finite products of orthogonal spectra to products (Proposition 1.24);
- for every finite family of orthogonal spectra, the canonical morphism from the wedge to the product is a stable equivalence (Proposition 1.24);
- stable equivalence of orthogonal spectra are stable under colimits of sequences of morphisms that are levelwise closed embeddings (Proposition 1.26);
- stable equivalence of orthogonal spectra are stable under arbitrary wedges, under finite products, under cobase change along levelwise h-cofibrations, under base change with levelwise Serre fibrations, under smash product with arbitrary based CW-complexes, and under maps from finite based CW-complexes (Proposition 1.27).

Definition 2.12 (Homotopy relation)**.** Two morphisms of orthogonal spectra $f_0, f_1\colon A \longrightarrow X$ are called *homotopic* if there is a morphism

$$H \;:\; A\wedge [0,1]_+ \;\longrightarrow\; X\ ,$$

called a *homotopy*, such that $f_0 = H\circ i_0$, and $f_1 = H\circ i_1$. The morphisms $i_j\colon A \longrightarrow A\wedge[0,1]_+$ for $j=0,1$ are the 'end point inclusions' which are given levelwise by $i_j(a) = a\wedge j$.

A homotopy between spectrum morphisms is really the same data as a collection of based homotopies between $f_0(V)$ and $f_1(V)\colon A(V) \longrightarrow X(V)$ for all inner product spaces V, compatible with the $O(V)$-actions and structure maps. In particular, homotopic morphisms induce the same map of homotopy groups.

Homotopies can equivalently be given in two adjoint forms. By the adjunction (2.9) a homotopy $H\colon A\wedge [0,1]_+ \longrightarrow X$ from f_0 to f_1 is adjoint to a morphism $\hat{H}\colon A \longrightarrow X^{[0,1]}$ such that $\mathrm{ev}_0\circ\hat{H} = f_0$ and $\mathrm{ev}_1\circ\hat{H} = f_1$ where $\mathrm{ev}_j\colon X^{[0,1]} \longrightarrow X$ for $j=0,1$ is given levelwise by evaluation at $j\in[0,1]$. Finally, the homotopy H

is also adjoint to a morphism of (unbased) spaces $[0,1] \longrightarrow \mathrm{map}(A,X)$, i.e., a path in the mapping space, to be discussed in Example 3.10 below. So two morphisms are homotopic if and only if they lie in the same path component of the mapping space $\mathrm{map}(A,X)$.

For morphisms of orthogonal spectra, 'homotopy' is an equivalence relation. We denote by $[A,X]$ the set of homotopy classes of morphisms from A to X, i.e., the classes under the equivalence relation generated by homotopy.

A morphism $f\colon A \longrightarrow B$ of orthogonal spectra is a *homotopy equivalence* if there exists a morphism $g\colon B \longrightarrow A$ such that gf and fg are homotopic to the respective identity morphisms. Hence every homotopy equivalence of orthogonal spectra is in particular levelwise a based homotopy equivalence of based spaces. So for morphisms of orthogonal or sequential spectra we have the implications

$$\text{homotopy equivalence} \implies \text{level equivalence} \implies \text{stable equivalence.}$$

In general, the reverse implications do not hold. However, every stable equivalence between Ω-spectra is a level equivalence.

Example 2.13 (Shift)**.** We let W be an inner product space and denote by

$$- \oplus W \ : \ \mathbf{O} \ \longrightarrow \ \mathbf{O}$$

the continuous functor given on objects by orthogonal direct sum with W, and on morphism spaces by

$$\mathbf{O}(U,V) \ \longrightarrow \ \mathbf{O}(U\oplus W, V\oplus W) \ , \quad (v,\varphi) \ \longmapsto \ ((v,0), \varphi\oplus W) \ .$$

The *Wth shift* of an orthogonal spectrum X is the composite

$$\mathrm{sh}^W X \ = \ X \circ (- \oplus W) \ . \tag{2.14}$$

In other words, the value of $\mathrm{sh}^W X$ at an inner product space V is

$$(\mathrm{sh}^W X)(V) \ = \ X(V\oplus W) \ .$$

The orthogonal group $O(V)$ acts through the monomorphism $-\oplus W\colon O(V) \longrightarrow O(V\oplus W)$. The structure map $\sigma^{\mathrm{sh}^W X}_{U,V}$ of $\mathrm{sh}^W X$ is the structure map $\sigma^X_{U,V\oplus W}$ of X. As an example, the shift of a suspension spectrum is another suspension spectrum, $\mathrm{sh}^W(\Sigma^\infty K) \cong \Sigma^\infty(S^W \wedge K)$. In the special case $W = \mathbb{R}$, we simply write $\mathrm{sh}\, X$ for $\mathrm{sh}^{\mathbb{R}} X$.

Since composition of functors is associative, the shift construction commutes on the nose with all constructions on orthogonal spectra that are given by post-composition with a continuous based functor as in Example 2.7. This applies in particular to smashing with and taking mapping space from a based space A, i.e.,

$$(\mathrm{sh}^W X)\wedge A \ = \ \mathrm{sh}^W(X\wedge A) \quad \text{and} \quad \mathrm{map}_*(A, \mathrm{sh}^W X) \ = \ \mathrm{sh}^W(\mathrm{map}_*(A,X)) \ .$$

So we can – and will – omit the parentheses in expressions such as $\mathrm{sh}^W X \wedge A$.

The shift construction is also transitive in the following sense. The values of $\mathrm{sh}^V(\mathrm{sh}^W X)$ and $\mathrm{sh}^{V\oplus W} X$ at an inner product space U are given by

$$(\mathrm{sh}^V(\mathrm{sh}^W X))(U) \ = \ X((U\oplus V)\oplus W)$$

and

$$(\mathrm{sh}^{V\oplus W} X)(U) \ = \ X(U\oplus (V\oplus W)) \ .$$

We use the effect of X on the associativity isomorphism

$$(U\oplus V)\oplus W \ \cong \ U\oplus(V\oplus W) \ , \quad ((u,v),w) \ \longmapsto \ (u,(v,w))$$

to identify these two spaces; then we abuse notation and write

$$\mathrm{sh}^V(\mathrm{sh}^W X) \ = \ \mathrm{sh}^{V\oplus W} X \ .$$

The suspension and the shift of an orthogonal spectrum X are related by a natural morphism

$$\lambda^V_X \ : \ X\wedge S^V \ \longrightarrow \ \mathrm{sh}^V X \ .$$

In level U, this is defined as $\lambda_X^V(U) = \sigma_{U,V}^{\mathrm{op}}$, the opposite structure map (2.4), i.e., the composite

$$X(U) \wedge S^V \xrightarrow{\text{twist}} S^V \wedge X(U) \xrightarrow{\sigma_{V,U}} X(V \oplus U) \xrightarrow{X(\tau_{V,U})} X(U \oplus V) = (\mathrm{sh}^V X)(U) \ .$$

In the special case $V = \mathbb{R}$ we abbreviate $\lambda_X^{\mathbb{R}}$ to $\lambda_X \colon X \wedge S^1 \longrightarrow \mathrm{sh}\, X$.

The λ-maps are transitive in the sense that for another inner product space W, the morphism $\lambda_X^{V\oplus W}$ coincides with the two composites in the commutative diagram:

$$\begin{tikzcd}
X \wedge S^{V\oplus W} \arrow[r, "\cong"] & X \wedge S^V \wedge S^W \arrow[r, "\lambda^W_{X\wedge S^V}"] \arrow[d, "\lambda_X^V \wedge S^W"'] & \mathrm{sh}^W X \wedge S^V \arrow[d, "\lambda^V_{\mathrm{sh}^W X}"] & \\
& \mathrm{sh}^V X \wedge S^W \arrow[r, "\mathrm{sh}^V(\lambda_X^W)"'] & \mathrm{sh}^V(\mathrm{sh}^W X) \arrow[r, equal] & \mathrm{sh}^{V\oplus W} X
\end{tikzcd}$$

The existence of the natural morphism $\lambda_X \colon X \wedge S^1 \longrightarrow \mathrm{sh}\, X$ is a feature of orthogonal spectra that is *not* shared by sequential spectra. Indeed, while $X \wedge S^1$ and $\mathrm{sh}\, X$ make perfect sense for sequential spectra, and while these constructions extend to endofunctors

$$- \wedge S^1 \ , \ \mathrm{sh} \ : \ \mathcal{S}p^{\mathbb{N}} \ \longrightarrow \ \mathcal{S}p^{\mathbb{N}} \ ,$$

there is no natural morphism from $X \wedge S^1$ to $\mathrm{sh}\, X$ in the context of sequential spectra. A popular mistake is the misconception that for a sequential spectrum X, the collection of based continuous maps

$$(X \wedge S^1)_n \ = \ X_n \wedge S^1 \ \xrightarrow{\ \tau\ } \ S^1 \wedge X \ \xrightarrow{\ \sigma_n\ } \ X_{1+n} \ = \ (\mathrm{sh}\, X)_n$$

were to form a morphism of sequential spectra – they do not!

Our next aim is to show that the morphism λ_X is in fact a stable equivalence. We let k be any integer and we observe that

$$\begin{aligned}
\pi_{1+k}(\mathrm{sh}\, X) \ &= \ \mathrm{colim}_n \ \pi_{n+1+k}((\mathrm{sh}\, X)(\mathbb{R}^n)) \\
&= \ \mathrm{colim}_n \ \pi_{n+1+k}(X(\mathbb{R}^{n+1})) \ = \ \mathrm{colim}_n \ \pi_{n+k}(X(\mathbb{R}^n)) \ = \ \pi_k(X) \ .
\end{aligned}$$

Proposition 2.15. *Let X be an orthogonal spectrum.*

(i) *Let $A \in O(n)$ be an orthogonal matrix, and let $f \colon S^{n+k} \longrightarrow X_n$ be a continuous based map. Then*

$$[X(A) \circ f] \ = \ \det(A) \cdot [f]$$

in the group in $\pi_k(X)$.

(ii) *For every integer k, the map*

$$\pi_{k+1}(X \wedge S^1) \ \xrightarrow{\pi_{1+k}(\lambda_X)} \ \pi_{1+k}(\mathrm{sh}\, X) \ = \ \pi_k(X)$$

is inverse to the suspension isomorphism up to the factor $(-1)^k$.

(iii) *The morphism*

$$\lambda_X \ : \ X \wedge S^1 \ \longrightarrow \ \mathrm{sh}\, X \qquad \textit{and its adjoint} \qquad \tilde{\lambda}_X \ : \ X \ \longrightarrow \ \Omega\, \mathrm{sh}\, X \ ,$$

are stable equivalences of orthogonal spectra.

Proof. (i) We abbreviate the determinant to $d = \det(A) \in \{\pm 1\}$, and we write $\delta \colon S^1 \longrightarrow S^1$ for the onepoint compactification of multiplication by d on $\mathbb{R}$; so δ is a based continuous map of degree d. Then we contemplate the commutative diagram:

$$\begin{tikzcd}
S^{1+n+k} \arrow[r, "S^1\wedge(X(A)\circ f)"] \arrow[d, "\delta\wedge S^{n+k}"'] & S^1 \wedge X_n \arrow[r, "\sigma_n"] \arrow[d, "\delta\wedge X(A^{-1})"] & X_{1+n} \arrow[d, "X\begin{pmatrix} d & 0 \\ 0 & A^{-1}\end{pmatrix}"] \\
S^{1+n+k} \arrow[r, "S^1\wedge f"'] & S^1 \wedge X_n \arrow[r, "\sigma_n"'] & X_{1+n}
\end{tikzcd}$$

The matrix $\left(\begin{smallmatrix} d & 0 \\ 0 & A^{-1}\end{smallmatrix}\right)$ has determinant 1, so it lies in the same path component of $O(1+n)$ as the identity matrix. Since $O(1+n)$ acts continuously on X_{1+n}, a path in $O(1+n)$ between these two matrices induces a homotopy from the map $X\left(\begin{smallmatrix} d & 0 \\ 0 & A^{-1}\end{smallmatrix}\right)$ to the identity of X_{1+n}. So the composite around the diagram through the upper right corner is homotopic to the stabilization of $X(A)\circ f$, and it thus represents the class $[X(A)\circ f]$. The composite around the lower left corner represents $d\cdot[f]$, so this proves the claim.

(ii) The composite

$$\pi_k(X) \xrightarrow{-\wedge S^1} \pi_{k+1}(X\wedge S^1) \xrightarrow{\pi_{k+1}(\lambda_X)} \pi_{k+1}(\mathrm{sh}\, X) \ = \ \pi_k(X)$$

sends the class represented by a based continuous map $f\colon S^{n+k}\longrightarrow X_n$ to the class of the composite

$$S^{n+k+1} \xrightarrow{f\wedge S^1} X_n\wedge S^1 \xrightarrow{\tau} S^1\wedge X_n \xrightarrow{\sigma_n} X_{1+n} \xrightarrow{X(\chi_{1,n})} X_{n+1}\ .$$

This composite equals the composite

$$S^{n+k+1} \xrightarrow{\chi_{n+k,1}} S^{1+n+k} \xrightarrow{S^1\wedge f} S^1\wedge X_n \xrightarrow{\sigma_n} X_{1+n} \xrightarrow{X(\chi_{1,n})} X_{n+1}\ ,$$

which represents the class

$$(-1)^{n+k}\cdot[X(\chi_{1,n})\circ\sigma_n(S^1\wedge f)] \ =_{\text{(i)}} \ (-1)^{n+k}\cdot\det(\chi_{1,n})\circ[\sigma_n(S^1\wedge f)] \ = \ (-1)^k\cdot[f]\ .$$

(iii) The suspension homomorphism $-\wedge S^1\colon \pi_k(X)\longrightarrow\pi_{k+1}(X\wedge S^1)$ is an isomorphism by Proposition 1.13. So the morphism λ_X induces an isomorphism on all homotopy groups by part (ii), and it is thus a stable equivalence. The adjoint $\tilde\lambda_X$ is then a stable equivalence by Corollary 1.14. □

We can now deduce various equivalent characterizations for stable equivalences. Some parts of this have already been shown in the previous propositions, but they are repeated here for easier reference.

Proposition 2.16. *For a morphism $f\colon A\longrightarrow B$ of orthogonal spectra the following are equivalent:*

(i) *the morphism f is a stable equivalence;*
(ii) *the mapping cone Cf of f has trivial homotopy groups;*
(iii) *the suspension $f\wedge S^1\colon A\wedge S^1\longrightarrow B\wedge S^1$ is a stable equivalence;*
(iv) *the shift $\mathrm{sh}\, f\colon\ \mathrm{sh}\, A\longrightarrow \mathrm{sh}\, B$ is a stable equivalence;*
(v) *the homotopy fiber $F(f)$ of f has trivial homotopy groups;*
(vi) *the loop $\Omega f\colon\Omega A\longrightarrow\Omega B$ is a stable equivalence.*

Proof. Conditions (i) and (ii) are equivalent by the long exact sequence of homotopy groups for a mapping cone, see Proposition 1.18. Conditions (i), (iii) and (vi) are equivalent because the suspension and loop constructions shift the homotopy groups, see Proposition 1.13. The natural morphism $\lambda_X\colon X\wedge S^1\longrightarrow \mathrm{sh}\, X$ is a stable equivalence by Proposition 2.15; so conditions (iii) and (iv) are equivalent. Conditions (i) and (v) are equivalent by the long exact sequence of homotopy groups for a homotopy fiber, see Proposition 1.22. □

3. Basic examples and constructions

Before developing any more theory, we give some examples of orthogonal spectra which represent prominent stable homotopy types. We discuss the sphere spectrum (Example 3.1), suspension spectra (Example 3.2), Eilenberg–Mac Lane spectra (Example 3.3), and Thom spectra (Example 3.6). It is a nice feature of orthogonal spectra that one can explicitly write down these examples in closed form with all the required symmetries.

Example 3.1 (Sphere spectrum)**.** The orthogonal *sphere spectrum* $\mathbb{S}$ is given by $\mathbb{S}(V)=S^V$ with functoriality by the map

$$\mathbf{O}(V,W)\wedge S^V \ \longrightarrow\ S^W\ , \quad (w,\varphi)\wedge v\ \longmapsto\ w+\varphi(v)\ .$$

Unraveling this shows that $O(V)$-acts on S^V as the onepoint compactification of the tautological action on V; and the structure maps

$$\sigma_{V,W}\ :\ S^V\wedge S^W\ \longrightarrow\ S^{V\oplus W}$$

are the canonical homeomorphisms. The sphere spectrum is isomorphic to the orthogonal spectrum $\mathbf{O}(0,-)$ represented by the 0-dimensional inner product space.

Example 3.2 (Suspension spectra)**.** Every pointed space K gives rise to an orthogonal *suspension spectrum* $\Sigma^\infty K$ with values

$$(\Sigma^\infty K)(V) \;=\; S^V \wedge K \ .$$

The functoriality is by the map

$$\mathbf{O}(V,W) \wedge S^V \wedge K \;\longrightarrow\; S^W \wedge K \ , \quad (w,\varphi) \wedge v \wedge k \;\longmapsto\; w + \varphi(v) \wedge k \ .$$

Unraveling this shows that $O(V)$-acts on $S^V \wedge K$ as the onepoint compactification of the tautological action on S^V and trivially on K. The structure map $\sigma_{V,W} : S^V \wedge S^W \wedge K \longrightarrow S^{V \oplus W} \wedge K$ is the smash product of the canonical homeomorphism with K. For example, the sphere spectrum $\mathbb{S}$ is isomorphic to the suspension spectrum $\Sigma^\infty S^0$.

Example 3.3 (Eilenberg–Mac Lane spectra, orthogonal version)**.** For an abelian group A, the *Eilenberg–Mac Lane spectrum* HA is defined at an inner product space V by

$$(HA)(V) \;=\; A[S^V] \ ,$$

the reduced A-linearization of the V-sphere. Let us review the linearization construction in some detail before defining the rest of the structure of the Eilenberg–Mac Lane spectrum.

For a general based space K, the underlying set of the A-linearization $A[K]$ is tensor product of A with the reduced free abelian group generated by the points of K. In other words, points of $A[K]$ are finite sums of points of K with coefficients in A, modulo the relation that all A-multiples of the basepoint are zero. The set $A[K]$ is topologized as a quotient space of the disjoint union of the spaces $A^n \times K^n$ (with discrete topology on A^n), via the surjection

$$\coprod_{n \geq 0} A^n \times K^n \;\longrightarrow\; A[K] \ , \quad (a_1, \dots, a_n, k_1, \dots, k_n) \;\mapsto\; \sum_{i=1}^n a_i \cdot k_i \ .$$

There is a natural map $\tilde{H}_n(K,A) \longrightarrow \pi_n(A[K],0)$ from the reduced singular homology groups of K with coefficients in A to the homotopy groups of the linearization: let $x = \sum_i a_i \cdot f_i$ be a singular chain of K with coefficients a_i in A, i.e., every $f_i : \nabla^n \longrightarrow K$ is a continuous map from the topological n-simplex. We use the abelian group structure of $A[K]$ and add the maps f_j pointwise and multiply by the coefficients, to get a single map $\underline{x} : \nabla^n \longrightarrow A[K]$, i.e., for $z \in \nabla^n$ we set

$$\underline{x}(z) \;=\; \sum_i a_i \cdot f_i(z) \ .$$

If the original chain x is a cycle in the singular chain complex, then the map $\underline{x}$ sends the boundary of the simplex to the neutral element 0 of $A[K]$. So $\underline{x}$ factors over a continuous based map $\nabla^n/\partial\nabla^n \longrightarrow A[K]$. After composing with a homeomorphism between the n-sphere and $\nabla^n/\partial\nabla^n$ this maps represents an element in the homotopy group $\pi_n(A[K],0)$. If K has the based homotopy type of a CW-complex, then the map $\tilde{H}_n(K,A) \longrightarrow \pi_n(A[K],0)$ is an isomorphism [ref]. In the special case $K = S^n$ this shows that the $A[S^n]$ has only one non-trivial homotopy group in dimension n, where it is isomorphic to A. In other words, $(HA)_n = A[S^n]$ is an Eilenberg–Mac Lane space of type (A,n).

Now we return to the definition of the Eilenberg–Mac Lane spectrum HA. The functoriality of HA is by the map

$$\mathbf{O}(V,W) \wedge A[S^V] \;\longrightarrow\; A[S^W] \ , \quad (w,\varphi) \wedge (\sum_i a_i v_i) \;\longmapsto\; \sum_i a_i \cdot (w + \varphi(v_i)) \ .$$

Unraveling this shows that $O(V)$-acts on $A[S^V]$ through its action on S^V and the functoriality of $A[-]$. And the structure map $\sigma_{V,W} : S^V \wedge (HA)(W) \longrightarrow (HA)(V \oplus W)$ is given by

$$S^V \wedge A[S^W] \longrightarrow A[S^{V \oplus W}] \ , \quad v \wedge \big(\sum_i a_i \cdot w_i\big) \;\longmapsto\; \sum_i \; a_i \cdot (v \wedge w_i) \ .$$

A non-trivial fact is that the adjoint structure maps

$$\tilde{\sigma}_{V,W} \ : \ A[S^W] \ \longrightarrow \ \mathrm{map}_*(S^V, A[S^{V\oplus W}])$$

are weak equivalences for all inner product spaces V and W. In particular, the adjoint structure maps

$$\tilde{\sigma}_n \ : \ (HA)_n = A[S^n] \ \longrightarrow \ \mathrm{map}_*(S^1, A[S^{1+n}]) = \Omega((HA)_{1+n})$$

are weak equivalences. Hence all maps in the colimit system defining $\pi_0(HA)$ are bijective, and so the canonical map

$$A \ = \ \pi_0(A[S^0]) \ \longrightarrow \ \mathrm{colim}_n \ \pi_n(A[S^n]) \ = \ \pi_0(HA) \tag{3.4}$$

is bijective. This map is also additive, and hence an isomorphism of abelian groups. More is true: the homotopy groups of the orthogonal spectrum HA are concentrated in dimension zero, i.e., the group $\pi_k(HA)$ is trivial for $k \neq 0$.

We shall see in Example 4.14 below that the Eilenberg–Mac Lane functor H can be extended to take rings to orthogonal ring spectra and modules to module spectra.

Eilenberg–Mac Lane spectra enjoy a special property: the n-th space $(HA)_n = (HA)(\mathbb{R}^n) = A[S^n]$ and the loop space of the next space $(HA)_{n+1}$ are both Eilenberg–Mac Lane spaces of type (A, n), and in fact the map $\tilde{\sigma}_n : (HA)_n \longrightarrow \Omega(HA)_{n+1}$ adjoint to the structure map is a weak equivalence for all $n \geq 0$. Spectra with this property play an important role in stable homotopy theory, and they deserve a special name:

Definition 3.5. An orthogonal spectrum is an Ω-*spectrum* if for all $n \geq 0$ the map $\tilde{\sigma}_n : X_n \longrightarrow \Omega X_{1+n}$ which is adjoint to the structure map $\sigma_n \colon S^1 \wedge X_n \longrightarrow X_{1+n}$ is a weak homotopy equivalence.

Example 3.6 (Thom spectra)**.** We introduce an orthogonal spectrum MO. For an inner product space V we let $\mathbf{L}(V, V^\infty)$ be the space of linear isometric embeddings from V into $V^\infty = \bigoplus_{n\geq 1} V$, with the weak topology as the union of the Stiefel manifold $\mathbf{L}(V, V^n)$. With this topology, $\mathbf{L}(V, V^\infty)$ is a closed subspace of the space $\mathrm{map}(V, V^\infty)$ of all continuous maps with the function space topology internal to the category $\mathbf{T}$, see for example [41, Proposition A.5 (ii)]. If V is non-zero, then V^∞ is infinite dimensional and the space $\mathbf{L}(V, V^\infty)$ is contractible, see for example [40, Proposition 1.2.21]. The orthogonal group $O(V)$ acts freely and continuously from the right on $\mathbf{L}(V, V^\infty)$ by precomposition, and the orbit space of this action identifies with the Grassmannian of $\dim(V)$-planes in V^∞, via the homeomorphism

$$\mathbf{L}(V, V^\infty)/O(V) \ \cong \ Gr_{\dim(V)}(V^\infty) \ , \quad \varphi \cdot O(V) \ \longmapsto \ \varphi(V) \ .$$

The tautological vector bundle over the Grassmannian corresponds to the vector bundle

$$\mathbf{L}(V, V^\infty) \times_{O(V)} V \ \longrightarrow \ \mathbf{L}(V, V^\infty)/O(V) \ , \quad [\varphi, v] \ \longmapsto \ \varphi \cdot O(V) \ .$$

We can thus form the Thom space of this vector bundle

$$MO(V) \ = \ \mathbf{L}(V, V^\infty)_+ \wedge_{O(V)} S^V \ .$$

The group $O(V)$ also acts continuously from the left on $L(V, V^\infty)$ through its diagonal action on V^∞; we give $MO(V)$ the induced action, i.e.,

$$A \cdot [\varphi, v] \ = \ [(A^\infty) \circ \varphi, v] \ .$$

The same construction gives an orthogonal spectrum MSO by dividing out only the action of the special orthogonal group $SO(V)$ of V. The Thom-Pontryagin construction provides homomorphisms $\mathbf{\Omega}^O_k \longrightarrow \pi_k(MO)$ from the group of bordism classes of k-dimensional smooth closed manifolds to the k-th homotopy group of the spectrum MO, and similarly for the other families of classical Lie groups. A celebrated theorem of Thom's [47] says that the Thom-Pontryagin map is an isomorphism. We intend to discuss these and other examples of Thom spectra in more detail in a later chapter.

Construction 3.7 (Free orthogonal spectra)**.** For every inner product space V, the evaluation functor

$$\mathrm{ev}_V \ : \ \mathcal{S}p \ \longrightarrow \ \mathbf{T}_*$$

at V has left adjoint

$$F_V \ : \ \mathbf{T}_* \ \longrightarrow \ \mathcal{S}p \ .$$

The value of the left adjoint at a based space K is

$$F_V K \ = \ \mathbf{O}(V,-) \wedge K \ ,$$

the objectwise smash product of the represented functor $\mathbf{O}(V,-) : \mathbf{O} \longrightarrow \mathbf{T}_*$ with K. The adjunction property is an instance of the enriched Yoneda lemma.

The free orthogonal spectrum $F_V K$ comes naturally with a right action of the orthogonal group $O(V)$ by precomposition; we will refer to this as the *'right action on the free coordinates'*. This right $O(V)$-action action is continuous and by automorphisms of orthogonal spectra. Moreover, the adjunction bijection

$$\mathbf{T}_*(K, X(V)) \ \cong \ \mathcal{S}p(F_V K, X) \ , \quad f \longmapsto \hat{f}$$

is natural for morphism of orthogonal spectra in X, so it is equivariant for the two natural left actions of $O(V)$.

We show in the next proposition that the free orthogonal spectrum $F_V K$ generated by a based space K in level V is stably equivalent to the V-fold loop of the suspension spectrum of K. If $\dim(V) = m$, this implies an isomorphism of homotopy groups

$$\pi_k(F_V K) \ \xrightarrow[\cong]{\varphi^V_*} \ \pi_k(\Omega^V(\Sigma^\infty K)) \cong \pi_{k+m}(\Sigma^\infty K)$$

to the stable homotopy groups K, shifted by the dimension of V.

Proposition 3.8. *Let V be an inner product space and K a based space. The morphism of orthogonal spectra*

$$\varphi^V \ : \ F_V K \ \longrightarrow \ \Omega^V(\Sigma^\infty K)$$

adjoint to the based continuous map

$$K \ \xrightarrow{\eta} \Omega^V(S^V \wedge K) \ = \ \Omega^V(\Sigma^\infty K)(V)$$

is a stable equivalence.

Construction 3.9 (Semifree orthogonal spectra)**.** There are somewhat 'less free' orthogonal spectra which start from a pointed $O(V)$-space L; we want to install L in level V, and then fill in the remaining data of an orthogonal spectrum as freely as possible. In other words, we claim that the forgetful *evaluation functor*

$$\mathrm{ev}_V \ : \ \mathcal{S}p \ \longrightarrow \ O(V)\text{-}\mathbf{T}_* \ , \quad X \longmapsto X(V)$$

has a left adjoint which we denote G_V; we refer to $G_V L$ as the *semifree orthogonal spectrum* generated by L in level V. (The evaluation functor ev_V also has a right adjoint, but we won't use it.) The spectrum $G_V L$ is explicitly given by

$$G_V L \ = \ \mathbf{O}(V,-) \wedge_{O(V)} L \ .$$

In other words: the value of $G_V L$ at W is $\mathbf{O}(V,W) \wedge_{O(V)} L$, the orbit space of $\mathbf{O}(V,W) \wedge L$ by the equivalence relation that equalizes the two $O(V)$-action on $\mathbf{O}(V,W)$ and L. The 'semifreeness' property of $G_V L$ is another instance of the enriched Yoneda lemma.

Example 3.10 (Mapping spaces)**.** There is a whole space of morphisms between two orthogonal spectra. For orthogonal spectra X and Y, every morphism $f : X \longrightarrow Y$ consists of a family of based maps $\{f(V) : X(V) \longrightarrow Y(V)\}_V$ indexed by inner product spaces V which satisfy some conditions. Every inner product space is isomorphic in the category $\mathbf{O}$ to $\mathbb{R}^n$ with the standard inner product, for some n. So a morphism $f : X \longrightarrow Y$ is already completely determined by the family of maps $\{f_n : X_n \longrightarrow Y_n\}_{n \geq 0}$. Said differently: the map

$$\mathcal{S}p(X,Y) \ \longrightarrow \ \prod_{n\geq 0} \mathbf{T}_*(X_n, Y_n)$$

sending f to the collection so its values at all $\mathbb{R}^n$ is injective. Moreover, the image of this evaluation map is a closed subset in the product of the mapping spaces $\prod_{n\geq 0} \mathrm{map}_*(X_n, Y_n)$; we topologize the set of morphisms $\mathcal{S}p(X,Y)$ of orthogonal spectra by giving it the subspace topology of the (compactly generated) product topology. We denote this mapping space by $\mathrm{map}(X,Y)$.

For this mapping space topology, the adjunction bijection

$$\mathcal{S}p(F_V, Y) \;\cong\; Y(V) \;, \quad f \;\longmapsto\; f(V)(0, \mathrm{Id}_V)$$

is in fact a homeomorphism

$$\mathrm{map}(F_V, Y) \;\cong\; Y(V) \;.$$

Furthermore, for a pointed space K and orthogonal spectra X and Y we have adjunction bijections

$$\mathrm{map}_*(K, \mathrm{map}(X,Y)) \;\cong\; \mathrm{map}(X\wedge K, Y) \;\cong\; \mathrm{map}(X, \mathrm{map}_*(K,Y)) \;,$$

where the first mapping space is taken in the category $\mathbf{T}_*$ of compactly generated based spaces.

We have associative and unital composition maps

$$\mathrm{map}(Y,Z)\wedge \mathrm{map}(X,Y) \;\longrightarrow\; \mathrm{map}(X,Z) \;.$$

Indeed, for orthogonal spectra of topological spaces this is just the observation that composition of morphisms is continuous for the mapping space topology.

4. Ring and module spectra

Definition 4.1. An *orthogonal ring spectrum* is an orthogonal spectrum R equipped continuous $(O(V)\times O(W))$-equivariant *multiplication maps*

$$\mu_{V,W} \;:\; R(V)\wedge R(W) \;\longrightarrow\; R(V\oplus W)$$

and a *unit* $\iota\in R(0)$ that satisfy the following two conditions:

(Associativity) For all inner product space U, V and W, the square

$$\begin{tikzcd} R(U)\wedge R(V)\wedge R(W) \arrow[r, "R(U)\wedge\mu_{V,W}"] \arrow[d, "\mu_{U,V}\wedge R(W)"'] & R(U)\wedge R(V\oplus W) \arrow[d, "\mu_{U,V\oplus W}"] \\ R(U\oplus V)\wedge R(W) \arrow[r, "\mu_{U\oplus V,W}"'] & R(U\oplus V\oplus W) \end{tikzcd}$$

commutes.

(Structure maps) For all inner product spaces V and W, the composite

$$S^V\wedge R(W) \xrightarrow{-\wedge\iota\wedge -} S^V\wedge R(0)\wedge R(W) \xrightarrow{\sigma_{V,0}\wedge R(W)} R(V\oplus 0)\wedge R(W)\cong R(V)\wedge R(W) \xrightarrow{\mu_{V,W}} R(V\oplus W)$$

is the structure map $\sigma_{V,W}$, and the composite

$$R(V)\wedge S^W \xrightarrow{-\wedge\iota\wedge -} R(V)\wedge R(0)\wedge S^W \xrightarrow{R(V)\wedge\sigma^{\mathrm{op}}_{0,W}} R(V)\wedge R(0\oplus W)\cong R(V)\wedge R(W) \xrightarrow{\mu_{V,W}} R(V\oplus W)$$

is the opposite structure map $\sigma^{\mathrm{op}}_{V,W}$.

An orthogonal ring spectrum R is *commutative* if the square

$$\begin{tikzcd} R(V)\wedge R(W) \arrow[r, "\mathrm{twist}"] \arrow[d, "\mu_{V\oplus W}"'] & R(W)\wedge R(V) \arrow[d, "\mu_{W,V}"] \\ R(V\oplus W) \arrow[r, "R(\tau_{V,W})"'] & R(W\oplus V) \end{tikzcd}$$

commutes for all inner product spaces V and W.

A *morphism* orthogonal ring spectra is a morphism $f : R \longrightarrow S$ of orthogonal spectra such that $f(0)(\iota^R)=\iota^S$, and $f(V\oplus W)\circ\mu^R_{V,W}=\mu^S_{V,W}\circ(f(V)\wedge f(W))$ for all inner product spaces V and W.

Remark 4.2.

- Special cases of the structure map conditions are that the composite
$$R(W) \xrightarrow{\iota\wedge -} R(0)\wedge R(W) \xrightarrow{\mu_{V,W}} R(0\oplus W)$$
coincides with the effect of R on the isometry $(0,-) : W \cong 0\oplus W$, and the composite
$$R(V) \xrightarrow{-\wedge\iota} R(V)\wedge R(0) \xrightarrow{\mu_{V,0}} R(V\oplus 0)$$
coincides with the effect of R on the isometry $(-,0) : V \cong V\oplus 0$, respectively.
- If the multiplication maps $\mu_{V,W} : R(V)\wedge R(W) \longrightarrow R(V\oplus W)$ are commutative, then the two structure map conditions are equivalent.
- By instances of the associativity and structure map axioms, the two composites
$$S^V \xrightarrow{-\wedge\iota} S^V\wedge R(0) \xrightarrow{\sigma_{V,0}} R(V\oplus 0) \xrightarrow[\cong]{R((v,0)\mapsto v)} R(V)$$
and
$$S^V \xrightarrow{\iota\wedge -} R(0)\wedge S^V \xrightarrow{\sigma^{\mathrm{op}}_{0,V}} R(0\oplus V) \xrightarrow[\cong]{R((0,v)\mapsto v)} R(V)$$
are equal. We will denote the common composite by $\iota_V : S^V \longrightarrow R(V)$ and also call it a *unit map* of the ring spectrum R.
- Since the (generalized) unit maps and the multiplication maps determine the structure maps of the underlying orthogonal spectrum of R, one could equivalently define orthogonal ring spectra as a collection of based $O(V)$-spaces $R(V)$ equipped with unit maps $\iota_V : S^V \longrightarrow R(V)$ and multiplication maps $\mu_{V,W} : R(V)\wedge R(W) \longrightarrow R(V\oplus W)$ that satisfy certain relations. The possibility of describing orthogonal ring spectra in this way is often convenient when presenting explicit examples.

Remark 4.3. We will show later that orthogonal ring spectra can also be interpreted as the monoid objects with respect to the symmetric monoidal *smash product* on the category of orthogonal spectra. This is very much analogous to the situation in classical algebra: typically, one first encounters rings as abelian groups equipped with another binary operation that is biadditive, associative, distributive and unital. Any typically, one realizes later that the category of rings is equivalent to the category of monoids with respect to the tensor product of abelian groups. Similar comments apply to the modules over ring spectra that we shall now introduce.

Definition 4.4. A *left module* over an orthogonal ring spectrum R is an orthogonal spectrum M equipped with continuous $(O(V)\times O(W))$-equivariant *action maps*
$$\alpha_{V,W} \;:\; R(V)\wedge M(W) \longrightarrow M(V\oplus W)$$
that satisfy the following two conditions:

(Associativity) For all inner product space U, V and W, the square
$$\begin{tikzcd}
R(U)\wedge R(V)\wedge M(W) \arrow[r, "R(U)\wedge\alpha_{V,W}"] \arrow[d, "\mu_{U,V}\wedge M(W)"'] & R(U)\wedge M(V\oplus W) \arrow[d, "\alpha_{U,V\oplus W}"] \\
R(U\oplus V)\wedge M(W) \arrow[r, "\alpha_{U\oplus V,W}"'] & M(U\oplus V\oplus W)
\end{tikzcd}$$
commutes.

(Unitality) For all inner product spaces V and W, the composite
$$S^V\wedge M(W) \xrightarrow{\iota_V\wedge -} R(V)\wedge M(W) \xrightarrow{\alpha_{V,W}} M(V\oplus W)$$
is the structure map $\sigma_{V,W}$.

A *morphism* of left R-modules is a morphism $f : M \longrightarrow N$ of orthogonal spectra such that and $f(V\oplus W)\circ\alpha^M_{V,W} = \alpha^N_{V,W}\circ(R(V)\wedge f(W))$ for all inner product spaces V and W. We denote the category of left R-modules by R mod .

The forgetful functors which associates to an orthogonal ring spectrum or module spectrum its underlying orthogonal spectrum have left adjoints. We intend to construct the left adjoints in after introducing the smash product of orthogonal spectra. The left adjoints associate to an orthogonal spectrum X the 'free R-module' $R \wedge X$ respectively the 'free orthogonal ring spectrum' TX generated by X, which we will refer to as the *tensor algebra*.

Construction 4.5. We let R be an orthogonal ring spectrum. The forgetful

$$R\text{-mod} \longrightarrow \mathcal{S}p$$

from left R-modules to orthogonal spectra preserves and reflects limits and colimits. An abstract justification for this is the fact – hopefully to be proved later – that the forgetful functor has both a left adjoint and a right adjoint.

Similarly, for every based space K, the adjoint functors

$$\mathcal{S}p \underset{\mathrm{map}_*(K,-)}{\overset{-\wedge K}{\rightleftarrows}} \mathcal{S}p$$

have a preferred lift to an adjoint functor pair on the category of left R-modules. For example, if M is an R-module, then the action maps for $M \wedge K$ are simply the maps

$$\begin{aligned} \alpha^{M\wedge K}_{V,W} \ = \ \alpha^M_{V,W} \wedge K \ : \ R(V) \wedge (M \wedge K)(W) \ &= R(V) \wedge M(W) \wedge K \\ &\longrightarrow M(V \oplus W) \wedge K \ = \ (M \wedge K)(V \oplus W) \ . \end{aligned}$$

Construction 4.6 (Shifting modules)**.** Shifting by an inner product space W lifts to an endofunctor of the category of R-modules. Indeed, if M is an R-module, we define an R-action on the shifted orthogonal spectrum $\mathrm{sh}^W M$ as the collection of maps

$$\begin{aligned} \alpha^{\mathrm{sh}^W M}_{U,V} = \alpha^M_{U,V\oplus W} \ : \ R(U) \wedge (\mathrm{sh}^W M)(V) \ &= R(U) \wedge M(V \oplus W) \\ &\longrightarrow M(U \oplus V \oplus W) \ = \ (\mathrm{sh}\, M)(U \oplus V) \ . \end{aligned}$$

Construction 4.7 (Multiplication on homotopy groups)**.** We let M be a left module over an orthogonal ring spectrum R: We introduce a pairing

$$\cdot \ : \ \pi_k(R) \times \pi_l(M) \ \longrightarrow \ \pi_{k+l}(M) \tag{4.8}$$

that makes the homotopy groups of M into a graded module over the homotopy groups of R. We let $f : S^{m+k} \longrightarrow R_m$ and $g : S^{n+l} \longrightarrow M_n$ represent classes in $\pi_k(R)$ and $\pi_l(M)$, respectively. We denote by $f \cdot g$ the composite

$$S^{m+k+n+l} \xrightarrow{f\wedge g} R_m \wedge M_n \xrightarrow{\alpha_{m,n}} M_{m+n}$$

and then define

$$[f] \cdot [g] \ = \ (-1)^{kn} \cdot [f \cdot g]$$

in the group $\pi_{k+l}(M)$.

The definition is clearly a generalization of the action of the stable stems on the homotopy groups of an orthogonal spectrum as defined in (1.7), recalling that $\pi_k^{\mathrm{s}} = \pi_k(\mathbb{S})$, and orthogonal spectra are canonically left modules over the sphere ring spectrum $\mathbb{S}$.

The multiplication maps $\mu_{V,W} : R(V) \wedge R(W) \longrightarrow R(V \oplus W)$ make R into a left module over itself; so for this R-module, the pairing specializes to an internal pairing

$$\cdot \ : \ \pi_k(R) \times \pi_l(R) \ \longrightarrow \ \pi_{k+l}(R) \ . \tag{4.9}$$

Proposition 4.10. *Let R be an orthogonal ring spectrum, and let M be a left R-module.*

(i) *The pairing*

$$\cdot \ : \ \pi_k(R) \times \pi_l(M) \ \longrightarrow \ \pi_{k+l}(M)$$

is well-defined and biadditive.

(ii) *Let* $1 \in \pi_0(R)$ *denote the homotopy class of the unit map* $\iota : S^0 \longrightarrow R_0$. *Then for every class* $y \in \pi_l(M)$ *the relation* $1 \cdot y = y$ *holds,*

(iii) *The following square commutes for all integers* j, k *and* l*:*

$$\begin{tikzcd}
\pi_j(R) \times \pi_k(R) \times \pi_l(M) \arrow[r, "\pi_l(R)\times\cdot"] \arrow[d, "\cdot\times\pi_m(M)"'] & \pi_j(R) \times \pi_{k+l}(M) \arrow[d, "\cdot"] \\
\pi_{j+k}(R) \times \pi_l(M) \arrow[r, "\cdot"'] & \pi_{j+k+l}(M)
\end{tikzcd}$$

(iv) *The pairings* (4.9) *make the homotopy groups of* R *into a graded ring with identity element* $1 \in \pi_0(R)$. *If* R *is commutative, then this product on* $\pi_*(R)$ *is graded-commutative.*

(v) *The pairings* (4.8) *make the homotopy groups of* M *into left module over the graded ring* $\pi_*(R)$.

(vi) *For every morphism* $\psi : M \longrightarrow N$ *of left* R*-modules, the homomorphism of graded abelian groups* $\psi_* : \pi_*(M) \longrightarrow \pi_*(N)$ *is* $\pi_*(R)$*-linear.*

(vii) *The collections of suspension and loop isomorphisms*

$$- \wedge S^1 \ : \ \pi_k(M) \ \longrightarrow \ \pi_{k+1}(M \wedge S^1) \qquad \text{and} \qquad \alpha \ : \ \pi_k(\Omega M) \ \longrightarrow \ \pi_{k+1}(M)$$

are $\pi_*(R)$*-linear.*

(viii) *Let* $f : M \longrightarrow N$ *be a homomorphism of left* R*-modules and let* Cf *and* Ff *be the mapping cone and homotopy fiber, respectively, of* f*, endowed with the natural* R*-action. Then the connecting homomorphisms*

$$\delta \ : \ \pi_{k+1}(Cf) \ \longrightarrow \ \pi_k(M) \qquad \text{and} \qquad \delta \ : \ \pi_{k+1}(N) \ \longrightarrow \ \pi_k(Ff)$$

defined in (1.17) *and* (1.21)*, respectively, are* $\pi_*(R)$*-linear. Hence the long exact sequences of homotopy groups of Propositions 1.18 and 1.22 are* $\pi_*(R)$*-linear.*

Proof. We check that the multiplication is well-defined. The following square commutes by naturality and the associativity and unitality properties of the multiplication and action maps:

$$\begin{tikzcd}
S^1 \wedge R_m \wedge M_n \arrow[r, "S^1\wedge\alpha_{m,n}"] \arrow[d, "\iota_1\wedge R_m\wedge M_n"] \arrow[dd, bend right, "\sigma_m\wedge M_n"'] & S^1 \wedge M_{m+n} \arrow[d, "\iota_1\wedge M_n"'] \arrow[dd, bend left, "\sigma_{m+n}"] \\
R_1 \wedge R_m \wedge M_n \arrow[r, "R_1\wedge\alpha_{m,n}"] \arrow[d, "\mu_{1,m}\wedge M_n"] & R_1 \wedge R_{m+n} \arrow[d, "\alpha_{1,m+n}"'] \\
R_{1+m} \wedge M_n \arrow[r, "\alpha_{1+m,n}"'] & M_{1+m+n}
\end{tikzcd}$$

So if we replace $f : S^{m+k} \longrightarrow R_m$ by its suspension $\sigma_m \circ (S^1 \wedge f) : S^{1+m+k} \longrightarrow R_{1+m}$, then

$$\begin{aligned}
(\sigma_m \cdot (S^1 \wedge f)) \cdot g \ &= \ \alpha_{1+m,n} \circ ((\sigma_m \cdot (S^1 \wedge f)) \wedge g) \\
&= \ \alpha_{1+m,n} \circ (\sigma_m \wedge M_m) \circ (S^1 \wedge f \wedge g) \\
&= \ \sigma_{m+n} \circ (S^1 \wedge \alpha_{m,n}(f \wedge g)) \ = \ \sigma_{m+n} \circ (S^1 \wedge (f \cdot g)) \ .
\end{aligned}$$

Since the sign in the formula (1.7) does not change, the resulting stable class is independent of the representative f of the stable class in $\pi_k(R)$.

Independence of the representative for $\pi_l(M)$ is slightly more subtle. The following diagram commutes by the associativity and structure map conditions on the multiplication and action maps, and the equivariance of the action map $\alpha_{1+m,n}$:

$$\begin{tikzcd}
S^1\wedge R_m\wedge M_n \arrow[rr, "\sigma_m\wedge M_n"] \arrow[dd, "\mathrm{twist}\wedge M_n"'] & & R_{1+m}\wedge M_n \arrow[d, "R(\chi_{1,m})\wedge M_n"'] \arrow[dr, "\alpha_{1+m,n}"] & \\
 & & R_{m+1}\wedge M_n \arrow[dr, "\alpha_{m+1,n}"'] & M_{1+m+n} \arrow[d, "M(\chi_{1,m}\oplus\mathbb{R}^n)"] \\
R_m\wedge S^1\wedge M_n \arrow[urr, bend left, "\sigma_m^{\mathrm{op}}\wedge M_n"] \arrow[r, "R_m\wedge\iota_1\wedge M_n"] \arrow[drr, bend right, "R_m\wedge\sigma_n"'] & R_m\wedge R_1\wedge M_n \arrow[ur, "\mu_{m,1}\wedge M_n"] \arrow[dr, "R_m\wedge\alpha_{1,n}"'] & & M_{m+1+n} \\
 & & R_m\wedge M_{1+n} \arrow[ur, "\alpha_{m,1+n}"'] &
\end{tikzcd}$$

If we replace $g : S^{n+l} \longrightarrow M_n$ by the representative $\sigma_n \circ (S^1 \wedge g) : S^{1+n+l} \longrightarrow M_{1+n}$, we now obtain

$$\begin{aligned}
f \cdot (\sigma_n \circ (S^1 \wedge g)) &= \alpha_{m,1+n} \circ (f \wedge (\sigma_n \circ (S^1 \wedge g))) \\
&= \alpha_{m,1+n} \circ (R_m \wedge \sigma_n) \circ (f \wedge S^1 \wedge g) \\
&= M(\chi_{1,m} \oplus \mathbb{R}^n) \circ \alpha_{1+m,n} \circ (\sigma_m \wedge M_n) \circ (\mathrm{twist}_{R_m,S^1}) \circ (f \wedge S^1 \wedge g) \\
&= M(\chi_{1,m} \oplus \mathbb{R}^n) \circ \alpha_{1+m,n} \circ (\sigma_m \wedge M_n) \circ (S^1 \wedge f \wedge g) \circ (\chi_{m+k,1} \wedge S^{n+l}) \\
&= M(\chi_{1,m} \oplus \mathbb{R}^n) \circ (\sigma_m(S^1 \wedge f) \cdot g) \circ (\chi_{m+k,1} \wedge S^{n+l})
\end{aligned}$$

By Proposition 2.15 (i), the map $M(\chi_{1,m} \oplus \mathbb{R}^n)$ induces multiplication by $(-1)^m$ on homotopy groups. This cancels part of the sign $(-1)^{m+k}$ that is the effect of precomposition with the shuffle permutation $\chi_{m+k,1} \wedge S^{n+l}$ on the left. So in the colimit $\pi_{k+l}(X)$ we have

$$[f \cdot (\sigma_n \circ (S^1 \wedge g))] \;=\; (-1)^k \cdot [\sigma_m(S^1 \wedge f) \cdot g)] \;=\; (-1)^k \cdot [f \cdot g] \;.$$

Since the dimension of $\sigma_n(S^1 \wedge g)$ is one more than the dimension of g, the extra factor $(-1)^k$ makes sure that product $[f] \cdot [g]$ as defined in (1.7) is independent of the representative of the stable class $[g]$.

The proof that the product is biadditive is the same as for action of the stable stems on the homotopy groups of any sequential spectrum in Example 1.6, based on the stability of pinch maps under smashing with a sphere.

Property (ii) is straightforward from the unitality property of the action of R on M: this requirements says in particular that for every continuous based map $g : S^{n+l} \longrightarrow M_n$; the composite

$$S^{n+l} \;\cong\; S^0 \wedge S^{n+l} \xrightarrow{\ \iota\wedge g\ } R_0 \wedge M_n \xrightarrow{\ \alpha_{0,n}\ } M_n$$

agrees with g. So $1 \cdot [g] = [\alpha_{0,n} \circ (\iota \wedge g)] = [g]$.

The associativity property is also fairly straightforward, and analogous to associativity of action of the stable stems in Example 1.6. Indeed, if $e : S^{p+j} \longrightarrow R_p$, $f : S^{m+k} \longrightarrow R_m$ and $g : S^{n+l} \longrightarrow M_n$ represent classes in $\pi_j(R)$, $\pi_k(R)$ and $\pi_l(M)$, respectively, then the diagram

$$\begin{tikzcd}
S^{p+j+m+k+n+l} \arrow[dr, "e\wedge f\wedge g"] \arrow[drrr, bend left, "e\wedge(f\cdot g)"] \arrow[ddrrrr, bend left=40, "e\cdot(f\cdot g)"] \arrow[ddrr, bend right, "(e\cdot f)\wedge g"'] \arrow[ddrrrr, bend right=60, "(e\cdot f)\cdot g"'] & & & & \\
 & R_p\wedge R_m\wedge M_n \arrow[rr, "R_p\wedge\alpha_{m,n}"'] \arrow[dr, "\mu_{p,m}\wedge M_n"] & & R_p\wedge M_{m+n} \arrow[dr, "\alpha_{p,m+n}"] & \\
 & & R_{p+m}\wedge M_n \arrow[rr, "\alpha_{p+m,n}"'] & & M_{p+m+n}
\end{tikzcd}$$

commutes. So

$$\begin{aligned}([e]\cdot[f])\cdot[g] \;&=\; (-1)^{jm}\cdot(-1)^{(j+k)n}\cdot[(e\cdot f)\cdot g] \\ &=\; (-1)^{kn}\cdot(-1)^{j(m+n)}\cdot[e\cdot(f\cdot g)] \;=\; [e]\cdot([f]\cdot[g])\ .\end{aligned}$$

(iv) Biadditivity, associativity and unitality of the pairing were established in parts (i), (ii) and (iii). The graded-commutativity is again shown by a similar argument as for the stable stems in Example 1.6. Indeed, we let $f : S^{m+k} \longrightarrow R_m$ and $g : S^{n+l} \longrightarrow R_n$ represent classes in $\pi_k(R)$ and $\pi_l(R)$. Then the following diagram commutes:

$$\begin{tikzcd} S^{m+k+n+l} \arrow[r, "f\wedge g"] \arrow[d, "\chi_{m+k,n+l}"'] & R_m\wedge R_n \arrow[r, "\mu_{m,n}"] \arrow[d, "\text{twist}"] & R_{m+n} \arrow[d, "R(\chi_{m,n})"] \\ S^{n+l+m+k} \arrow[r, "g\wedge f"'] & R_n\wedge R_m \arrow[r, "\mu_{n+m}"'] & R_{n+m} \end{tikzcd}$$

commutes. The coordinate permutation $\chi_{m+k.n+l}$ of $S^{m+k+n+l}$ induces the sign $(-1)^{(m+k)(n+l)}$; and the map $R(\chi_{m,n})$ induces multiplication by $(-1)^{mn}$ on homotopy groups, by Proposition 2.15 (i). This yields the relation

$$[f]\cdot[g] \;=\; (-1)^{kn}\cdot[f\cdot g] \;=\; (-1)^{kl+lm}\cdot[g\cdot f] \;=\; (-1)^{kl}\cdot[g]\cdot[f]\ .$$

in the group $\pi_{k+l}(R)$.

(v) Biadditivity, associativity and unitality of the action of $\pi_*(R)$ on $\pi_*(M)$ were established in parts (i), (ii) and (iii).

Property (vi) is almost obvious: if $f : S^{m+k} \longrightarrow R_m$ and $g : S^{n+l} \longrightarrow M_n$ represent classes in $\pi_k(R)$ and $\pi_l(M)$, then

$$\begin{aligned} f\cdot(\psi_n\circ g) \;&=\; \alpha_{m,n}\circ(f\wedge(\psi_n\circ g)) \;=\; \alpha_{m,n}\circ(R_m\wedge\psi_n)\circ(f\wedge g) \\ &=\; \psi_{m+n}\circ\alpha_{m,n}\circ(f\wedge g) \;=\; \psi_{m+n}\circ(f\cdot g)\ .\end{aligned}$$

Hence

$$[f]\cdot\psi_*[g] \;=\; (-1)^{kn}\cdot[f\cdot(\psi_n\circ g)] \;=\; (-1)^{kn}\cdot[\psi_{m+n}\circ(f\cdot g)] \;=\; \psi_*([f]\cdot[g])$$

in the group $\pi_{k+l}(N)$.

Property (vii) is also straightforward from the definitions. For the suspension homomorphisms we let $f : S^{m+k} \longrightarrow R_m$ and $g : S^{n+l} \longrightarrow M_n$ represent classes in $\pi_k(R)$ and $\pi_l(M)$, respectively, then

$$(f\cdot g)\wedge S^1 \;=\; (\alpha_{m,n}\circ(f\wedge g))\wedge S^1 \;=\; (\alpha_{m,n}\wedge S^1)\circ(f\wedge g\wedge S^1)) \;=\; f\cdot(g\wedge S^1)\ .$$

We have used that the action map $\alpha_{m,n}^{M\wedge S^1} : R_m\wedge(M\wedge S^1)_n$ is the map $\alpha^M_{m,n}\wedge S^1 : R_m\wedge M_n\wedge S^1$. We conclude that

$$([f]\cdot[g])\wedge S^1 \;=\; (-1)^{kn}\cdot[(f\cdot g)\wedge S^1] = \;(-1)^{kn}\cdot[f\cdot(g\wedge S^1)] \;=\; [f]\cdot([g]\wedge S^1)$$

in the group $\pi_{k+l+1}(M\wedge S^1)$. The $\pi_*(R)$-linearity of the loop isomorphisms can be proved by similarly explicit manipulations, or by exploiting that the loop isomorphism coincides with the composite

$$\pi_k(\Omega M) \xrightarrow{-\wedge S^1} \pi_{k+1}((\Omega M)\wedge S^1) \xrightarrow{\epsilon_*} \pi_{k+1}(M)\ .$$

The suspension homomorphism for ΩM is $\pi_*(R)$-linear by the above. The adjunction counit $\epsilon : (\Omega M)\wedge S^1 \longrightarrow M$ is a homomorphism of R-modules, so its effect on homotopy groups is $\pi_*(R)$-linear.

(viii) The connecting homomorphism $\delta : \pi_{k+1}(Cf) \longrightarrow \pi_k(M)$ is defined as the composite of the effect of the morphism $p_* : \pi_{k+1}(Cf) \longrightarrow \pi_{k+1}(M)$ and the inverse of the suspension homomorphism. The map p_* is $\pi_*(R)$-linear by part (vi); the suspension isomorphism is $\pi_*(R)$-linear by part (vii), hence its inverse is $\pi_*(R)$-linear, too. So the connecting homomorphism for the mapping cone is $\pi_*(R)$-linear. The argument for the connecting homomorphism of the homotopy fiber is dual. □

4.1. **Examples of ring spectra.**

Example 4.11 (The orthogonal sphere ring spectrum)**.** The orthogonal sphere spectrum $\mathbb{S}$ from Example 3.1 is a commutative orthogonal ring spectrum with respect to the canonical homeomorphisms $\mu_{V,W}$: $S^V \wedge S^W \cong S^{V\oplus W}$ as multiplication maps, and the identities as unit maps $\iota_V : S^V \longrightarrow \mathbb{S}(V)$. The forgetful functor from the category of left $\mathbb{S}$-modules to the category of orthogonal spectra is an isomorphism of categories. In this sense, $\mathbb{S}$-modules 'are' orthogonal spectra.

Example 4.12 (Spherical monoid rings)**.** We let M be a topological monoid. The multiplication and unit of M induces the structure of orthogonal ring spectrum on the unreduced suspension spectrum $\Sigma^\infty_+ M$, as follows; The unit is the point $0 \wedge 1 \in S^0 \wedge M_+$, where $0 \in S^0 = \{0, \infty\}$ is the non-basepoint and $1 \in M$ is the multiplicative unit. The multiplication map $\mu_{V,W}$ is given by the composite

$$\begin{aligned}(\Sigma^\infty_+ M)(V) \wedge (\Sigma^\infty_+ M)(W) \ &= S^V \wedge M_+ \wedge S^W \wedge M_+ \\ &\xrightarrow[\cong]{S^V \wedge \mathrm{twist} \wedge M_+} (S^V \wedge S^W) \wedge (M \times M)_+ \xrightarrow{\mu_{V,W} \wedge \mathrm{mult.}} S^{V\oplus W} \wedge M_+ \ = \ (\Sigma^\infty_+ M)(V \oplus W) \ .\end{aligned}$$

We write $\mathbb{S}M$ for the resulting orthogonal ring spectrum and refer to it as the *spherical monoid ring* of the topological monoid M. If the multiplication of M is commutative, then so is the one of $\mathbb{S}M$. A left $\mathbb{S}M$-action on an orthogonal spectrum amounts to the same data as a continuous left action of the monoid M by endomorphisms of orthogonal spectra.

Example 4.13 (Monoid ring spectra)**.** We let R be an orthogonal ring spectrum R and M a topological monoid. We define an orthogonal ring spectrum RM by $RM = R \wedge M_+$, i.e., the levelwise smash product with M with a disjoint basepoint added. The unit map is the point $\iota \wedge 1 \in R(0) \wedge M_+$, there $1 \in M$ is the multiplicative unit. The multiplication map $\mu_{V,W}$ is given by the composite

$$(R(V) \wedge M_+) \wedge (R(W) \wedge M_+) \cong (R(V) \wedge R(W)) \wedge (M \times M)_+ \xrightarrow{\mu_{V,W} \wedge \mathrm{mult.}} R(V \oplus W) \wedge M_+ \ .$$

In the special case when $R = \mathbb{S}$ is the sphere ring spectrum, then RM becomes the spherical monoid ring of Example 4.12. If both R and M are commutative, then so is RM. A left RM-module amounts to the same data as a left R-module together with a continuous left action of the monoid M by R-linear endomorphisms.

Example 4.14 (Eilenberg–MacLane ring spectra)**.** We let A be a ring, associative and unital, but not necessarily commutative. We let M be a left A-module. Then A and M have underlying additive abelian groups, which have associated orthogonal Eilenberg–MacLane spectra HA and HM as introduced in Example 3.3. The A-action on M induced an HA-action on HM as follows: for inner product spaces V and W, the action map is given by

$$\begin{aligned}\alpha_{V,W} \ : \ (HA)(V) \wedge (HM)(W) = A[S^V] \wedge M[S^W] \ &\longrightarrow \ M[S^{V\oplus W}] = (HM)(V \oplus W) \\ (\sum_i a_i \cdot v_i) \wedge (\sum_j m_i \cdot w_i) \quad &\longmapsto \quad \sum_{i,j} (a_i m_j) \cdot (v_i \wedge w_j) \ .\end{aligned}$$

We also define unit maps

$$\iota_V : S^V \longrightarrow A[S^V] = (HA)(V) \qquad \text{by} \qquad \iota_V(v) \ = \ 1 \cdot v \ .$$

This data makes HA into an orthogonal ring spectrum, and HM into a left HA-module spectrum. Moreover, if the multiplication of A is commutative, then so is the multiplication of HA. And the isomorphism (3.4) between A and $\pi_0(HA)$ is a ring isomorphism, relative to which $M \cong \pi_0(HM)$ is an isomorphism of left A-modules.

Example 4.15 (Matrix ring spectra)**.** If R is an orthogonal ring spectrum and $m \geq 1$ we define the orthogonal ring spectrum $M_m(R)$ of $(m \times m)$*-matrices* over R by

$$M_m(R) \ = \ \mathrm{map}_*(m_+, R \wedge m_+) \ .$$

Here $m_+ = \{0, 1, \dots, m\}$ with basepoint 0, and so $M_m(R)$ is a m-fold product of a m-fold coproduct (wedge) of copies of R. So 'elements' of $M_m(R)$ are more like matrices which in each row have at most one nonzero entry. The multiplication

$$\mu_{V,W} \ : \ \mathrm{map}_*(m_+, R(V) \wedge m_+) \ \wedge \ \mathrm{map}_*(m_+, R(W) \wedge m_+) \ \longrightarrow \ \mathrm{map}_*(m_+, R(V \oplus W) \wedge m_+)$$

sends $f \wedge g$ to the composite

$$m_+ \xrightarrow{f} R(V) \wedge m_+ \xrightarrow{R(V)\wedge g} R(V) \wedge R(W) \wedge m_+ \xrightarrow{\mu_{V,W} \wedge m_+} R(V \oplus W) \wedge m_+ \ .$$

Example E.7 is devoted to constructing an isomorphism of graded rings $\pi_*(M_m(R)) \cong M_m(\pi_*(R))$, i.e., the topological matrix construction realizes the algebraic matrix construction.

Example 4.16 (Opposite ring spectrum)**.** For every orthogonal ring spectrum R we can define the *opposite* ring spectrum R^{op} by keeping the same spaces, orthogonal group actions and unit maps, but with new multiplication $\mu^{\mathrm{op}}_{V,W}$ on R^{op} given by the composite

$$\begin{aligned} R^{\mathrm{op}}(V) \wedge R^{\mathrm{op}}(W) \ &= \ R(V) \wedge R(W) \xrightarrow{\text{twist}} R(W) \wedge R(V) \\ &\xrightarrow{\mu_{W,V}} \ R(W \oplus V) \ \xrightarrow{R(\tau_{W,V})} \ R(V \oplus W) \ = \ R^{\mathrm{op}}(V \oplus W) \ . \end{aligned}$$

By definition, an orthogonal ring spectrum R is commutative if and only if $R^{\mathrm{op}} = R$. If we were to define the category of right modules over R, we would find that it is isomorphic to the category of left modules over R^{op}.

For example, we have $(RM)^{\mathrm{op}} = (R^{\mathrm{op}})(M^{\mathrm{op}})$ for the monoid ring spectrum (Example 4.12) of a topological monoid M and its opposite, and $(HA)^{\mathrm{op}} = H(A^{\mathrm{op}})$ for the Eilenberg–Mac Lane ring spectrum (Example 4.14) of an ordinary ring A and its opposite.

By the centrality of the unit, the underlying orthogonal spectra of R and R^{op} are equal (not just isomorphic), hence R and R^{op} have the same (not just isomorphic) and homotopy groups. The purpose of Example E.8 is to show that

$$\pi_*(R^{\mathrm{op}}) \ = \ (\pi_*(R))^{\mathrm{op}}$$

(again equality) as graded rings, where the right hand side is the graded-opposite ring, i.e., the graded abelian group $\pi_*(R)$ with new product $x \cdot_{\mathrm{op}} y = (-1)^{kl} \cdot y \cdot x$ for $x \in \pi_k(R)$ and $y \in \pi_l(R)$.

Example 4.17 (Thom ring spectra)**.** In Example 3.6 we introduced the orthogonal Thom spectrum MO. Its value at an inner product space V is

$$MO(V) \ = \ \mathbf{L}(V, V^\infty)_+ \wedge_{O(V)} S^V \ ,$$

the Thom space of the tautological vector bundle over the Grassmannian of $\dim(V)$-planes over V^∞. We extend this to a commutative orthogonal ring spectrum as follows. If W is another inner product space, we define a multiplication map

$$\mu_{V,W} \ : \ MO(V) \wedge MO(W) \ \longrightarrow \ MO(V \oplus W)$$

by

$$\begin{aligned} (\mathbf{L}(V, V^\infty)_+ \wedge_{O(V)} S^V) \wedge (\mathbf{L}(W, W^\infty)_+ \wedge_{O(W)} S^W) \ &\longrightarrow \\ &\mathbf{L}(V \oplus W, (V \oplus W)^\infty)_+ \wedge_{O(V \oplus W)} S^{V \oplus W} \\ [\varphi, v] \wedge [\psi, w] \ &\longmapsto \ [\kappa_{V,W} \circ (\varphi \oplus \psi), \ v \oplus w] \end{aligned}$$

Here $\kappa_{V,W} : V^\infty \oplus W^\infty \cong (V \oplus W)^\infty$ is the shuffling isometry

$$\kappa_{V,W}((v_1, v_2, \dots), (w_1, w_2, \dots)) \ = \ ((v_1, w_1), (v_2, w_2), \dots) \ .$$

The multiplication maps are associative and commutative, and they are unital with respect to the maps

$$\iota_V = [i_V, -] \ : \ S^V \ \longrightarrow \ \mathbf{L}(V, V^\infty)_+ \wedge_{O(V)} S^V = MO(V)$$

where $i_V(v) = (v, 0, 0, \dots) : V \longrightarrow V^\infty$ is the isometric embedding as the first summand.

Example 4.18. The Thom spectrum MO of the previous example admits a 'periodization', i.e., a $\mathbb{Z}$-graded commutative orthogonal ring spectrum MOP whose degree 0 component is MO, and whose degree 1 component contains a unit of dimension 1. The definition of MOP is the same as for MO, with the only difference that we take Thom spaces over the full Grassmannian of all subspaces, of all dimensions, in V^∞. In more detail, we define

$$MOP(V) \ = \ \bigvee_{k\geq 0} \mathbf{L}(\mathbb{R}^k, V^\infty)_+ \wedge_{O(k)} S^k \ .$$

In much the same way as for MO, this is the Thom space of the tautological vector bundle – of non-constant rank! –

$$\coprod_{k\geq 0} \mathbf{L}(\mathbb{R}^k, V^\infty) \times_{O(k)} \mathbb{R}^k \ \longrightarrow \ \coprod_{k\geq 0} Gr_k(V^\infty) \ , \quad [\varphi, v] \ \longmapsto \ \varphi(\mathbb{R}^k) \ .$$

The structure maps, multiplication and unit maps are defined in the same way as for MO.

The orthogonal spectrum MOP is $\mathbb{Z}$-graded, with m-th homogeneous summand

$$MOP^{[m]}(V) \ = \ \mathbf{L}(\mathbb{R}^{\dim(V)+m}, V^\infty)_+ \wedge_{O(\dim(V)+m)} S^{\dim(V)+m} \ .$$

Then $MOP^{[0]} = MO$ and

$$MOP \ = \ \bigvee\nolimits_{m\in\mathbb{Z}} MOP^{[m]}$$

as orthogonal spectra.

We want to explain the periodicity of the ring spectrum MOP. For the following argument it will be convenient to use a homeomorphic description of the tautological vector bundle – and hence its Thom space – as the subspace of $V^\infty \times Gr(V^\infty)$ consisting of those pairs (v, L) with $v \in L$. The homeomorphism from the previous description takes

$$[\varphi, x] \ \in \ \mathbf{L}(\mathbb{R}^n, V^\infty) \times_{O(n)} \mathbb{R}^n$$

to $(\varphi(x), \varphi(\mathbb{R}^n))$.

We let $t \in \pi_{-1}(MOP^{[-1]})$ be the class represented by the point

$$(0, \{0\}) \ \in \ Th(Gr_0(\mathbb{R}^\infty)) \ = \ MOP^{[-1]}(\mathbb{R}) \ = \ (MOP^{[-1]})_1 \ .$$

We let $\sigma \in \pi_1(MOP^{[1]})$ be the class represented by the map

$$S^2 \ \longrightarrow \ Th(Gr_2(\mathbb{R}^\infty)) \ = \ MOP^{[1]}(\mathbb{R}) \ = \ (MOP^{[1]})_1 \ , \quad x \ \longmapsto \ (x, \mathbb{R}^2) \ .$$

Here $\mathbb{R}^2$ denotes the 2-dimensional subspace of $\mathbb{R}^\infty$ spanned by the first two coordinates. As the next proposition shows, MOP is periodic in the sense that t is a unit in the graded ring $\pi_*(MOP)$, with inverse σ.

Theorem 4.19. (i) *The relation $t \cdot \sigma = 1$ holds in $\pi_0(MOP)$.*

(ii) *The relation $2 = 0$ holds in $\pi_0(MO)$. In particular, all homotopy groups of MO and MOP are $\mathbb{F}_2$-vector spaces.*

(iii) *For every $m \geq 0$, the orthogonal spectrum $MOP^{[m]}$ is stably equivalent to $MO \wedge S^m$; for every $m \leq 0$, the orthogonal spectrum $MOP^{[m]}$ is stably equivalent to $\Omega^{-m}(MO)$.*

Proof. (i) The class $t \cdot \sigma$ is represented by the composite

$$S^2 \ \xrightarrow{x\mapsto(0,\{0\})\wedge(x,\mathbb{R}^2)} \ MOP^{[-1]}(\mathbb{R}) \wedge MOP^{[1]}(\mathbb{R}) \ \xrightarrow{\mu_{\mathbb{R},\mathbb{R}}} \ MOP^{[0]}(\mathbb{R}\oplus\mathbb{R})$$

where the first map is the smash product of the defining representatives for t and σ. Expanding the definition of $\mu_{\mathbb{R}.\mathbb{R}}$ identifies this composite as the map

$$S^2 \ \longrightarrow \ MOP^{[0]}(\mathbb{R}\oplus\mathbb{R}) \ , \quad (x, y) \ \longmapsto \ ((0, x, 0, y), (0 \oplus \mathbb{R}) \oplus (0 \oplus \mathbb{R})) \ .$$

This differs from the representative of the unit $1 \in \pi_0(MOP)$ by the action of the linear isometry

$$\mathbb{R}^4 \ \longrightarrow \ \mathbb{R}^4 \ , \quad (a, b, c, d) \ \longmapsto \ (b, d, c, a) \ .$$

This isometry has determinant 1, and is this homotopic in $O(4)$ to the identity. So the representatives of $t \cdot \sigma$ and 1 are homotopic, and we conclude that $t \cdot \sigma = 1$ in $\pi_0(MOP)$.

(ii) The relation $2 = 0$ is now a formal consequence: let R be any commutative orthogonal ring spectrum that has an invertible element $t \in \pi_k(R)$ in an *odd* degree k. The graded-commutativity of the multiplication (Proposition 4.10 (iv)) yields $t^2 = -t^2$, and hence $2t^2 = 0$. Since t is invertible, multiplication with t is injective, so $2 = 0$ in $\pi_0(R)$. The graded multiplication in particular makes the group $\pi_l(R)$ into a module over the ring $\pi_0(R)$, so all the groups $\pi_l(R)$ are $\mathbb{F}_2$-vector spaces.

(iii) Since the class $t \in \pi_{-1}(MOP^{[-1]})$ is invertible by part (i), multiplication by it is an isomorphism

$$t \cdot - \ : \ \pi_{k+1}(MOP^{[k+1]}) \xrightarrow{\cong} \pi_k(MOP^{[k]}) \ ;$$

the inverse is given by multiplication by $\sigma \in \pi_1(MOP^{[1]})$. We will now show that multiplication by t is realized, in a certain precise way, by a periodicity morphism $j : MOP \longrightarrow \mathrm{sh}(MOP)$: the value at an inner product space V is the map

$$\begin{array}{rcccl} j(V) \ : \ MOP(V) & \longrightarrow & MOP(V \oplus \mathbb{R}) & = & \mathrm{sh}(MOP)(V) \\ (x, L) & \longmapsto & (i^\infty(x), i^\infty(L)) & & \end{array}$$

induced by the linear isometric embedding $i^\infty : V^\infty \longrightarrow (V \oplus \mathbb{R})^\infty$ with $i : V \longrightarrow V \oplus \mathbb{R}$ the embedding as the first summand. The morphism j is even a homomorphism of left MOP-module spectra. The morphism j is homogeneous of degree -1 in terms of the $\mathbb{Z}$-grading of MOP, i.e., it restricts to a morphism of orthogonal spectra

$$j \ : \ MOP^{[m+1]} \longrightarrow \mathrm{sh}(MOP^{[m]})$$

where m is any integer.

The map $j(V)$ factors as the composite

$$MOP(V) \xrightarrow{-\wedge(0,\{0\})} MOP(V) \wedge MOP(\mathbb{R}) \xrightarrow{\mu_{V,\mathbb{R}}} MOP(V \oplus \mathbb{R}) \ = \ \mathrm{sh}(MOP)(V) \ .$$

Since the point $(0, \{0\})$ of $MOP(\mathbb{R})$ represents the class t, this shows that the map

$$j_* \ : \ \pi_{k+1}(MOP^{[m+1]}) \longrightarrow \pi_{k+1}(\mathrm{sh}(MOP^{[m]})) \ = \ \pi_k(MOP^{[m]})$$

is multiplication by the class t. In particular, j_* is an isomorphism, and hence j is stable equivalence from $MOP^{[m+1]}$ to $\mathrm{sh}(MOP^{[m]})$. By Proposition 2.15, $MOP^{[m+1]}$ is then also stably equivalent to $MOP^{[m]} \wedge S^1$. For $m \geq 0$, the orthogonal spectrum $MOP^{[m]}$ is thus stably equivalent to $MO \wedge S^m$, by induction. For $m \geq 0$, the orthogonal spectrum MO is stably equivalent $MOP^{[-m]} \wedge S^m$. Hence $\Omega^{-m}(MO)$ is stably equivalent $\Omega^{-m}(MOP^{[-m]} \wedge S^m)$, which is stably equivalent to $MOP^{[-m]}$ by Proposition 1.13. $\square$

Example 4.20 (Complex Thom spectra)**.** We define another Thom spectrum MU, the unitary (or complex) analog of MO. Closely related, strictly commutative ring spectrum models for this homotopy type have been discussed in various places, for example [31], [15, Example 5.8], [44, Appendix A] or [8, Section 8]. The Thom ring spectra MU, MSU and MSp representing unitary, special unitary or symplectic bordism have to be handled slightly differently from real Thom spectra such as MO in the previous example. The point is that MU and MSU are most naturally indexed on 'even spheres', i.e., one-point compactifications of complex vector spaces, and MSp is most naturally indexed on spheres of dimensions divisible by 4. However, a small variation gives MU, MSU and MSp as commutative orthogonal ring spectra, as we shall now explain. The complex cobordism spectrum MU plays an important role in stable homotopy theory because of its relationship to the theory of formal groups laws. There is also a periodic version MUP of MU, a $\mathbb{Z}$-graded commutative ring spectrum whose k-th summands is stably equivalent to a $2k$-fold suspension of MU.

We let $V_\mathbb{C} = \mathbb{C} \otimes_\mathbb{R} V$ denote the complexification of a real vector space V. If V is endowed with a euclidean inner product, then $V_\mathbb{C}$ inherits a hermitian inner product by the $\mathbb{R}$-bilinear extension of

$$(\lambda \otimes v, \mu \otimes w) \ = \ \bar{\lambda} \cdot \mu \cdot \langle v, w \rangle \ ,$$

for $\lambda, \mu \in \mathbb{C}$ and $v, w \in V$. Given two hermitian inner product spaces U and U', we write $\mathbf{L}^\mathbb{C}(U, U')$ for the space of $\mathbb{C}$-linear isometric embeddings. If W and W' are finite dimensional, we give $\mathbf{L}^\mathbb{C}(W, W')$ the topology as a complex Stiefel manifold. If W' is infinite dimensional, we give $\mathbf{L}^{\mathbb{C}}(W,W')$ the resulting weak topology.

We first consider the collection of pointed spaces $\overline{MU}$ with

$$\overline{MU}(V) \;=\; \mathbb{L}^{\mathbb{C}}(V_{\mathbb{C}},V_{\mathbb{C}}^{\infty})_+ \wedge_{U(V_{\mathbb{C}})} S^{V_{\mathbb{C}}} \ ,$$

the Thom space of the complex vector bundle

$$\mathbb{L}^{\mathbb{C}}(V_{\mathbb{C}},V_{\mathbb{C}}^{\infty}) \times_{U(V_{\mathbb{C}})} V_{\mathbb{C}} \;\longrightarrow\; Gr^{\mathbb{C}}_{\dim(V)}(V_{\mathbb{C}}^{\infty}) \ .$$

Here $U(V_{\mathbb{C}}) = \mathbf{L}^{\mathbb{C}}(V_{\mathbb{C}},V_{\mathbb{C}})$ is the unitary group of the hermitian inner product space $V_{\mathbb{C}}$. The $O(V)$-action arises from the diagonal action on $V_{\mathbb{C}}^{\infty}$, similarly as in the case of MO above.

There are $O(V)\times O(W)$-equivariant multiplication maps

$$\begin{aligned} \bar{\mu}_{V,W} \ : \ \overline{MU}(V)\wedge\overline{MU}(W) \ &\longrightarrow \quad \overline{MU}(V\oplus W) \\ [\varphi,x]\wedge[\psi,y] \quad &\longmapsto \ [{}^{\kappa_{V,W}}(\varphi\oplus\psi),(x,y)] \end{aligned}$$

where ${}^{\kappa_{V,W}}(\varphi\oplus\psi)$ denotes the 'conjugate' of $\varphi\oplus\psi : V_{\mathbb{C}}\oplus W_{\mathbb{C}} \longrightarrow V_{\mathbb{C}}^{\infty}\oplus W_{\mathbb{C}}^{\infty}$ by the preferred $\mathbb{C}$-linear isometries

$$V_{\mathbb{C}}\oplus W_{\mathbb{C}} \;\cong\; (V\oplus W)_{\mathbb{C}} \qquad \text{and} \qquad V_{\mathbb{C}}^{\infty}\oplus W_{\mathbb{C}}^{\infty} \;\cong\; (V\oplus W)_{\mathbb{C}}^{\infty} \ .$$

There is a unit map $\iota_0 : S^0 \longrightarrow \overline{MU}(0)$, but instead of a unit maps from the sphere S^V, we instead have a unit maps

$$\bar{\iota}_V \ : \ S^{V_{\mathbb{C}}} \;\longrightarrow\; \overline{MU}(V) \ , \quad v\mapsto\ (i_V, i_V(v)) \ ,$$

where $i_V : V_{\mathbb{C}} \longrightarrow V_{\mathbb{C}}^{\infty}$ is the embedding as the first summand. Thus we do *not* end up with an orthogonal spectrum since we only get structure maps $S^{V_{\mathbb{C}}}\wedge\overline{MU}(W) \longrightarrow \overline{MU}(V\oplus W)$ involving even-dimensional spheres. In other words, $\overline{MU}$ has the structure of what could be called an 'even ring spectrum'.

In order to get an honest orthogonal ring spectrum, we have to tweak the construction somewhat, by 'looping with imaginary spheres'. We define

$$MU(V) \;=\; \mathrm{map}_*(S^{iV},\overline{MU}(V)) \ ,$$

the iV-loop space of the previously defined Thom space $\overline{MU}(V)$, where iV is the $\mathbb{R}$-subspace of $V_{\mathbb{C}}$ consisting of the vectors $i\otimes v$ for $v\in V$. A commutative multiplication is given by

$$\mu_{V,W} \ : \ MU(V)\wedge MU(W) \;\longrightarrow\; MU(V\oplus W) \ , \quad f\wedge g \;\longmapsto\; f\cdot g \ .$$

Here $f : S^{iV} \longrightarrow \overline{MU}(V)$, $g : S^{iW} \longrightarrow \overline{MU}(W)$, and $f\cdot g$ denotes the composite

$$S^{i(V\oplus W)} \;\cong\; S^{iV}\wedge S^{iW} \;\xrightarrow{f\wedge g}\; \overline{MU}(V)\wedge\overline{MU}(W) \xrightarrow{\bar{\mu}_{V,W}} \overline{MU}(V\oplus W) \ ,$$

The unit maps $\iota_V : S^V \longrightarrow MU(V)$ are adjoint to

$$S^V\wedge S^{iV} \;\cong\; S^{V_{\mathbb{C}}} \;\xrightarrow{\bar{\iota}_V}\; \overline{MU}(V) \ .$$

These multiplication maps unital, associative and commutative, and make MU into a commutative orthogonal ring spectrum.

The homotopy groups of MU are given by

$$\pi_k(MU) \;= \mathrm{colim}_n\ \pi_{n+k}(\mathrm{map}_*(S^n,\overline{MU}(\mathbb{R}^n))) \;\cong\; \mathrm{colim}_n\ \pi_{2n+k}(\overline{MU}(\mathbb{R}^n)) \ ;$$

by the unitary analog of Thom's celebrated theorem, they are isomorphic to the ring of cobordism classes of stably almost complex k-manifolds.

Essentially the same construction gives an orthogonal spectrum MSU. The symplectic bordism and MSp can also be handled similarly, but one need to replace the field of complex numbers by the skew-field of quaternions.

5. Cofibration categories

We will eventually introduce the stable homotopy category as the localization of the category of orthogonal spectra at the class of stable equivalences. Most of the formal properties of the stable homotopy category are shared by the derived category of any orthogonal ring spectrum, so we will develop the theory in this generality, see Section 6 below.

If all we had was the relative category of orthogonal spectra and stable equivalences, it would be hard to prove anything useful about its localization, the stable homotopy category. This is where some kind of additional structure comes in handy. We will use *cofibration categories*. This notion was first introduced and studied (in the dual formulation) by Brown [7] under the name 'categories of fibrant objects'. Closely related sets of axioms have been explored by various authors, compare Remark 5.2.

Definition 5.1. A *cofibration category* is a category $\mathcal{C}$ equipped with two classes of morphisms, called *cofibrations* and *weak equivalences*, respectively, that satisfy the following axioms (C1)–(C4).

(C1) All isomorphisms are cofibrations and weak equivalences. Cofibrations are stable under composition. The category $\mathcal{C}$ has an initial object and every morphism from an initial object is a cofibration.

(C2) Given two composable morphisms f and g in $\mathcal{C}$, if two of the three morphisms f, g and gf are weak equivalences, then so is the third.

(C3) Given a cofibration $i : A \longrightarrow B$ and any morphism $f : A \longrightarrow C$, there exists a pushout square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, tail, "i"'] & C \arrow[d, tail, "j"] \\ B \arrow[r, "j"'] & D \end{tikzcd}$$

in $\mathcal{C}$ and the morphism j is a cofibration. If additionally i is a weak equivalence, then so is j.

(C4) Every morphism in $\mathcal{C}$ can be factored as the composite of a cofibration followed by a weak equivalence.

An *acyclic cofibration* is a morphism that is both a cofibration and a weak equivalence.

We will often decorate cofibrations by a tail at the arrow, as in $\rightarrowtail$; will often denote weak equivalences by a tilde over the arrow, as in $\xrightarrow{\sim}$; hence acyclic cofibrations come with a tail and a tilde, as in $\overset{\sim}{\rightarrowtail}$.

We record some elementary consequences of the axioms:

- In a cofibration category a coproduct $B \vee C$ of any two objects in $\mathcal{C}$ exists by (C3) with A an initial object, and the canonical morphisms from B and C to $B \vee C$ are cofibrations.
- If $i : A \longrightarrow B$ and $i' : A' \longrightarrow B'$ are cofibrations, so is their coproduct $i \amalg i' : A \amalg A' \longrightarrow B \amalg B'$. Indeed, applying (C3) to the two pushout squares

$$\begin{tikzcd} A \arrow[r, tail, "i"] \arrow[d, tail] & B \arrow[d, tail] \\ A \amalg A' \arrow[r, tail, "i \amalg A'"'] & B \amalg A' \end{tikzcd} \qquad \begin{tikzcd} A' \arrow[r, tail, "i'"] \arrow[d, tail] & B' \arrow[d, tail] \\ B \amalg A' \arrow[r, tail, "B \amalg i'"'] & B \amalg B' \end{tikzcd}$$

shows that $i \amalg A'$ and $B \amalg i'$ are cofibrations, hence so is their composite, by (C1). The same argument shows that whenever i and i' are acyclic cofibrations, so is $i \amalg i'$.

The *homotopy category* of a cofibration category is a localization at the class of weak equivalences, i.e., a functor $\gamma : \mathcal{C} \longrightarrow \mathrm{Ho}(\mathcal{C})$ that takes all weak equivalences to isomorphisms and is initial among such functors. The homotopy category always exists if one is willing to pass to a larger universe. To get a locally small homotopy category (i.e., have 'small hom-sets'), additional assumptions are necessary; one possibility is to assume that $\mathcal{C}$ has 'enough fibrant objects', compare Remark 5.14. We recall some basic facts about the homotopy category of a cofibration category in Theorem 5.13.

Remark 5.2. The above notion of cofibration category is due to K. S. Brown [7]. More precisely, Brown introduced 'categories of fibrant objects', and the axioms (C1)–(C4) are equivalent to the dual of the axioms (A)–(E) of Part I.1 in [7]. The concept of a cofibration category is a substantial generalization of Quillen's notion of a 'closed model category' [36]: from a Quillen model category one obtains a cofibration category by restricting to the full subcategory of cofibrant objects and forgetting the class of fibrations.

Cofibration categories are closely related to 'categories with cofibrations and weak equivalences' in the sense of Waldhausen [52]. In fact, a category with cofibrations and weak equivalences that also satisfies the *saturation axiom* [52, 1.2] and the *cylinder axiom* [52, 1.6] is in particular a cofibration category as in Definition 5.1. Further relevant references on closely related axiomatic frameworks are Baues' monograph [3] and Cisinski's article [9]. Radulescu-Banu's extensive paper [37] is the most comprehensive source for basic results on cofibration categories and, among other things, contains a survey of the different kinds of cofibration categories and their relationships.

A property that we will frequently use is the following *gluing lemma*. A proof of the gluing lemma can be found in Lemma 1.4.1 (1) of [37].

Proposition 5.3 (Gluing lemma)**.** *Let $\mathcal{C}$ be a cofibration category. Consider a commutative $\mathcal{C}$-diagram*

$$\begin{tikzcd} A \arrow[d, "\sim"'] & B \arrow[l, tail, "i"'] \arrow[r] \arrow[d, "\sim"'] & C \arrow[d, "\sim"] \\ A' & B' \arrow[l, tail, "i'"] \arrow[r] & C' \end{tikzcd}$$

such that i and i' are cofibrations and all three vertical morphisms are weak equivalences. The the induced morphism between the horizontal pushouts $A \cup_B C \longrightarrow A' \cup_{B'} C'$ is a weak equivalence.

A special case of the gluing lemma is particularly important. Let $i : A \longrightarrow B$ be a cofibration, and let $f : B \longrightarrow C$ be a weak equivalence. Applying the gluing lemma to the commutative diagram

$$\begin{tikzcd} A \arrow[d, equal] & B \arrow[l, tail, "i"'] \arrow[r, equal] \arrow[d, equal] & B \arrow[d, "\sim"', "f"] \\ A & B \arrow[l, tail, "i"] \arrow[r, "f"'] & C \end{tikzcd}$$

yields that the morphism g in the pushout square

$$\begin{tikzcd} B \arrow[r, "f", "\sim"'] \arrow[d, tail, "i"'] & C \arrow[d, tail] \\ A \arrow[r, "\sim", "g"'] & A \cup_B C \end{tikzcd}$$

is a weak equivalence, too.

5.1. **The homotopy relation.**

One key feature that makes the homotopy category of a cofibration category more manageable that an arbitrary localization is a 'calculus of fractions' for morphisms in the homotopy category. We will develop this part of the theory now.

Definition 5.4. Let A be an object of a cofibration category $\mathcal{C}$. A *cylinder object* for A is a quadruple (I, i_0, i_1, p) consisting of an object I, morphisms $i_0, i_1 : A \longrightarrow I$ and a weak equivalence $p : I \longrightarrow A$ satisfying $pi_0 = pi_1 = \mathrm{Id}_A$ and such that $i_0 + i_1 : A \amalg A \longrightarrow I$ is a cofibration.

Two morphisms $f, g : A \longrightarrow Z$ in a cofibration category are *homotopic* if there exists a cylinder object (I, i_0, i_1, p) for A and a morphism $H : I \longrightarrow Z$ (the *homotopy*) such that $f = Hi_0$ and $g = Hi_1$.

Every object has a cylinder object: axiom (C4) lets us factor the fold map $\mathrm{Id}+\mathrm{Id} : A \amalg A \longrightarrow A$ as a cofibration $i_0 + i_1 : A \amalg A \longrightarrow I$ followed by a weak equivalence $p : I \longrightarrow A$.

Since the morphism p in a cylinder object is a weak equivalence, $\gamma(p)$ is an isomorphism in $\mathrm{Ho}(\mathcal{C})$ and so $\gamma(i_0) = \gamma(i_1)$ since they share $\gamma(p)$ as common left inverse. So if f and g are homotopic via H, then $\gamma(f) = \gamma(H)\gamma(i_0) = \gamma(H)\gamma(i_1) = \gamma(g)$. In other words: homotopic morphisms become equal in the homotopy category. The converse is not true in general, but part (ii) of the 'calculus of fractions' (Theorem 5.13) says that the converse is true up to post-composition with a weak equivalence.

Proposition 5.5. *Let A and Z be objects in a cofibration category $\mathcal{C}$.*

(i) *'Homotopy' is an equivalence relation on the set of morphisms $\mathcal{C}(A, Z)$.*
(ii) *Postcomposition with any $\mathcal{C}$-morphism $Z \longrightarrow \bar{Z}$ preserves the homotopy relation.*
(iii) *Let $f, g : A \longrightarrow Z$ be homotopic, and let $\varphi : \bar{A} \longrightarrow A$ be any $\mathcal{C}$-morphism. Then there is an acyclic cofibration $s : Z \longrightarrow Z'$ such that the two morphisms $sf\varphi, sg\varphi : A \longrightarrow Z'$ are homotopic; moreover, the homotopy can be witnessed by any cylinder object for $\bar{A}$.*
(iv) *Let $f, g : A \longrightarrow Z$ and $\tau : \bar{A} \longrightarrow A$ be $\mathcal{C}$-morphism such that τ is a weak equivalence. If $f\tau, g\tau : \bar{A} \longrightarrow Z$ are homotopic, then f and g are homotopic.*

Proof. (i) For reflexivity we let (I, i_0, i_1, p) be any cylinder object for A. Then for any morphism $f : A \longrightarrow Z$, the morphism $fp : I \longrightarrow Z$ is a homotopy from f to itself.

For symmetry we let $H : I \longrightarrow Z$ be a homotopy from $f : A \longrightarrow Z$ to $g : A \longrightarrow Z$, based on some cylinder object (I, i_0, i_1, p). Interchanging the roles of i_0 and i_1 yields another cylinder object (I, i_1, i_0, p) for A. The same morphism $H : I \longrightarrow Z$ is now a homotopy from g to f based on this new cylinder object.

As a preparation for the transitivity relation we explain how two cylinder objects (I, i_0, i_1, p) and (J, j_0, j_1, q) for A can be glued into a third cylinder object $(I \cup_A J, l_0, l_1, r)$. We define $I \cup_A J$ by a choice of pushout

$$\begin{tikzcd}
A \arrow[r, tail, "j_0"] \arrow[d, tail, "i_1"'] & J \arrow[d, tail, "b"] \\
I \arrow[r, tail, "a"'] & I \cup_A J
\end{tikzcd}$$

Such pushout exists because i_1 and j_0 are cofibrations. Since i_1 and j_0 are acyclic cofibrations, so are a and b, by (C3). We define

$$l_0 \;=\; ai_0 \qquad \text{and} \qquad l_1 \;=\; bj_1 \ .$$

The universal property of the pushout provides a unique morphism $r : I \cup_A J \longrightarrow Z$ such that

$$ra \;=\; p \qquad \text{and} \qquad rb \;=\; q \ .$$

Then

$$rl_0 \;=\; rai_0 \;=\; pi_1 \;=\; \mathrm{Id}_A \ ,$$

and similarly $rl_1 = \mathrm{Id}_A$. Since $a : I \longrightarrow I \cup_A J$ and $p : Z \longrightarrow A$ are weak equivalences, the 2-out-of-3 axiom (C2) and the relation $ra = p$ show that r is a weak equivalence.

Since the coproduct of two cofibrations is a cofibration, the pushout square

$$\begin{tikzcd}
A \amalg A \amalg A \amalg A \arrow[r, tail, "i_0+i_1+j_0+j_1"] \arrow[d, "A\amalg\nabla\amalg A"'] & I \amalg J \arrow[d] \\
A \amalg A \amalg A \arrow[r, tail, "l_0+l_{1/2}+l_1"'] & I \cup_A J
\end{tikzcd}$$

shows that the lower horizontal morphism is a cofibration. Since embedding $A \amalg A \longrightarrow A \amalg A \amalg A$ as the first and last summand is a cofibration, too; we conclude that $l_0 + l_1 : A \amalg A \longrightarrow I \cup_A J$ is a cofibration. This concludes the proof that the quadruple $(I \cup_A J, l_0, l_1, r)$ is indeed a cylinder object for A.

For transitivity we consider three morphisms $f, g, h : A \longrightarrow Z$, a homotopy $H : I \longrightarrow Z$ from f to g based on a cylinder object (I, i_0, i_1, p), and a homotopy $K : J \longrightarrow Z$ from g to h based on the cylinder object (J, j_0, j_1, q). Because

$$Hi_1 \;=\; g \;= Kj_0 \;:\; A \;\longrightarrow\; Z \;,$$

the universal property of the pushout provides a unique morphism $H \cup K : I \cup_A J \longrightarrow Z$ such that

$$(H \cup K)a \;=\; H \qquad \text{and} \qquad (H \cup K)b \;=\; K \;.$$

Then

$$(H \cup K)l_0 \;=\; (H \cup K)ai_0 \;=\; Hi_0 \;=\; f \;,$$

and similarly $(H \cup K)l_1 = h$. In other words: $H \cup K$ is a homotopy from f to h.

(ii) Given a homotopy $H : I \longrightarrow Z$ between two morphisms $f : A \longrightarrow Z$ to $g : A \longrightarrow Z$ and any morphism $\psi : Z \longrightarrow \bar{Z}$, then ψH is a homotopy, based on the same cylinder object, from ψf to ψg. So 'homotopy' is stable under postcomposition.

(iii) We let (I, i_0, i_1, p) and (J, j_0, j_1, q) be cylinder objects for A and $\bar{A}$, respectively. We start with a preliminary construction that fixes the defect that cylinder objects we not assumed to be functorial. The left vertical morphism in the commutative square

$$\begin{tikzcd}
\bar{A} \amalg \bar{A} \arrow[r, "i_0\varphi + i_1\varphi"] \arrow[d, "j_0+j_1"', tail] & I \arrow[d, "p", tail] \\
J \arrow[r, "\varphi q"'] & A
\end{tikzcd}$$

is a cofibration; so a pushout of the initial part of the diagram exists. We apply the factorization axiom (C4) to the morphism

$$(\varphi q) \cup p \;:\; J \cup_{\bar{A} \amalg \bar{A}} I \;\longrightarrow\; A \;;$$

we obtain a cofibration and a weak equivalence

$$\bar{\varphi} \cup t \;:\; J \cup_{\bar{A} \amalg \bar{A}} I \;\longrightarrow\; I' \qquad \text{and} \qquad p' \;:\; I' \;\longrightarrow\; A$$

whose composite is $(\varphi q) \cup p$. We define

$$i'_0 \;=\; ti_0 \;: A \;\longrightarrow\; I' \qquad \text{and} \qquad i'_1 \;=\; ti_1 \;: A \;\longrightarrow\; I' \;;$$

then the following diagram commutes:

(5.6)
$$\begin{tikzcd}
\bar{A} \amalg \bar{A} \arrow[rr, "\varphi \amalg \varphi"] \arrow[d, "j_0+j_1"', tail] & & A \amalg A \arrow[dl, "i'_0+i'_1"', tail] \arrow[dr, "i_0+i_1", tail] & \\
J \arrow[r, "\bar{\varphi}"] \arrow[d, "q"', "\sim"] & I' \arrow[dr, "p'"', "\sim"] & & I \arrow[ll, "t"', "\sim", tail] \arrow[dl, "p", "\sim"'] \\
\bar{A} \arrow[rr, "\varphi"'] & & A &
\end{tikzcd}$$

We claim that the quadruple $(I', i'_0.i'_1, p')$ is a new cylinder object for A. Because the morphisms p and p' are weak equivalences, so is $t : I \longrightarrow I'$ by 2-out-of-3. Because $j_0 + j_1 : \bar{A} \amalg \bar{A} \longrightarrow J$ is a cofibration, so is the canonical morphism $I \longrightarrow J \cup_{\bar{A} \amalg \bar{A}} I$. Since the morphism $\bar{\varphi} \cup t : J \cup_{\bar{A} \amalg \bar{A}} I \longrightarrow I'$ is a cofibration by design, we conclude that $t : I \longrightarrow I'$ is a cofibration. Because $i_0 + i_1 : A \amalg A \longrightarrow I$ is a cofibration, so is

$$i'_0 + i'_1 \;=\; t \circ (i_0 + i_1) \;:\; A \amalg A \longrightarrow I' \;.$$

The upshot of this discussion is that we have constructed a new cylinder object $(I', i'_0.i'_1, p')$ for A, an acyclic cofibration $t : I \longrightarrow I'$, and a morphism $\bar{\varphi} : J \longrightarrow I'$ such that the diagram (5.6) commutes.

Now we prove part (iii). We let $H : I \longrightarrow Z$ be a homotopy from $f : A \longrightarrow Z$ to $g : A \longrightarrow Z$, based on the cylinder object (I, i_0, i_1, p). We define the acyclic cofibration $s : Z \longrightarrow Z'$ by a choice of pushout:

$$\begin{tikzcd} I \arrow[r, tail, "t", "\sim"'] \arrow[d, "H"'] & I' \arrow[d, "\kappa"] \\ Z \arrow[r, tail, "\sim", "s"'] & Z' \end{tikzcd}$$

Then

$$\kappa\bar{\varphi}j_0 \;=\; \kappa i_0'\varphi \;=\; \kappa t i_0 \varphi \;=\; sHi_0\varphi \;=\; sf\varphi \ ,$$

and similarly $\kappa\bar{\varphi}j_1 = sg\varphi$. In other words, the morphism $\kappa\bar{\varphi} : J \longrightarrow Z'$ is a homotopy from $sf\varphi$ to $sg\varphi$.

(iv) Since $f\tau$ is homotopic to $g\tau$, there is a cylinder object (J, j_0, j_1, q) for $\bar{A}$ and a homotopy $H : J \longrightarrow Z$ from $f\tau$ to $g\tau$. We form a pushout

$$\begin{tikzcd} \bar{A} \amalg \bar{A} \arrow[r, tail, "j_0+j_1"] \arrow[d, "\tau\amalg\tau"', "\sim"] & J \arrow[d, "\sim"', "\bar{\tau}"] \\ A \amalg A \arrow[r, tail, "i_0+i_1"'] & I \end{tikzcd}$$

The morphism $i_0 + i_1 : A \amalg A \longrightarrow I$ is then a cofibration because $j_0 + j_1$ is; and the morphism $\bar{\tau}$ is a weak equivalence because $\tau \amalg \tau$ is, by the gluing lemma. The universal property of the pushout provides a unique morphism $p : I \longrightarrow A$ such that

$$pi_0 \;=\; pi_1 \;=\; \mathrm{Id}_A \qquad \text{and} \qquad p\bar{\tau} \;=\; \tau q \ .$$

Because τ, $\bar{\tau}$ and q are weak equivalences, so is p, by 2-out-of-3. We conclude that (I, i_0, i_1, p) is a cylinder object for A.

The relations

$$f\tau \;=\; Hj_0 \qquad \text{and} \qquad g\tau \;=\; Hj_1$$

and the universal property of the pushout provide a unique morphism $K : I \longrightarrow Z$ such that

$$Ki_0 \;=\; f \ , \quad Ki_1 \;=\; g \qquad \text{and} \qquad K\bar{\tau} \;=\; H \ .$$

In particular, $\bar{K}$ is a homotopy from f to g. □

5.2. **Localization by fractions.** We can now exhibit a localization of a cofibration category at the class of weak equivalences, by a construction that resembles the definition of fractions.

Construction 5.7. We let $\mathcal{C}$ be a cofibration category. We define a category $\mathrm{Ho}(\mathcal{C})$ with the same objects as $\mathcal{C}$. Morphisms in $\mathrm{Ho}(\mathcal{C})$ from A to B are equivalence classes of pairs (f, τ) consisting of $\mathcal{C}$-morphisms $f : A \longrightarrow Z$ and $\tau : B \longrightarrow Z$ with the same target, and such that τ is an acyclic cofibration. Two such pairs (f, τ) and (f', τ') are *equivalent* if there are acyclic cofibrations $a : Z \longrightarrow \bar{Z}$ and $b : Z' \longrightarrow \bar{Z}$ such that the following diagram commutes up to homotopy:

$$\begin{tikzcd} & Z \arrow[d, tail, "a"', "\sim"] & \\ A \arrow[ur, "f"] \arrow[dr, "f'"'] & \bar{Z} & B \arrow[ul, tail, "\tau"', "\sim"] \arrow[dl, tail, "\sim", "\tau'"'] \\ & Z' \arrow[u, tail, "b", "\sim"'] & \end{tikzcd}$$

We write $(f, \tau) \approx (f', \tau')$ for this relation.

Proposition 5.8. *The relation $\approx$ is an equivalence relation.*

Proof. The relation $\approx$ is clearly reflexive and symmetric, because the homotopy relation is reflexive and symmetric. But we need to argue that it is also transitive.

We suppose that $(f,\tau) \approx (f',\tau')$ via two acyclic cofibrations $a : Z \longrightarrow \bar{Z}$ and $b : Z' \longrightarrow \bar{Z}$, i.e., such that af is homotopic to $bf' : A \longrightarrow \bar{Z}'$, and $a\tau$ is homotopic to $b\tau'$. And we suppose that also $(f',\tau') \approx (f'',\tau'')$ via two acyclic cofibrations $a' : Z' \longrightarrow \bar{Z}'$ and $b' : Z'' \longrightarrow \bar{Z}'$, i.e., such that $a'f'$ is homotopic to $b'f'' : A \longrightarrow \bar{Z}'$, and $a'\tau'$ is homotopic to $b'\tau''$.

Because b and a' are acyclic cofibrations, we can choose a pushout:

$$\begin{tikzcd} Z \arrow[r, tail, "a'", "\sim"'] \arrow[d, tail, "b"', "\sim"] & \bar{Z}' \arrow[d, tail, "\beta", "\sim"'] \\ \bar{Z} \arrow[r, tail, "\sim", "\alpha"'] & E \end{tikzcd}$$

Moreover, the morphisms α and β are also acyclic cofibrations. The homotopy relation is compatible with postcomposition, so

$$\alpha a f \sim \alpha b f' = \beta a' f' \sim \beta b' f'' \qquad \text{and} \qquad \alpha a \tau \sim \alpha b \tau' = \beta a' \tau' \sim \beta b' \tau'' \ .$$

Since the homotopy relation is transitive, the acyclic cofibrations $\alpha a : Z \longrightarrow E$ and $\beta b' : Z'' \longrightarrow E$ witness that $(f,\tau) \approx (f'',\tau'')$. □

Construction 5.9. We continue with the definition of the category $\mathrm{Ho}(\mathcal{C})$. Morphisms in $\mathrm{Ho}(\mathcal{C})$ from A to B are equivalence classes under the relation $\approx$ of pairs $(f : A \longrightarrow Z, \tau : B \longrightarrow Z)$ such that τ is an acyclic cofibration. We write

$$\tau\backslash f \ : \ A \ \longrightarrow \ B$$

for the equivalence class of the pair (f,τ).

Now we define composition in the category $\mathrm{Ho}(\mathcal{C})$. We consider two pairs of morphisms (f,τ) and (g,σ) that represent morphisms $\tau\backslash f : A \longrightarrow B$ and $\sigma\backslash g : B \longrightarrow C$ in $\mathrm{Ho}(\mathcal{C})$. Because $\tau : B \longrightarrow Z$ is an acyclic cofibration, there is a pushout in $\mathcal{C}$:

$$\begin{tikzcd} B \arrow[r, "g"] \arrow[d, tail, "\tau"', "\sim"] & Y \arrow[d, tail, "\psi", "\sim"'] \\ Z \arrow[r, "\varphi"'] & W \end{tikzcd} \tag{5.10}$$

Moreover, the morphism ψ is an acyclic cofibration, too. We then define the composite by

$$(\sigma\backslash g) \circ (\tau\backslash f) \ = \ (\psi\sigma)\backslash(\varphi f) \ . \tag{5.11}$$

Theorem 5.12. *Let $\mathcal{C}$ be a cofibration category.*

(i) *The composition* (5.11) *is well-defined and makes* $\mathrm{Ho}(\mathcal{C})$ *into a category.*
(ii) *The assignments $\gamma(A) = A$ and $\gamma(f) = \mathrm{Id}\backslash f$ define a functor $\gamma : \mathcal{C} \longrightarrow \mathrm{Ho}(\mathcal{C})$.*
(iii) *For every acyclic cofibration $\tau : B \longrightarrow Z$ in $\mathcal{C}$, the morphism $\gamma(\tau)$ is an isomorphism with inverse $\tau\backslash \mathrm{Id}_Z$, and the relation*
$$\tau\backslash f \ = \ \gamma(\tau)^{-1} \circ \gamma(f)$$
holds for all $\mathcal{C}$-morphisms $f : A \longrightarrow Z$.
(iv) *The functor $\gamma : \mathcal{C} \longrightarrow \mathrm{Ho}(\mathcal{C})$ takes weak equivalences to isomorphisms.*
(v) *The functor $\gamma : \mathcal{C} \longrightarrow \mathrm{Ho}(\mathcal{C})$ is a localization of $\mathcal{C}$ at the class of weak equivalence.*

Proof. (i) We start by observing that the composition (5.11) does not depend on the choice of pushout (5.10). Indeed, any two choices of pushout are canonically isomorphic, and the resulting pairs $(\varphi f, \psi\sigma)$ are then $\approx$-equivalent via isomorphisms.

The equivalence relation $\approx$ is generated by three 'elementary' instances:

(1) for every acyclic cofibration $a : Z \longrightarrow \bar{Z}$, the pair (f,τ) is equivalent to $(af, a\tau)$;
(2) for all pairs of homotopic morphisms $f, f' : A \longrightarrow Z$, the pair (f,τ) is equivalent to (f',τ);

(3) for all pairs of homotopic acyclic cofibrations $\tau, \tau' : B \longrightarrow Z$, the pair (f, τ) is equivalent to (f, τ').

So it suffices to show that pre- and postcomposition is compatible with each of these elementary relations.

We start with postcomposition by the morphism represented by a pair (g, σ), for a $\mathcal{C}$-morphisms $g : B \longrightarrow Y$ and an acyclic cofibration $\sigma : C \longrightarrow Y$.

Relation (1): We choose two pushout squares in $\mathcal{C}$:

$$\begin{tikzcd} B \arrow[r, "g"] \arrow[d, tail, "\tau"', "\sim"] & Y \arrow[d, tail, "\psi", "\sim"'] \\ Z \arrow[r, "\varphi"'] \arrow[d, tail, "a"', "\sim"] & W \arrow[d, tail, "\lambda", "\sim"'] \\ \bar{Z} \arrow[r, "\kappa"'] & V \end{tikzcd}$$

The composite is then a pushout, too. So

$$(\sigma\backslash g) \circ ((a\tau)\backslash(af)) \;=\; (\lambda\psi\sigma)\backslash(\kappa a f) \;=\; (\lambda\psi\sigma)\backslash(\lambda\varphi f) \;=\; (\psi\sigma)\backslash(\varphi f) \;=\; (\sigma\backslash g) \circ (\tau\backslash f) \ .$$

Relation (2): We choose a pushout (5.10). Postcomposition preserves the homotopy relation; because $f, f' : A \longrightarrow Z$ are homotopic, so are $\varphi f, \varphi f' : A \longrightarrow W$. Hence $(\varphi f, \psi\sigma) \approx (\varphi f', \psi\sigma)$.

Relation (3): we choose three pushouts

$$\begin{tikzcd} & B \arrow[r, tail, "\tau", "\sim"'] \arrow[d, "g"'] & Z \arrow[d, "\varphi"] \\ B \arrow[r, "g"] \arrow[d, tail, "\tau'"', "\sim"] & Y \arrow[r, tail, "\sim", "\psi"'] \arrow[d, tail, "\psi'", "\sim"'] & W \arrow[d, tail, "\alpha", "\sim"'] \\ Z \arrow[r, "\varphi'"'] & W' \arrow[r, tail, "\sim", "\beta"'] & V \end{tikzcd}$$

Because τ and τ' are acyclic cofibrations, so are the morphisms ψ, ψ', α and β. Postcomposition preserves the homotopy relation; since τ and τ' are homotopic, we conclude that

$$\alpha\varphi\tau \;=\; \alpha\psi g \;=\; \beta\psi' g \;=\; \beta\varphi'\tau' \;\sim\; \beta\varphi'\tau \ .$$

Because τ is a weak equivalence, Proposition 5.5 (iv) shows that $\alpha\varphi : Z \longrightarrow V$ is homotopic to $\beta\varphi'$.

Unfortunately, precomposition with $f : A \longrightarrow Z$ need not preserve the homotopy relation. However, Proposition 5.5 (iii) provides an acyclic cofibration $s : V \longrightarrow V'$ such that $s\alpha\varphi f : A \longrightarrow V'$ is homotopic to $s\beta\varphi' f$. Moreover,

$$s\alpha\psi\sigma \;=\; s\beta\psi'\sigma \;:\; C \;\longrightarrow\; V' \ .$$

So the acyclic cofibrations $s\alpha : W \longrightarrow V'$ and $s\beta : W' \longrightarrow V'$ witness that $(\varphi f, \psi\sigma) \approx (\varphi' f, \psi'\sigma)$.

Now we turn to precomposition by the morphism represented by a pair (e, ν), for a $\mathcal{C}$-morphism $e : E \longrightarrow X$ and an acyclic cofibration $\nu : A \longrightarrow X$.

Relation (1): We choose two pushout squares

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, tail, "\nu"', "\sim"] & Z \arrow[r, "a", "\sim"'] \arrow[d, tail, "\xi", "\sim"'] & \bar{Z} \arrow[d, tail, "\zeta", "\sim"'] \\ X \arrow[r, "\chi"'] & U \arrow[r, "\mu"'] & V \end{tikzcd}$$

The composite is then a pushout, too. So

$$((a\tau)\backslash(af)) \circ (\nu\backslash e) \;=\; (\zeta a\tau)\backslash(\mu\chi e) \;=\; (\mu\xi\tau)\backslash(\mu\chi e) \;=\; (\xi\tau)\backslash(\chi e) \;=\; (\tau\backslash f) \circ (\nu\backslash e) \ .$$

Relation (2): we choose three pushouts

$$\begin{tikzcd}
& A \arrow[r, tail, "\nu", "\sim"'] \arrow[d, "f"'] & X \arrow[d, "\chi"] \\
A \arrow[r, "f'"] \arrow[d, tail, "\nu"', "\sim"] & Z \arrow[r, tail, "\sim", "\xi"'] \arrow[d, tail, "\sim"', "\xi'"] & U \arrow[d, tail, "\sim"', "\alpha"] \\
X \arrow[r, "\chi'"'] & U' \arrow[r, tail, "\sim", "\beta"'] & V
\end{tikzcd}$$

Because ν is an acyclic cofibration, so are the morphisms ξ, ξ', α and β. Postcomposition preserves the homotopy relation; since f is homotopic to f', we conclude that

$$\alpha\chi\nu \;=\; \alpha\xi f \;=\; \beta\xi' f \;\sim\; \beta\xi' f' \;=\; \beta\chi'\nu \;.$$

Because ν is a weak equivalence, Proposition 5.5 (iv) shows that $\alpha\chi : X \longrightarrow V$ is homotopic to $\beta\chi'$. Postcomposition preserves the homotopy relation, so also

$$\alpha\chi e \;\sim\; \beta\chi' e \;.$$

Moreover,

$$\alpha\xi\tau \;=\; \beta\xi'\tau \;:\; B \;\longrightarrow\; V \;.$$

So the acyclic cofibrations $\alpha : U \longrightarrow V$ and $\beta : U' \longrightarrow V$ witness that $(\chi e, \xi\tau) \approx (\chi' e, \xi'\tau)$.

Relation (3): We choose a pushout

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, tail, "\nu"', "\sim"] & Z \arrow[d, tail, "\sim"', "\xi"] \\
X \arrow[r, "\chi"'] & U
\end{tikzcd}$$

Postcomposition preserves the homotopy relation. So if $\tau, \tau' : B \longrightarrow Z$ are homotopic, so are $\xi\tau, \xi\tau' : B \longrightarrow U$. Hence $(\chi e, \xi\tau) \approx (\chi e, \xi\tau')$.

Now that we know that composition in $\mathrm{Ho}(\mathcal{C})$ is well-defined, we can easily check that it is associative. We consider three pairs (e, ν), (f, τ) and (g, σ) that represent composable fractions. We choose three pushouts

$$\begin{tikzcd}
& B \arrow[r, "g"] \arrow[d, tail, "\tau"', "\sim"] & Y \arrow[d, tail, "\sim"', "\psi"] \\
A \arrow[r, "f"] \arrow[d, tail, "\nu"', "\sim"] & Z \arrow[r, "\varphi"'] \arrow[d, tail, "\sim"', "\xi"] & W \arrow[d, tail, "\sim"', "\lambda"] \\
X \arrow[r, "\chi"'] & U \arrow[r, "\mu"'] & V
\end{tikzcd}$$

Then the two composite squares are pushouts, too. So

$$\begin{aligned}
(\sigma\backslash g) \circ ((\tau\backslash f) \circ (\nu\backslash e)) \;&=\; (\sigma\backslash g) \circ ((\xi\tau)\backslash(\chi e)) \;=\; (\lambda\psi\sigma)\backslash(\mu\chi e) \\
&=\; ((\psi\sigma)\backslash(\varphi f)) \circ (\nu\backslash e) \;=\; ((\sigma\backslash g) \circ (\tau\backslash f)) \circ (\nu\backslash e) \;.
\end{aligned}$$

For every object A, the fraction $\mathrm{Id}_A\backslash\mathrm{Id}_A$ is clearly a two-sided unit for composition.

(ii) The proof that γ is indeed a functor is straightforward.

(iii) To calculate the composite $(\tau\backslash \mathrm{Id}_Z) \circ \gamma(\tau) = (\tau\backslash \mathrm{Id}_Z) \circ (\mathrm{Id}_Z \backslash\tau)$ we must choose a pushout of two instances of the identity of Z; the square consisting of four instances of Id_Z does the job, and it yields the relation

$$(\tau\backslash \mathrm{Id}_Z) \circ \gamma(\tau) \;=\; (\tau\backslash \mathrm{Id}_Z) \circ (\mathrm{Id}_Z \backslash\tau) \;=\; \tau\backslash\tau \;=\; \mathrm{Id}_B \backslash \mathrm{Id}_B \;.$$

In order to calculate the other composite, we choose a pushout

$$\begin{tikzcd} B \arrow[r, tail, "\tau", "\sim"'] \arrow[d, tail, "\tau"', "\sim"] & Z \arrow[d, tail, "\sim"', "\psi_1"] \\ Z \arrow[r, tail, "\sim", "\psi_0"'] & Z \cup_B Z \end{tikzcd}$$

Because $\psi_0\tau = \psi_1\tau$ and τ is a weak equivalence, Proposition 5.5 (iv) shows that ψ_0 and ψ_1 are homotopic. The acyclic cofibration $\psi_1 : Z \longrightarrow Z \cup_B Z$ thus witnesses that $(\mathrm{Id}_Z, \mathrm{Id}_Z) \approx (\psi_0, \psi_1)$. So

$$\gamma(\tau) \circ (\tau\backslash \mathrm{Id}_Z) \ = \ (\mathrm{Id}_Z \backslash \tau) \circ (\tau\backslash \mathrm{Id}_Z) \ = \ \psi_1\backslash\psi_0 \ = \ \mathrm{Id}_Z \backslash \mathrm{Id}_Z \ .$$

(iv) We **claim** that every $\mathcal{C}$-morphism $f : A \longrightarrow B$ admits a factorization $f = qj$ such that $j : A \longrightarrow Z$ is a cofibration and the morphism $q : Z \longrightarrow B$ is left inverse to an acyclic cofibration $r : B \longrightarrow Z$. To prove the claim we use an abstract version of the mapping cylinder factorization. We choose a cylinder object (I, i_0, i_1, p) for A, and form a pushout

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, tail, "i_1"', "\sim"] & B \arrow[d, tail, "\sim"', "r"] \\ I \arrow[r, "\varphi"'] & I \cup_f B = Z \end{tikzcd}$$

The universal property of the pushout provides a unique morphism $q : Z \longrightarrow B$ such that $q\varphi = fp$ and $qr = \mathrm{Id}_B$. In particular, q is left inverse to the acyclic cofibration r. We define $j = \varphi i_0 : A \longrightarrow Z$. Then

$$qj \ = \ q\varphi i_0 \ = \ fpi_1 \ = \ f \ .$$

Moreover, the square

$$\begin{tikzcd} A \amalg B \arrow[r, "\mathrm{Id}_A \amalg f"] \arrow[d, tail, "i_0+i_1"'] & A \amalg B \arrow[d, tail, "j+r"] \\ I \arrow[r, "\varphi"'] & I \cup_f B = Z \end{tikzcd}$$

is a pushout; so $j + r : A \amalg B \longrightarrow Z$ is a cofibration. Since the canonical morphism $A \longrightarrow A \amalg B$ is a cofibration, too, so is the morphism j.

Now we can prove part (iv). We factor the given weak equivalence $f : A \longrightarrow B$ as provided by the above claim, so that $f = qj$, and $qr = \mathrm{Id}_B$ for an acyclic cofibration $r : B \longrightarrow Z$. Because f and q are weak equivalences, so is j; hence j is an acyclic cofibration. Thus $\gamma(r)$ and $\gamma(j)$ are isomorphisms by part (iii). Because

$$\gamma(q) \circ \gamma(r) \ = \ \gamma(qr) \ = \ \mathrm{Id}_B \ ,$$

the morphism $\gamma(q)$ is an isomorphism, and inverse to $\gamma(r)$. So

$$\gamma(f) \ = \ \gamma(q) \circ \gamma(j) \ = \ \gamma(r)^{-1} \circ \gamma(j) \ = \ r\backslash j$$

is an isomorphism.

(v) We let $F : \mathcal{C} \longrightarrow \mathcal{D}$ be any functor that takes weak equivalences to isomorphisms. We have to show that F factors uniquely through the functor $\gamma : \mathcal{C} \longrightarrow \mathrm{Ho}(\mathcal{C})$.

Claim: let $f, g : A \longrightarrow B$ be homotopic $\mathcal{C}$-morphisms. Then $F(f) = F(g)$. Indeed, there is a cylinder object (I, i_0, i_1, p) for A, and a homotopy $H : I \longrightarrow B$ such that

$$Hi_0 \ = \ f \qquad \text{and} \qquad Hi_1 \ = \ g \ .$$

Because p is a weak equivalence, the morphism $F(p) : FI \longrightarrow FA$ is an isomorphism. Because $F(p)$ is an isomorphism and

$$F(p) \circ F(i_0) \ = \ F(p \circ i_0) \ = \ F(\mathrm{Id}_A) \ = \ F(p \circ i_1) \ = \ F(p) \circ F(i_1) \ ,$$

we conclude that $F(i_0) = F(i_1)$. Hence

$$F(f) \;=\; F(H)\circ F(i_0) \;=\; F(H)\circ F(i_1) \;=\; F(g)\ .$$

Now we can prove the uniqueness property of a localization. Let $G : \mathrm{Ho}(\mathcal{C}) \longrightarrow \mathcal{D}$ be any functor such that $G\circ\gamma = F$. Because γ is the identity on objects, G agrees with F on objects. On morphisms we have

$$G(\tau\backslash f) \;=\; G(\gamma(\tau)^{-1}\circ\gamma(f)) \;=\; G(\gamma(\tau))^{-1}\circ G(\gamma(f)) \;=\; F(\tau)^{-1}\circ F(f)\ .$$

So the effect of G on morphisms is determined by the effect of F on morphisms.

For the existence part of a localization, we define $G : \mathrm{Ho}(\mathcal{C}) \longrightarrow \mathcal{D}$ on objects by $G(A) = F(A)$. On morphisms, we define

$$G(\tau\backslash f) \;=\; F(\tau)^{-1}\circ F(f)\ .$$

To show that this is well-defined, we suppose that $(f,\tau) \approx (f',\tau')$. Since the equivalence relation $\approx$ is generated by the three elementary instances, it suffices to check these elementary cases.

For the relation (1) we let $a : Z \longrightarrow \bar{Z}$ is any acyclic cofibration. Then $F(a)$ is an isomorphism, and hence

$$F(a\tau)^{-1}\circ F(af) \;=\; F(\tau)^{-1}\circ F(a)^{-1}\circ F(a)\circ F(f) \;=\; F(\tau)^{-1}\circ F(f)\ .$$

Relations (2) and (3) are also fine because F takes the same value on homotopic morphisms.

Now we argue that G is indeed a functor. Clearly $G(1_A\backslash 1_A) = \mathrm{Id}_{F(A)}$, so G preserves identities. Now we let (f,τ) and (g,σ) represent two composable morphism $\tau\backslash f : A \longrightarrow B$ and $\sigma\backslash g : B \longrightarrow C$ in $\mathrm{Ho}(\mathcal{C})$. We choose a pushout square (5.10). Then

$$F(\varphi)\circ F(\tau) \;=\; F(\psi)\circ F(g)\ .$$

Because τ and ψ are weak equivalences, $F(\tau)$ and $F(\psi)$ are isomorphisms, and thus

$$F(\psi)^{-1}\circ F(\varphi) \;=\; F(g)\circ F(\tau)^{-1}\ .$$

Hence

$$\begin{aligned} G((\sigma\backslash g)\circ(\tau\backslash f)) \;&=\; G((\psi\sigma)\backslash(\varphi f)) \;=\; F(\psi\sigma)^{-1}\circ F(\varphi f)\\ &=\; F(\sigma)^{-1}\circ F(\psi)^{-1}\circ F(\varphi)\circ F(f)\\ &=\; F(\sigma)^{-1}\circ F(g)\circ F(\tau)^{-1}\circ F(f) \;=\; G(\sigma\backslash g)\circ G(\tau\backslash f))\ . \end{aligned}$$

□

The specific construction of the localization $\gamma : \mathcal{C} \longrightarrow \mathrm{Ho}(\mathcal{C})$ immediately implies the following corollary, which we will refer to as a 'calculus of fractions' for $\mathrm{Ho}(\mathcal{C})$. The theorem states the main properties of the localization functor without reference to the specific construction.

Theorem 5.13 (Calculus of fractions)**.** *Let $\mathcal{C}$ be a cofibration category and $\gamma : \mathcal{C} \longrightarrow \mathrm{Ho}(\mathcal{C})$ a localization at the class of weak equivalences. Then:*

(i) *Every morphism in $\mathrm{Ho}(\mathcal{C})$ is a 'left fraction', i.e., is of the form $\gamma(\tau)^{-1}\circ\gamma(f)$, where f and τ are $\mathcal{C}$-morphisms with the same target, and τ is an acyclic cofibration.*

(ii) *Given two morphisms $f, g : A \longrightarrow B$ in $\mathcal{C}$, then $\gamma(f) = \gamma(g)$ in $\mathrm{Ho}(\mathcal{C})$ if and only if there is an acyclic cofibration $s : B \longrightarrow \bar{B}$ such that sf and sg are homotopic.*

Proof. Part (i) is a restatement of the relation $\tau\backslash f = \gamma(\tau)^{-1}\circ\gamma(f)$.

(ii) If there is an acyclic cofibration $s : B \longrightarrow \bar{B}$ with $sf \sim sg$, then

$$\gamma(s)\circ\gamma(f) \;=\; \gamma(sf) \;=\; \gamma(sg) \;=\; \gamma(s)\circ\gamma(g)\ .$$

We have exploited that every functor that inverts weak equivalences takes the same value on homotopic morphisms, compare the proof of Theorem 5.12. Because s is a weak equivalence, the morphism $\gamma(s)$ is an isomorphism, and hence $\gamma(f) = \gamma(g)$.

Conversely, the relation $\gamma(f) = \gamma(g)$ means that $(f, \mathrm{Id}_B) \approx (g, \mathrm{Id}_B)$. So there are homotopic acyclic cofibrations $a, b : B \longrightarrow \bar{Z}$ such that af is homotopic to bg. We choose a pushout in $\mathcal{C}$:

$$\begin{tikzcd} B \arrow[r, tail, "b", "\sim"'] \arrow[d, tail, "a"', "\sim"] & \bar{Z} \arrow[d, tail, "\psi", "\sim"'] \\ \bar{Z} \arrow[r, tail, "\varphi"'] & W \end{tikzcd}$$

Postcomposition preserves the homotopy relation, so we conclude that

$$\varphi a \ = \ \psi b \ \sim \ \psi a \ .$$

Because a is a weak equivalence, Proposition 5.5 (iv) shows that φ and ψ are homotopic. Proposition 5.5 (iii) provides an acyclic cofibration $t : W \longrightarrow \bar{B}$ such that

$$t\varphi af \sim t\psi af \ .$$

Because the homotopy relation is transitive, the relations

$$t\varphi af \ \sim \ t\psi af \ \sim \ t\psi bg \ = \ t\varphi ag$$

show that $s = t\varphi a : B \longrightarrow \bar{B}$ can serve as the desired acyclic cofibration. □

Remark 5.14. On the face of it, the homotopy category of a cofibration category raises set-theoretic issues: in general the hom-'sets' in $\mathrm{Ho}(\mathcal{C})$ may not be small, but rather proper classes. One way to deal with this is to work with universes in the sense of Grothendieck; the homotopy category of a cofibration category then always exists in a larger universe. Another way to address the set theory issues is to restrict attention to those cofibration categories that have 'enough fibrant objects'. An object of a cofibration category $\mathcal{C}$ is *fibrant* if every acyclic cofibration out of it has a retraction. If the object B is fibrant, then the map $\gamma : \mathcal{C}(A, B) \longrightarrow \mathrm{Ho}(\mathcal{C})(A, B)$ given by the localization functor is surjective: an arbitrary morphism from A to B in $\mathrm{Ho}(\mathcal{C})$ is of the form $\gamma(s)^{-1}\gamma(a)$ for some acyclic cofibration $s : B \longrightarrow Z$. Since B is fibrant, there is a retraction $r : Z \longrightarrow B$ with $rs = \mathrm{Id}_B$, and then $\gamma(s)^{-1}\gamma(a) = \gamma(ra)$. Moreover, if two $\mathcal{C}$-morphisms $f, g : A \longrightarrow B$ become equal after applying the functor γ, then there is an acyclic cofibration $s : B \longrightarrow \bar{B}$ such that sf is homotopic to sg. Composing with any retraction to s shows that f is already homotopic to g. So the map $\mathcal{C}(A, B)/\text{homotopy} \longrightarrow \mathrm{Ho}(\mathcal{C})(A, B)$ sending the class of f to $\gamma(f)$, is bijective. We say that the cofibration category $\mathcal{C}$ has *enough fibrant objects* if, for every object X, there is a weak equivalence $r : X \longrightarrow Z$ with fibrant target. For example, if $\mathcal{C}$ is the collection of cofibrant objects in an ambient Quillen model category, then it has enough fibrant objects.

If $r : X \longrightarrow Z$ is a weak equivalence with fibrant target, then for every other object A the two maps

$$\mathrm{Ho}(\mathcal{C})(A, X) \ \xrightarrow{\gamma(r)_*} \ \mathrm{Ho}(\mathcal{C})(A, Z) \ \xleftarrow{\ \gamma\ } \ \mathcal{C}(A, Z)/\text{homotopy}$$

are bijective, so the morphisms $\mathrm{Ho}(\mathcal{C})(A, X)$ form a set (as opposed to a proper class). So, if $\mathcal{C}$ has enough fibrant objects, then the homotopy category $\mathrm{Ho}(\mathcal{C})$ has small hom-sets (or is 'locally small').

Now that we have a good handle on the homotopy category of a cofibration category, we can use the calculus of fractions to derive additional desirable properties. The homotopy category will typically have only very few limits and colimits. But it always has finite coproducts, and general coproducts and finite products are inherited from the cofibration category.

Proposition 5.15. *Let $\mathcal{C}$ be a cofibration category.*

(i) *The localization functor $\gamma : \mathcal{C} \longrightarrow \mathrm{Ho}(\mathcal{C})$ preserves finite coproducts. In particular, the homotopy category $\mathrm{Ho}(\mathcal{C})$ has finite coproducts.*
(ii) *Let I be any set. Suppose that $\mathcal{C}$ has I-indexed coproducts, and that the classes of cofibrations and acyclic cofibrations are closed under I-indexed coproducts. Then the localization functor $\gamma : \mathcal{C} \longrightarrow \mathrm{Ho}(\mathcal{C})$ preserves I-indexed coproducts. In particular, the homotopy category $\mathrm{Ho}(\mathcal{C})$ has I-indexed coproducts.*

Proof. (i) A cofibration category always has finite coproducts, and cofibrations and acyclic cofibrations are closed under finite coproducts. So part (i) is a special case of part (ii).

(ii) Since the localization functor γ can be arranged to be the identity on objects, we drop γ in front of objects to simplify the notation. Now we consider an I-indexed family $\{X_i\}_{i\in I}$ of $\mathcal{C}$-objects. We denote a coproduct of the family by $\bigvee_{i\in I} X_i$, and we write $\kappa_j : X_j \longrightarrow \bigvee_{i\in I} X_i$ for the universal morphisms. We must show that for every $\mathcal{C}$-object Y, the map

$$\mathrm{Ho}(\mathcal{C})(\bigvee_{i\in I} X_i, Y) \longrightarrow \prod_{j\in I} \mathrm{Ho}(\mathcal{C})(X_j, Y) , \quad \psi \longmapsto (\psi\circ\gamma(\kappa_j))_{j\in I} \tag{5.16}$$

is bijective. For surjectivity we let $(\psi_j : X_j \longrightarrow Y)$ be any I-indexed family of morphisms in $\mathrm{Ho}(\mathcal{C})$. By the calculus of left fractions, we can write

$$\psi_j \;=\; \gamma(s_j)^{-1}\circ\gamma(f_j)$$

for some families of $\mathcal{C}$-morphisms $f_j : X_j \longrightarrow W_j$ and $s_j : Y \longrightarrow W_j$ such that the morphisms s_j are acyclic cofibrations. We choose a coproduct of the family $\{W_i\}_{i\in I}$ and a coproduct of the constant family $\{Y\}_{i\in I}$ of copies of Y. Then we form the $\mathcal{C}$-morphisms

$$\bigvee_{i\in I} X_i \xrightarrow{\bigvee f_i} \bigvee_{i\in I} W_i \xleftarrow[\sim]{\bigvee s_i} \bigvee_{i\in I} Y \xrightarrow{\nabla} Y ,$$

where ∇ denotes the fold morphism. Since coproducts of acyclic cofibrations are acyclic cofibrations, the middle morphism is an acyclic cofibration. So we can form the morphism

$$\gamma(\nabla)\circ\gamma(\bigvee s_i)^{-1}\circ\gamma(\bigvee f_i) \;:\; \bigvee_{i\in I} X_i \;\longrightarrow\; Y$$

in the homotopy category. Then for every $j\in J$, the following diagram commutes:

$$\begin{tikzcd}
X_j \arrow[r, "f_j"] \arrow[d, "\kappa_j"'] & W_j \arrow[d, "\kappa_j"'] & Y \arrow[l, tail, "s_j"', "\sim"] \arrow[d, "\kappa_j"'] \arrow[dr, equal] & \\
\bigvee_{i\in I} X_i \arrow[r, "\bigvee f_i"'] & \bigvee_{i\in I} W_i & \bigvee_{i\in I} Y \arrow[l, tail, "\sim", "\bigvee s_i"'] \arrow[r, "\nabla"'] & Y
\end{tikzcd}$$

Hence

$$\begin{aligned}
\gamma(\nabla)\circ\gamma(\bigvee s_i)^{-1}\circ\gamma(\bigvee f_i)\circ\gamma(\kappa_j) \;&=\; \gamma(\nabla)\circ\gamma(\bigvee s_i)^{-1}\circ\gamma((\bigvee f_i)\circ\kappa_j)\\
&=\; \gamma(\nabla)\circ\gamma(\bigvee s_i)^{-1}\circ\gamma(\kappa_j)\circ\gamma(f_j)\\
&=\; \gamma(\nabla)\circ\gamma(\kappa_j)\circ\gamma(s_j)^{-1}\circ\gamma(f_j) \;=\; \gamma(s_j)^{-1}\circ\gamma(f_j) \;=\; \psi_j \ .
\end{aligned}$$

So the map (5.16) sends the morphism $\gamma(\nabla)\circ\gamma(\bigvee s_i)^{-1}\circ\gamma(\bigvee f_i)$to the original family $(\psi_j)_{j\in I}$, and thus the map (5.16) is surjective.

For injectivity we consider two morphisms $\psi,\psi' : \bigvee_{i\in I} X_i \longrightarrow Y$ in $\mathrm{Ho}(\mathcal{C})$ such that $\psi\circ\gamma(\kappa_j) = \psi'\circ\gamma(\kappa_j)$ for all $j\in I$. We start with the special case where $\psi = \gamma(f)$ and $\psi' = \gamma(f')$ for two $\mathcal{C}$-morphisms $f,f' : \bigvee_{i\in I} X_i \longrightarrow Y$. Because

$$\gamma(f\kappa_j) \;=\; \psi\circ\gamma(\kappa_j) \;=\; \psi'\circ\gamma(\kappa_j) = \gamma(f'\kappa_j) ,$$

the calculus of left fractions provides acyclic cofibrations $t_j : Y \longrightarrow \bar{Y}_j$ such that $t_j f\kappa_j : X_i \longrightarrow \bar{Y}_j$ is homotopic to $t_j f'\kappa_j : X_j \longrightarrow \bar{Y}_j$ for every $j\in I$. We choose a pushout:

$$\begin{tikzcd}
\bigvee_{i\in I} Y \arrow[r, "\nabla"] \arrow[d, tail, "\bigvee t_i"', "\sim"] & Y \arrow[d, tail, "t", "\sim"'] \\
\bigvee_{i\in I} \bar{Y}_i \arrow[r, "\nabla'"'] & Y'
\end{tikzcd}$$

Since coproducts of acyclic cofibrations are acyclic cofibrations, the left vertical morphism is an acyclic cofibration, and hence so is the right vertical morphism $t : Y \longrightarrow Y'$.

For each $j \in I$, we choose a cylinder object Z_j of X_j and a homotopy $H_j : Z_j \longrightarrow \bar{Y}_j$ from $t_j f \kappa_j$ to $t_j f' \kappa_j$. Since coproducts preserve cofibrations and acyclic cofibrations, the coproduct $\bigvee_{i\in I} Z_i$ is a cylinder object for $\bigvee_{i\in I} X_i$, where we leave the additional data of a cylinder object implicit. Moreover, the composite

$$\bigvee_{i\in I} Z_i \xrightarrow{\bigvee H_i} \bigvee_{i\in I} \bar{Y}_i \xrightarrow{\nabla'} Y'$$

is then a homotopy from tf to tf'. We conclude that $\gamma(tf) = \gamma(tf')$ in $\mathrm{Ho}(\mathcal{C})$. Since t is a weak equivalence, $\gamma(t)$ is an isomorphism in $\mathrm{Ho}(\mathcal{C})$, and so $\gamma(f) = \gamma(f')$. This proves injectivity in the special case.

Now we treat the general case, and we let $\psi, \psi' : \bigvee_{i\in I} X_i \longrightarrow Y$ be arbitrary morphisms in $\mathrm{Ho}(\mathcal{C})$ such that $\psi \circ \gamma(\kappa_j) = \psi' \circ \gamma(\kappa_j)$ for all $j \in I$. The calculus of left fractions provides $\mathcal{C}$-morphisms $f : \bigvee_{i\in I} X_i \longrightarrow W$, $f' : \bigvee_{i\in I} X_i \longrightarrow W'$, $s : Y \longrightarrow W$ and $s' : Y \longrightarrow W'$ such that s and s' are acyclic cofibrations and such that

$$\psi \;=\; \gamma(s)^{-1} \circ \gamma(f) \qquad \text{and} \qquad \psi' \;=\; \gamma(s')^{-1} \circ \gamma(f') \;.$$

We choose a pushout:

$$\begin{tikzcd} Y \arrow[r, tail, "s", "\sim"'] \arrow[d, tail, "s'"', "\sim"] & W \arrow[d, tail, "t", "\sim"'] \\ W' \arrow[r, tail, "\sim", "t'"'] & V \end{tikzcd}$$

Then t and t' are acyclic cofibrations because s and s' are. We now obtain the relation

$$\gamma(tf) \circ \gamma(\kappa_j) \;=\; \gamma(t) \circ \gamma(s) \circ \psi \circ \gamma(\kappa_j) \;=\; \gamma(t') \circ \gamma(s') \circ \psi' \circ \gamma(\kappa_j) \;=\; \gamma(t'f') \circ \gamma(\kappa_j)$$

for every $j \in I$. The special case treated above lets us conclude that $\gamma(tf) = \gamma(t'f')$. Thus

$$\gamma(ts) \circ \psi \;=\; \gamma(tf) \;=\; \gamma(t'f') \;=\; \gamma(t's') \circ \psi' \;.$$

Because the morphism $\gamma(ts) = \gamma(t's')$ is an isomorphism, also $\psi = \psi'$. This completes the proof. □

Theorem 5.17. *Let $\mathcal{C}$ be a cofibration category.*

(i) *The localization functor $\gamma : \mathcal{C} \longrightarrow \mathrm{Ho}(\mathcal{C})$ preserves terminal objects.*
(ii) *Let I be any set. Suppose that $\mathcal{C}$ has I-indexed products, and that the class of weak equivalences is stable under I-indexed products. Then the localization functor $\gamma : \mathcal{C} \longrightarrow \mathrm{Ho}(\mathcal{C})$ preserves I-indexed products. In particular, $\mathrm{Ho}(\mathcal{C})$ has I-indexed products.*

Proof. (i) Logically speaking, part (i) is a special case of part (ii) for $I = \emptyset$. However, part (i) can easily be proved directly, as follows. We let $*$ be a terminal object of $\mathcal{C}$. We write $p_A : A \longrightarrow *$ for the unique morphism from a $\mathcal{C}$-object A to $*$. By the calculus of fractions, any morphism from A to $*$ in $\mathrm{Ho}(\mathcal{C})$ is of the form $\gamma(s)^{-1} \circ \gamma(f)$ for some $\mathcal{C}$-morphism $f : A \longrightarrow C$ and a weak equivalence $s : * \longrightarrow C$. Because $*$ is terminal, the unique morphism $p_C : C \longrightarrow *$ is right inverse to s, and hence also a weak equivalence. Hence

$$\gamma(s)^{-1} \circ \gamma(f) \;=\; \gamma(p_C) \circ \gamma(f) \;=\; \gamma(p_C \circ f) \;=\; \gamma(p_A) \;.$$

So $\gamma(p_A)$ is the only element of $\mathrm{Ho}(A, *)$, and hence $*$ is also a terminal object of $\mathrm{Ho}(\mathcal{C})$.

(ii) We consider any I-indexed family $\{X_i\}_{i\in I}$ of $\mathcal{C}$-objects. We let $\prod_{i\in I} X_i$ denote a product of the family, with projections $p_j : \prod_{i\in I} X_i \longrightarrow X_j$. We must show that for every $\mathcal{C}$-object A, the map

$$\mathrm{Ho}(\mathcal{C})(A, \prod_{i\in I} X_i) \;\longrightarrow\; \prod_{j\in I} \mathrm{Ho}(\mathcal{C})(A, X_j) \;, \quad \psi \;\longmapsto\; (\gamma(p_j) \circ \psi)_{j\in I} \tag{5.18}$$

is bijective.

For surjectivity we consider an I-family of morphism $\psi_j : A \longrightarrow X_j$ in $\mathrm{Ho}(\mathcal{C})$. The calculus of fractions lets us write $\psi_j = \gamma(s_j)^{-1} \circ \gamma(f_j)$ for some $\mathcal{C}$-morphisms $f_j : A \longrightarrow Z_j$ and weak equivalences $s_j : X_j \longrightarrow Z_j$.

The morphism $\prod_{i\in I} s_i : \prod_{i\in I} X_i \longrightarrow \prod_{i\in I} Z_i$ is then a weak equivalence by hypothesis. We claim that the map (5.18) takes the morphism

$$\gamma(\prod_{i\in I} s_i)^{-1} \circ \gamma((f_i)_{i\in I}) \ : \ A \ \longrightarrow \ \prod_{i\in I} X_i$$

to the given family $(\psi_i)_{i\in I}$. Indeed, the the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "(f_i)_{i\in I}"] \arrow[dr, bend right, "f_j"'] & \prod_{i\in I} Z_i \arrow[d, "p_j"] & \prod_{i\in I} X_i \arrow[l, "\prod s_i"', "\sim"] \arrow[d, "p_j"] \\ & Z_j & X_j \arrow[l, "\sim", "s_j"'] \end{tikzcd}$$

So

$$\gamma(p_j) \circ \gamma(\prod_{i\in I} s_i)^{-1} \circ \gamma((f_i)_{i\in I}) \ = \ \gamma(s_j)^{-1} \circ \gamma(p_j) \circ \gamma((f_i)_{i\in I}) \ = \ \gamma(s_j)^{-1} \circ \gamma(f_j) \ = \ \psi_j \ .$$

For injectivity we consider two morphisms $\varphi, \psi : A \longrightarrow \prod_{i\in I} X_i$ in $\mathrm{Ho}(\mathcal{C})$ such that

$$\gamma(p_j) \circ \varphi \ = \ \gamma(p_j) \circ \psi \tag{5.19}$$

for all $j \in I$. We must show that then $\varphi = \psi$. We start with the special case where $\varphi = \gamma(f)$ and $\psi = \gamma(g)$ for $\mathcal{C}$-morphisms $f, g : A \longrightarrow \prod_{i\in I} X_i$. Then

$$\gamma(p_j \circ f) \ = \ \gamma(p_j) \circ \gamma(f) \ = \ \gamma(p_j) \circ \gamma(g) \ = \ \gamma(p_j \circ g)$$

for all $j \in I$. The calculus of fractions provides weak equivalences $s_j : X_j \longrightarrow \bar{X}_j$ such that $s_j p_j f : A \longrightarrow \bar{X}_j$ is homotopic to $s_j p_j g$. We let

$$H^j \ : \ I^j \ \longrightarrow \ \bar{X}_j$$

be a homotopy, based on some cylinder object $(I^j, \iota_0^j, \iota_1^j, p^j)$ for A, that witnesses $s_j p_j f \sim s_j p_j g$.

Because the morphisms $p^j : I^j \longrightarrow A$ are weak equivalences, so is $\prod_{i\in I} p^i : \prod_{i\in I} I^i \longrightarrow \prod_{i\in I} A$. Hence $\gamma(\prod_{i\in I} p^i)$ is an isomorphism. The morphisms

$$\prod_{i\in I} \iota_0^i \ : \ \prod_{i\in I} A \ \longrightarrow \ \prod_{i\in I} I^i \qquad \text{and} \qquad \prod_{i\in I} \iota_1^i \ : \ \prod_{i\in I} A \ \longrightarrow \ \prod_{i\in I} I^i$$

are both right inverse to $\prod_{i\in I} p^i$; so the morphisms $\gamma(\prod_{i\in I} \iota_0^i)$ and $\gamma(\prod_{i\in I} \iota_1^i)$ are both right inverse to the isomorphism $\gamma(\prod_{i\in I} p^i)$. Hence

$$\gamma(\prod_{i\in I} \iota_0^i) \ = \ \gamma(\prod_{i\in I} \iota_1^i) \ .$$

We write $\Delta : A \longrightarrow \prod_{i\in I} A$ for the diagonal morphism, i.e., $p_i \circ \Delta = \mathrm{Id}_A$ for all $i \in I$. Then

$$\begin{aligned} \gamma(\prod s_i) \circ \gamma(f) \ &= \ \gamma(\prod (s_i p_i f)) \circ \gamma(\Delta) \\ &= \ \gamma(\prod (H^i \circ \iota_0^i)) \circ \gamma(\Delta) \\ &= \ \gamma(\prod H^i) \circ \gamma(\prod \iota_0^i) \circ \gamma(\Delta) \\ &= \ \gamma(\prod H^i) \circ \gamma(\prod \iota_1^i) \circ \gamma(\Delta) \\ &= \ \gamma(\prod (H^i \circ \iota_1^i)) \circ \gamma(\Delta) \\ &= \ \gamma(\prod (s_i p_i g)) \circ \gamma(\Delta) \ = \ \gamma(\prod s_i) \circ \gamma(g) \ . \end{aligned}$$

Because the morphisms s_j are all weak equivalences, their product is also a weak equivalence. Hence $\gamma(\prod_{i\in I} s_i)$ is an isomorphism, and we conclude that $\gamma(f) = \gamma(g)$.

Now we let $\varphi, \psi : A \longrightarrow \prod_{i\in I} X_i$ be arbitrary morphisms in $\mathrm{Ho}(\mathcal{C})$ that satisfy (5.19). The calculus of fractions provides $\mathcal{C}$-morphisms $f : A \longrightarrow Z$ and $g : A \longrightarrow Z'$ and acyclic cofibrations $s : \prod_{i\in I} X_i \longrightarrow Z$ and $t : \prod_{i\in I} X_i \longrightarrow Z'$ such that

$$\varphi \;=\; \gamma(s)^{-1} \circ \gamma(f) \quad \text{and} \qquad \psi \;=\; \gamma(t)^{-1} \circ \gamma(g) \ .$$

We choose a pushout:

$$\begin{tikzcd} \prod_{i\in I} X_i \arrow[r, tail, "s", "\sim"'] \arrow[d, tail, "t"', "\sim"] & Z \arrow[d, tail, "\bar{t}", "\sim"'] \\ Z' \arrow[r, tail, "\sim", "\bar{s}"'] & V \end{tikzcd}$$

Then $\bar{s}$ and $\bar{t}$ are acyclic cofibrations because s and t are. So

$$u \;=\; \bar{s}t \;=\; \bar{t}s \;:\; \prod_{i\in I} X_i \;\longrightarrow\; V$$

is an acyclic cofibration, too. For every $j \in I$ we choose a pushout:

$$\begin{tikzcd} \prod_{i\in I} X_i \arrow[r, tail, "u", "\sim"'] \arrow[d, "p_j"'] & V \arrow[d, "q_j"] \\ X_j \arrow[r, tail, "\sim", "v_j"'] & \bar{X}_j \end{tikzcd}$$

We write

$$F \;:\; A \;\longrightarrow\; \prod_{i\in I} \bar{X}_i \qquad \text{and} \qquad G \;:\; A \;\longrightarrow\; \prod_{i\in I} \bar{X}_i$$

for the morphisms with components

$$\bar{p}_j \circ F \;=\; q_j\bar{t}f \;:\; A \;\longrightarrow\; \bar{X}_j \qquad \text{and} \qquad \bar{p}_j \circ G \;=\; q_j \circ \bar{s}g \;:\; A \;\longrightarrow\; \bar{X}_j \ .$$

Then

$$\begin{aligned} \gamma(\bar{p}_j) \circ \gamma(F) \;&=\; \gamma(q_j\bar{t}f) \;=\; \gamma(q_j) \circ \gamma(\bar{t}) \circ \gamma(s) \circ \varphi \;=\; \gamma(q_j) \circ \gamma(u) \circ \varphi \\ &=\; \gamma(v_j) \circ \gamma(p_j) \circ \varphi \;=\; \gamma(v_j) \circ \gamma(p_j) \circ \psi \\ &=\; \gamma(q_j) \circ \gamma(u) \circ \psi \;=\; \gamma(q_j) \circ \gamma(\bar{s}) \circ \gamma(t) \circ \psi \\ &=\; \gamma(q_j) \circ \gamma(\bar{s}) \circ \gamma(g) \;=\; \gamma(q_j\bar{s}g) \;=\; \gamma(\bar{p}_j) \circ \gamma(G) \ . \end{aligned}$$

The special case proved in the previous paragraph thus shows that $\gamma(F) = \gamma(G)$.

Because all the morphisms $v_j : X_j \longrightarrow \bar{X}_j$ are weak equivalences, so is their product $\prod_{i\in I} v_i : \prod_{i\in I} X_i \longrightarrow \prod_{i\in I} \bar{X}_i$. The following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[drr, bend right, "F"'] & Z \arrow[d, tail, "\bar{t}"', "\sim"] & \prod_{i\in I} X_i \arrow[l, tail, "\sim"', "s"] \arrow[dl, tail, "u"', "\sim"] \arrow[d, "\sim"', "\prod v_i"] \\ & V \arrow[r, "(q_j)_{j\in I}"] & \prod_{i\in I} \bar{X}_i \end{tikzcd}$$

So

$$\varphi \;=\; \gamma(s)^{-1} \circ \gamma(f) \;=\; \gamma(\prod_{i\in I} v_i)^{-1} \circ \gamma(F) \ ,$$

and similarly $\psi = \gamma(\prod_{i\in I} v_i)^{-1} \circ \gamma(G)$. Hence $\varphi = \psi$, as claimed. □

6. The stable homotopy category

In this section we introduce the stable homotopy category as the localization of the category of orthogonal spectra at the class of stable equivalences. Most of the formal properties of the stable homotopy category are shared by the derived category of any orthogonal ring spectrum, so we will develop the theory in this generality. Since the category of orthogonal spectra is isomorphic to $\mathbb{S}$-mod, this is always included as the special case of the sphere ring spectrum $\mathbb{S}$.

Definition 6.1. The *derived category* $\mathcal{D}(R)$ of an orthogonal ring spectrum R is the localization of the category of R-modules at the class of stable equivalences.

We write

$$\gamma \ : \ R\text{-mod} \longrightarrow \mathcal{D}(R)$$

for the localization functor. In other words, γ is initial among all functor from R-mod that take stable equivalences to isomorphisms. We will exhibit a cofibration structure on the category of R-modules that has the stable equivalences as its weak equivalences, see Theorem 6.8. So the theory of cofibration categories provides a construction of $\mathcal{D}(R)$ via the 'calculus of fractions', see Theorem 5.12.

The special case of the sphere spectrum deserves explicit mentioning:

Definition 6.2. The *stable homotopy category* $\mathcal{SH}$ is the localization of the category of orthogonal spectra at the class of stable equivalences.

While is can generally be difficult to explicitly describe morphism sets in a localization of a category at a random class of morphisms, we can already calculate certain morphisms sets in $\mathcal{D}(R)$ using only the universal property as a localization, see the following Theorem 6.4.

Construction 6.3. We let E be a sequential spectrum, and we let A be a based space. We define a set $E\{A\}$ by

$$E\{A\} \ = \ \mathrm{colim}_n \, [S^n \wedge A, E_n]_* \ ,$$

where $[-,-]_*$ denotes the set of based homotopy classes of based continuous maps. The colimit is taken along the maps

$$[S^n \wedge A, E_n]_* \xrightarrow{S^1\wedge -} [S^1 \wedge S^n \wedge A, S^1 \wedge E_n]_* \xrightarrow{(\sigma_n)_*} [S^{1+n} \wedge A, E_{1+n}]_* \ .$$

Some comments about the construction $E\{A\}$ are in order.

- The construction $E\{A\}$ is covariantly functorial in the sequential spectrum E, and contravariantly functorial in the based space A.
- For $n \geq 2$, the set $[S^n \wedge A, E_n]_*$ has a natural abelian group structure by 'pinch sum', using any pinch map of S^n. In other words, if $f, g : S^n \wedge A \longrightarrow X$ represent two classes, then their sum is represented by the composite

  $$S^n \wedge A \xrightarrow{\text{pinch}} (S^n \vee S^n) \wedge A \ \cong \ (S^n \vee A) \wedge (S^n \vee A) \xrightarrow{f+g} X \ .$$

  The stabilization maps are homomorphisms, so the colimit $E\{A\}$ inherits an abelian group structure that is natural in E and in A.
- For $A = S^k$, the preferred homeomorphisms $S^n \wedge S^k \cong S^{n+k}$ let us identify $[S^n \wedge S^k, E_n]_*$ with $[S^{n+k}, E_n]_* = \pi_{n+k}(E_n)$; under these bijections, the stabilization maps coincide with the stabilization maps that define the stable homotopy group $\pi_k(E)$. So in the colimit over n, we obtain a natural isomorphism

  $$E\{S^k\} \ \cong \ \pi_k(E) \ .$$
- Under the adjunction bijections

  $$[S^n \wedge A, E_n]_* \ = \ [S^n, \mathrm{map}_*(A, E_n)]_* \ = \ \pi_n(\mathrm{map}_*(A, E)_n)$$

  the stabilization maps coincide with the stabilization maps that define the stable homotopy group $\pi_0(\mathrm{map}_*(A, E))$. So we obtain another natural isomorphism

  $$E\{A\} \ \cong \ \pi_0(\mathrm{map}_*(A, E)) \ .$$

Given an orthogonal ring spectrum R and a based space A, we define the *tautological class*

$$\iota_A \ \in \ (R \wedge A)\{A\}$$

as the class represented by the based continuous map

$$\iota_0 \wedge - \ : \ S^0 \wedge A \ \longrightarrow \ R(0) \wedge A \ ,$$

where $\iota_0 : S^0 \longrightarrow R(0)$ is the unit map of the ring spectrum structure.

Theorem 6.4. *Let R be an orthogonal ring spectrum, and let A be a based space that admits the structure of a finite CW-complex.*

(i) *The functor*

$$(-)\{A\} \ : \ R\text{-mod} \ \longrightarrow \ (\text{sets})$$

takes stable equivalences to bijections.

(ii) *Let $\Phi : R\text{-mod} \longrightarrow (\text{sets})$ be a functor that takes stable equivalences to bijections. Then evaluation at the tautological ι_A is a bijection*

$$\mathrm{Nat}_{R\text{-mod}\to(\text{sets})}((-)\{A\}, \Phi) \ \longrightarrow \ \Phi(R \wedge A) \ , \quad \tau \ \longmapsto \ \tau(\iota_A) \ .$$

(iii) *The pair $(R \wedge A, \iota_A)$ represents the functor*

$$(-)\{A\} \ : \ \mathcal{D}(R) \ \longrightarrow \ (\text{sets}) \ , \quad M \ \longmapsto \ M\{A\} \ .$$

Proof. (i) If A admits the structure of a finite CW-complex, then $\mathrm{map}_*(A, -)$ preserves stable equivalences by Proposition 1.27 (vii). So the functor

$$\mathcal{S}p^{\mathbb{N}} \ \longrightarrow \ \mathcal{A}b \ , \quad E \ \longmapsto \ \pi_0(\mathrm{map}_*(A, E))$$

takes stable equivalences to isomorphisms. Because $E\{A\}$ is naturally isomorphic to $\pi_0(\mathrm{map}_*(A, E))$, this proves the first claim.

(ii) To show that the evaluation map is injective we show that every natural transformation $\tau : (-)\{A\} \longrightarrow \Phi$ is determined by the element $\tau_{R\wedge A}(\iota_A)$. We let M be any R-module and $f : S^n \wedge A \longrightarrow M_n$ a representative for a class in $M\{A\}$. The adjoint of f is then a based continuous map $f^\flat : A \longrightarrow \Omega^n M_n$. There is thus a unique morphism of R-modules

$$f^\sharp \ : \ R \wedge A \ \longrightarrow \ \Omega^n \, \mathrm{sh}^n \, M$$

such that the composite

$$A \ \xrightarrow{\iota\wedge -} \ R(0) \wedge A \ \xrightarrow{f^\sharp(0)} \ (\Omega^n \, \mathrm{sh}^n \, M)(0) = \Omega^n M_n$$

is $f^\flat$. The value of $f^\sharp$ at an inner product space V is the composite

$$\begin{aligned} R(V) \wedge A \ \xrightarrow{R(V)\wedge f^\flat} \ & R(V) \wedge (\Omega^n M_n) \ = \ R(V) \wedge (\Omega^n M)(\mathbb{R}^n) \\ & \xrightarrow{\alpha_{V,\mathbb{R}^n}} \ \Omega^n M(V \oplus \mathbb{R}^n) \ = \ (\Omega^n \, \mathrm{sh}^n \, M)(V) \ . \end{aligned}$$

This morphism then satisfies

$$f^\sharp\{A\}(\iota_A) \ = \ \tilde{\lambda}^n_M\{A\}[f] \qquad \text{in } \ (\Omega^n \, \mathrm{sh}^n \, M)\{A\} \ ,$$

where $\tilde{\lambda}^n_M : M \longrightarrow \Omega^n \, \mathrm{sh}^n \, M$ is the stable equivalence discussed in Proposition 2.15. The diagram

$$\begin{tikzcd} (R \wedge A)\{A\} \arrow[r, "f^\sharp\{A\}"] \arrow[d, "\tau_{R\wedge A}"'] & (\Omega^n \, \mathrm{sh}^n \, M)\{A\} \arrow[d, "\tau_{\Omega^n \, \mathrm{sh}^n \, M}"] & M\{A\} \arrow[l, "\tilde{\lambda}^n_M\{A\}"', "\cong"] \arrow[d, "\tau_M"] \\ \Phi(R \wedge A) \arrow[r, "\Phi(f^\sharp)"'] & \Phi(\Omega^n \, \mathrm{sh}^n \, M) & \Phi(M) \arrow[l, "\cong", "\Phi(\tilde{\lambda}^n_M)"'] \end{tikzcd}$$

commutes and the two right horizontal maps are bijective. So

$$\Phi(f^\sharp)(\tau_{R\wedge A}(\iota_A)) \ = \ \tau_{\Omega^n \, \mathrm{sh}^n \, M}(f^\sharp\{A\}(\iota_A)) \ = \ \tau_{\Omega^n \, \mathrm{sh}^n \, M}(\tilde{\lambda}^n_M\{A\}[f]) \ = \ \Phi(\tilde{\lambda}^n_M)(\tau_M[f]) \ .$$

Since $\Phi(\tilde{\lambda}^n_M)$ is bijective, this proves that $\tau_M[f]$ is determined by the value of τ on the tautological class ι_A.

It remains to construct, for every element $y \in \Phi(R \wedge A)$, a natural transformation $\tau : (-)\{A\} \longrightarrow \Phi$ with $\tau_{R\wedge A}(\iota_A) = y$. The previous paragraph dictates what to do: we represent any given class in $M\{A\}$ by a continuous based map $f : S^n \wedge A \longrightarrow M_n$. Then we set

$$\tau_M[f] \ = \ \Phi(\tilde{\lambda}^n_M)^{-1}(\Phi(f^\sharp)(y)) \ .$$

We verify that the element $\tau_M[f]$ is independent of the representative for the class $[f]$. To this end we need to show that $\tau_M[f]$ does not change if we either replace f by a homotopic map, or if we stabilize it. If $\bar{f} : S^n \wedge A \longrightarrow M_n$ is homotopic to f, then the morphism $\bar{f}^\sharp$ is homotopic to $f^\sharp$ via a homotopy of morphisms of R-modules

$$K \ : \ R \wedge A \wedge [0,1]_+ \ \longrightarrow \Omega^n \operatorname{sh}^n M \ .$$

The morphism $q : R\wedge A\wedge[0,1]_+ \longrightarrow R\wedge A$ that maps $[0,1]$ to a single point is a homotopy equivalence, hence a stable equivalence, of R-modules. So $\Phi(q)$ is a bijection. The two embeddings $i_0, i_1 : R\wedge A \longrightarrow R\wedge A\wedge[0,1]_+$ as the endpoints of the interval are right inverse to q, so $\Phi(q) \circ \Phi(i_0) = \Phi(q) \circ \Phi(i_1) = \mathrm{Id}$. Since $\Phi(q)$ is bijective, $\Phi(i_0) = \Phi(i_1)$. Hence

$$\Phi(\bar{f}^\sharp) \ = \ \Phi(K \circ i_0) \ = \ \Phi(K) \circ \Phi(i_0) \ = \ \Phi(K) \circ \Phi(i_1) \ = \ \Phi(K \circ i_1) \ = \ \Phi(f^\sharp) \ .$$

This shows that $\tau_M[f]$ does not change if we modify f by a homotopy.

Now we replace the representative by its stabilization $\sigma_n(S^1 \wedge f) : S^{1+n} \wedge A \longrightarrow M_{1+n}$. We define a morphism of R-modules

$$\kappa \ : \ \Omega^n \operatorname{sh}^n M \ \longrightarrow \ \Omega^{1+n} \operatorname{sh}^{1+n} M$$

at an inner product space V as the map

$$\kappa(V) \ : \ \Omega^n M(V \oplus \mathbb{R}^n) \ \longrightarrow \ \Omega^{1+n} M(V \oplus \mathbb{R}^{1+n})$$

that sends $h : S^n \longrightarrow M(V \oplus \mathbb{R}^n)$ to the composite

$$S^{1+n} \xrightarrow{S^1\wedge h} S^1 \wedge M(V \oplus \mathbb{R}^n) \xrightarrow{\sigma_{\mathbb{R},V\oplus\mathbb{R}^n}} M(\mathbb{R} \oplus V \oplus \mathbb{R}^n) \xrightarrow{M(\tau_{\mathbb{R},V\oplus\mathbb{R}^n})} M(V \oplus \mathbb{R}^{1+n}) \ .$$

We claim that the following diagram commutes:

$$\begin{tikzcd} R \wedge A \arrow[r, "f^\sharp"] \arrow[dr, bend right, "(\sigma_n(S^1\wedge f))^\sharp"'] & \Omega^n \operatorname{sh}^n M \arrow[d, "\kappa"'] & M \arrow[l, "\tilde{\lambda}^n_M"'] \arrow[dl, bend left, "\tilde{\lambda}^{1+n}_M"] \\ & \Omega^{1+n} \operatorname{sh}^{1+n} M & \end{tikzcd}$$

For the left part, it suffices to check commutativity at the zero inner product space and after precomposition with $\iota_0 \wedge - : A \longrightarrow R_0 \wedge A$. This amounts to the commutativity (directly visible upon inspection of all definitions) of the triangle

$$\begin{tikzcd} A \arrow[r, "f^\flat"] \arrow[dr, bend right, "(\sigma_n(S^1\wedge f))^\flat"'] & \Omega^n M_n \arrow[d, "\kappa(0)"] \\ & \Omega^{1+n} M_{1+n} \end{tikzcd}$$

For the right part, we must verify the commutativity of the following triangles:

$$\begin{tikzcd} \Omega^n M(V \oplus \mathbb{R}^n) \arrow[d, "\kappa(V)"'] & M(V) \arrow[l, "\tilde{\lambda}^n_M(V)"'] \arrow[dl, bend left, "\tilde{\lambda}^{1+n}_M(V)"] \\ \Omega^{1+n} M(V \oplus \mathbb{R}^{1+n}) & \end{tikzcd}$$

Upon adjoint the various sphere coordinates, this becomes the commutativity of the outer part of the following diagram:

$$\begin{tikzcd}
S^1 \wedge M(V) \wedge S^n \arrow[rr, "\mathrm{twist} \wedge S^n"] \arrow[d, "S^1 \wedge \sigma^{\mathrm{op}}_{V,\mathbb{R}^n}"'] \arrow[dr, "\sigma_{\mathbb{R},V} \wedge S^n"] & & M(V) \wedge S^1 \wedge S^n \arrow[d, "\sigma^{\mathrm{op}}_{V,\mathbb{R}} \wedge S^n"'] \arrow[dd, bend left, "\sigma^{\mathrm{op}}_{V,\mathbb{R}^{1+n}}"] \\
S^1 \wedge M(V \oplus \mathbb{R}^n) \arrow[dr, "\sigma_{\mathbb{R},V\oplus\mathbb{R}^n}"'] & M(\mathbb{R} \oplus V) \wedge S^n \arrow[r, "M(\tau_{\mathbb{R},V}) \wedge S^n"'] \arrow[d, "\sigma^{\mathrm{op}}_{\mathbb{R}\oplus V,\mathbb{R}^n}"'] & M(V \oplus \mathbb{R}) \wedge S^n \arrow[d, "\sigma^{\mathrm{op}}_{V\oplus\mathbb{R},\mathbb{R}^n}"'] \\
& M(\mathbb{R} \oplus V \oplus \mathbb{R}^n) \arrow[r, "M(\tau_{\mathbb{R},V\oplus\mathbb{R}^n})"'] & M(V \oplus \mathbb{R}^{1+n})
\end{tikzcd}$$

Anyhow, the commutativity of the previous diagram yields the relation

$$\Phi(\tilde{\lambda}^n_M)^{-1} \circ \Phi(f^\sharp) \ = \ \Phi(\tilde{\lambda}^{1+n}_M)^{-1} \circ \Phi(\kappa) \circ \Phi(f^\sharp) \ = \ \Phi(\tilde{\lambda}^{1+n}_M)^{-1} \circ \Phi((\sigma_n(S^1 \wedge f))^\sharp) \ ,$$

and hence the class $\tau_M[f]$ remains unchanged upon stabilization of f.

Now we know that $\tau_M[f]$ is independent of the choice of representative for the class x, and it remains to show that τ is natural. But this is straightforward: if $\psi : M \longrightarrow N$ is a morphism of R-modules and $f : S^n \wedge A \longrightarrow M_n$ a representative for a class in $M\{A\}$, then $\psi_n \circ f : S^n \wedge A \longrightarrow N_n$ represents the class $\psi\{A\}[f]$. Moreover, the following diagram of R-modules commutes:

$$\begin{tikzcd}
R \wedge A \arrow[r, "f^\sharp"] \arrow[dr, bend right, "(\psi_n \circ f)^\sharp"'] & \Omega^n \operatorname{sh}^n M \arrow[d, "\Omega^n \operatorname{sh}^n \psi"] & M \arrow[l, "\tilde{\lambda}^n_M"'] \arrow[d, "\psi"] \\
& \Omega^n \operatorname{sh}^n N & N \arrow[l, "\tilde{\lambda}^n_N"]
\end{tikzcd}$$

So naturality follows:

$$\begin{aligned}
\tau_N(\psi\{A\}[f]) \ &= \ (\Phi(\tilde{\lambda}^n_N)^{-1} \circ \Phi((\psi_n \circ f)^\sharp))(y) \\
&= \ (\Phi(\tilde{\lambda}^n_N)^{-1} \circ \Phi(\Omega^n \operatorname{sh}^n \psi) \circ \Phi(f^\sharp))(y) \\
&= \ (\Phi(\psi) \circ \Phi(\tilde{\lambda}^n_M)^{-1} \circ \Phi(f^\sharp))(y) \ = \ \Phi(\psi)(\tau_M[f]) \ .
\end{aligned}$$

Finally, the class ι_A is represented by $\iota_0 \wedge A : S^0 \wedge A \longrightarrow R_0 \wedge A$, and $(\iota_0 \wedge A)^\sharp$ is the identity of $R \wedge A$. Hence $\tau_{R\wedge A}(\iota_A) = \Phi(\mathrm{Id}_{R\wedge A})(y) = y$.

(iii) We will implicitly identify natural transformations between functors that invert stable equivalences with natural transformations between the induced functors on the derived category. This is legitimate because precomposition with $\gamma : R\text{-mod} \longrightarrow \mathcal{D}(R)$ is an isomorphism of categories from $\mathrm{Fun}(\mathcal{D}(R), \mathcal{C})$ to the full subcategory of $\mathrm{Fun}(R\text{-mod}, \mathcal{C})$ spanned by the functors that invert stable equivalences.

We apply part (ii) to the functor $\mathcal{D}(R)(R \wedge A, -) \circ \gamma : R\text{-mod} \longrightarrow (\text{sets})$. We obtain a unique natural transformation

$$\tau \ : \ (-)\{A\} \ \Longrightarrow \ \mathcal{D}(R)(R \wedge A, -) \circ \gamma$$

such that $\tau_{R\wedge A}(\iota_A) = \mathrm{Id}_{R\wedge A}$. On the other hand, the Yoneda lemma provides a unique natural transformation

$$j \ : \ \mathcal{D}(R)(R \wedge A, -) \ \longrightarrow \ (-)\{A\}$$

such that $j_{R\wedge A}(\mathrm{Id}_{R\wedge A}) = \iota_A$. The composite $j \circ \tau$ then satisfies

$$(\tau \circ j)_{R\wedge A}(\mathrm{Id}_{R\wedge A}) \ = \ \tau_{R\wedge A}(\iota_A) \ = \ \mathrm{Id}_{R\wedge A} \ .$$

So again by the Yoneda lemma, $\tau \circ j : (-)\{A\} \Longrightarrow (-)\{A\}$ is the identity natural transformation. Similarly, the composite $j \circ \tau$ satisfies

$$(j \circ \tau)_{R\wedge A}(\iota_A) \ = \ j_{R\wedge A}(\mathrm{Id}_{R\wedge A}) \ = \ \iota_A \ .$$

So the uniqueness clause in part (i) shows that $j \circ \tau : \mathcal{D}(R)(R \wedge A, -) \Longrightarrow \mathcal{D}(R)(R \wedge A, -)$ is the identity natural transformation. $\square$

Example 6.5 (Representing π_n)**.** Theorem 6.4 (iii) effectively says that we can identify the morphism group $\mathcal{D}(R)(R \wedge A, M)$ in the derived category of a ring spectrum R with the group $M\{A\}$ that was defined in concrete terms from the based space A and the spaces in the underlying sequential spectrum of M.

For $A = S^k$, the group $M\{S^k\}$ identifies with the homotopy group $\pi_k(M)$, in a way that identifies the tautological class ι_{S^k} with the class $1 \wedge S^k \in \pi_k(R \wedge S^k)$, the image of the multiplicative unit $1 \in \pi_0(R)$ under the iterated suspension isomorphism $- \wedge S^k : \pi_0(R) \cong \pi_k(R \wedge S^k)$. So for $A = S^k$, Theorem 6.4 (iii) specializes to a bijection

$$\mathcal{D}(R)(R \wedge S^k, M) \cong \pi_k(M) , \quad \psi \longmapsto \psi_*(1 \wedge S^k) .$$

In the special case $R = \mathbb{S}$ of the sphere ring spectrum, $R \wedge S^k$ becomes the suspension spectrum $\Sigma^\infty S^k$. So in this case we obtain a bijection

$$\mathcal{SH}(\Sigma^\infty S^k, X) \cong \pi_k(X)$$

that is natural in the orthogonal spectrum X.

To establish further properties of $\mathcal{SH}$ and $\mathcal{D}(R)$ we will exploit that the stable equivalences of orthogonal spectra and R-modules participate in certain cofibration structures.

Definition 6.6. A morphism of orthogonal spectra $i : A \longrightarrow B$ is an *h-cofibration* if it has the following homotopy extension property. For every morphism of orthogonal spectra $\varphi : B \longrightarrow X$ and every homotopy $H : A \wedge [0,1]_+ \longrightarrow X$ such that $H_0 = \varphi i$, there is a homotopy $\bar{H} : B \wedge [0,1]_+ \longrightarrow X$ such that $H_0 = \varphi$ and $\bar{H} \circ (i \wedge [0,1]_+) = H$.

There is a universal test case for the homotopy extension problem, namely when X is the pushout:

$$\begin{tikzcd} A \arrow[r, "{(-,0)}"] \arrow[d, "i"'] & A \wedge [0,1]_+ \arrow[d, "H"] \\ B \arrow[r, "\varphi"'] & B \cup_i (A \wedge [0,1]_+) \end{tikzcd}$$

So a morphism $i : A \longrightarrow B$ is an h-cofibration if and only if the canonical morphism

$$B \cup_i (A \wedge [0,1]_+) \longrightarrow B \wedge [0,1]_+$$

has a retraction. Also, the adjunction between $-\wedge[0,1]_+$ and $(-)^{[0,1]}$ lets us rewrite any homotopy extension data (φ, H) in adjoint form as a commutative square:

$$\begin{tikzcd} A \arrow[r, "\hat{H}"] \arrow[d, "i"'] & X^{[0,1]} \arrow[d, "\mathrm{ev}_0"] \\ B \arrow[r, "\varphi"'] & X \end{tikzcd}$$

A solution to the homotopy extension problem is adjoint to a lifting, i.e., a morphism $\lambda : B \longrightarrow X^{[0,1]}$ such that $\lambda i = \hat{H}$ and $\mathrm{ev}_0 \circ \lambda = \varphi$. So a morphism $i : A \longrightarrow B$ is an h-cofibration if and only if it has the left lifting property with respect to the evaluation morphisms $\mathrm{ev}_0 : X^{[0,1]} \longrightarrow X$ for all orthogonal spectra X.

The three equivalent characterizations of h-cofibrations quickly imply various closure properties: every class of morphisms that can be characterized by the left lifting property with respect to some other class has the closure properties listed, see Exercise E.9.

Proposition 6.7. (i) *The class of h-cofibrations of orthogonal spectra is closed under retracts, cobase change, coproducts and sequential compositions.*

(ii) *For every orthogonal spectrum X, the unique morphism $* \longrightarrow X$ from the initial orthogonal spectrum is an h-cofibration.*

(iii) *Every h-cofibration of orthogonal spectra is levelwise an h-cofibration of based spaces.*

Theorem 6.8. *For every orthogonal ring spectrum R, the category of R-modules is a cofibration category with respect to the morphisms that are h-cofibrations or stable equivalences of underlying orthogonal spectra, respectively.*

Proof. Most of the axioms are straightforward from the definitions. For (C1) we note that clearly, all isomorphisms are h-cofibrations and stable equivalences; the constant functor $\mathbf{O} \longrightarrow \mathbf{T}_*$ with value a one-point space has a unique structure of R-module that makes it an initial object. And the unique morphism from an initial orthogonal spectrum to any other orthogonal spectrum is an h-cofibration by Proposition 6.7 (ii).

The class of stable equivalences clearly satisfies the 2-out-of-3-property (C2). The h-cofibrations are closed under cobase change by Proposition 6.7 (i); and h-cofibrations that are simultaneously stable equivalences are stable under cobase change by Proposition 1.27. So axiom (C3) holds.

Finally, a morphism $X \longrightarrow Y$ of R-modules factors in the category R-mod as the composite of the mapping cylinder inclusion $-\wedge 0 : X \longrightarrow X \wedge [0,1]_+ \cup_f Y$, followed by the projection $X \wedge [0,1]_+ \cup_f Y \longrightarrow Y$ to the 'end' of the cylinder. This projection is a homotopy equivalence of orthogonal spectra, and hence a stable equivalence. The following square is a pushout:

$$\begin{tikzcd} X \amalg X \arrow[r, "\mathrm{Id}_X \amalg f"] \arrow[d, tail, "(-\wedge 0)+(-\wedge 1)"'] & X \amalg Y \arrow[d, tail] \\ X \wedge [0,1]_+ \arrow[r] & X \wedge [0,1]_+ \cup_f Y \end{tikzcd}$$

The left vertical morphism is an h-cofibration, hence so is the right vertical morphism. Because the summand inclusion $X \longrightarrow X \amalg Y$ is an h-cofibration, too, we conclude that the mapping cylinder inclusion $- \wedge 0 : X \longrightarrow X \wedge [0,1]_+ \cup_f Y$ is an h-cofibration. This verifies the factorization axiom (C4). □

Remark 6.9. The stable equivalences of R-modules can be complemented in many different ways into a cofibration category, i.e., there are several other choices of classes of cofibrations that also satisfy the axioms of Definition 6.7 in conjunction with the stable equivalences. Two examples are:

- The h-cofibrations of R-modules, i.e., morphisms that have the homotopy extensions property internal to the category R-mod.
- The cofibrations in the model category structure on R-mod from [30, Theorem 12.1].

Since the homotopy category – and also the underlying ∞-category – of a cofibration category are independent (up to equivalence) of the class of cofibrations, the choice of cofibrations is not particularly important, and is mostly a matter of taste and convenience.

Example 6.10. We let R be an orthogonal ring spectrum, and we consider the cofibration structure on the category of left R-modules from Theorem 6.8. For every R-module M the inclusions of the endpoints of the interval and the unique map from $[0,1]$ to a one-point space induce a factorization

$$M \vee M \xrightarrow{i_0+i_1} M \wedge [0,1]_+ \xrightarrow{p} M$$

as an h-cofibration of R-modules followed by a homotopy equivalence of R-modules.

So the 'cylinder' $M \wedge [0,1]_+$ is indeed a cylinder object in the abstract sense of Definition 5.4, and morphisms of R-modules that are homotopic in the 'concrete' sense (where a homotopy is a morphism defined on $M \wedge [0,1]_+$) are also homotopic in the 'abstract' sense (i.e., where a homotopy is a morphism defined on a general cylinder object).

One should beware, however, that the converse is *not* true: if f and g are homotopic in the abstract sense of Definition 5.4, then there need not be a 'classical homotopy' defined on $M \wedge [0,1]_+$.

With enough of the theory of cofibration categories available from Section 5, we can fairly easily show that the stable homotopy category, and more generally the derived category of any orthogonal ring spectrum, is an additive category, i.e., there is a natural commutative groups structure on the homomorphism sets such that composition is biadditive. This definition makes it sound as if 'additive category' is extra structure on a category (namely the addition on morphism sets), but in fact, 'additive category' is really a property of a category (namely having finite sums which are isomorphic to products). So we present the construction of the addition on hom-sets in this generality.

Definition 6.11. A category $\mathcal{C}$ is *preadditive* if it has a zero object, finite coproducts, and for every pair of objects X and Y the morphisms $p_1 = \mathrm{Id} + 0 : X \amalg Y \longrightarrow X$ and $p_2 = 0 + \mathrm{Id} : X \amalg Y \longrightarrow Y$ make $X \amalg Y$ into a product of X and Y, where '0' is the unique morphism which factors through a zero object.

In other words, we demand that for every object A the map

$$\mathcal{C}(A, X \amalg Y) \longrightarrow \mathcal{C}(A, X) \times \mathcal{C}(A, Y) , \quad f \mapsto (p_1 f, p_2 f)$$

is a bijection.

Example 6.12. The category of abelian groups is preadditive, and so is the category of left modules over any ring. The category of abelian monoids is preadditive.

Construction 6.13. Let $\mathcal{C}$ be a preadditive category. We can define a binary operation on the morphism set $\mathcal{C}(A, X)$ for every pair of objects A and X. Given morphisms $a, b : A \longrightarrow X$ we let $a \perp b : A \longrightarrow X \amalg X$ be the unique morphism such that $p_1(a \perp b) = a$ and $p_2(a \perp b) = b$. Then we define $a + b : A \longrightarrow X$ as $\nabla(a \perp b)$ where $\nabla = \mathrm{Id} + \mathrm{Id} : X \amalg X \longrightarrow X$ is the fold morphism.

Proposition 6.14. *Let $\mathcal{C}$ be a preadditive category.*

(i) *For every pair of objects A and X of $\mathcal{C}$ the binary operation $+$ makes the set $\mathcal{C}(A, X)$ of morphisms into an abelian monoid with the zero morphism as neutral element. Moreover, the monoid structure is natural for all morphisms in both variables, or, equivalently, composition is biadditive.*

(ii) *If, moreover, the shearing morphism $p_1 \perp \nabla : X \amalg X \longrightarrow X \amalg X$ is an isomorphism, then the abelian monoid $\mathcal{C}(A, X)$ has additive inverse, i.e., is an abelian group, for every object A.*

(iii) *Let $F, G : \mathcal{C} \longrightarrow \mathcal{A}b\mathcal{M}on$ be two functors that preserve zero objects. Suppose moreover that the functor G preserves finite coproducts. Then every natural transformation between the underlying set-valued functors of F and G is automatically additive.*

(iv) *Let $G : \mathcal{C} \longrightarrow \mathcal{A}b\mathcal{M}on$ be a functor that preserves zero objects and finite coproducts. Then for all objects A and X of $\mathcal{C}$ and every element $a \in G(A)$ the evaluation map*

$$\mathcal{C}(A, X) \longrightarrow G(X) , \quad f \longmapsto G(f)(a)$$

is a monoid homomorphism.

Proof. (i) The proof is lengthy, but completely formal. For the associativity of '+' we consider three morphisms $a, b, c : A \longrightarrow X$. Then $a + (b + c)$ respectively $(a + b) + c$ are the two upper and lower composites around the commutative diagram:

$$\begin{tikzcd}
 & & X \amalg X \arrow[dr, "\nabla"] & \\
A \arrow[r, "a\perp b\perp c"] \arrow[urr, bend left, "a\perp(b+c)"] \arrow[drr, bend right, "(a+b)\perp c"'] & X \amalg X \amalg X \arrow[ur, "\mathrm{Id}\amalg\nabla"] \arrow[dr, "\nabla\amalg\mathrm{Id}"] & & X \\
 & & X \amalg X \arrow[ur, "\nabla"'] &
\end{tikzcd}$$

The commutativity is a consequence of two elementary facts: first, $b \perp a = (a \perp b)\tau$ where $\tau : X \amalg X \longrightarrow X \amalg X$ is the automorphism which interchanges the two factors; this follows from $p_1\tau = p_2$ and $p_2\tau = p_1$. Second, the fold morphism is commutative, i.e., $\nabla\tau = \nabla : X \amalg X \longrightarrow X$. Altogether we get

$$a + b \;=\; \nabla(a \perp b) \;=\; \nabla\tau(a \perp b) \;=\; \nabla(b \perp a) \;=\; b + a \,.$$

As before we denote by $0 \in \mathcal{C}(A,X)$ the unique morphism which factors through a zero object. Then we have $a\perp 0 = i_1 a$ in $\mathcal{C}(A, X \amalg X)$ where $i_1 : X \longrightarrow X \amalg X$ is the embedding as the first factor. Hence $a + 0 = \nabla(a\perp 0) = \nabla i_1 a = a$; by commutativity we also have $0 + a = a$.

Now we verify naturality of the addition on $\mathcal{C}(A,X)$ in A and X. To check $c(a+b) = ca + cb$ for $a, b : A \longrightarrow X$ and $c : X \longrightarrow Y$ we consider the commutative diagram

$$\begin{tikzcd}
A \arrow[r, "a\perp b"'] \arrow[d, "a\perp b"'] \arrow[rr, bend left, "ca\perp cb"] & X \amalg X \arrow[r, "c\amalg c"'] \arrow[d, "\nabla"] & Y \amalg Y \arrow[d, "\nabla"] \\
X \amalg X \arrow[r, "\nabla"'] & X \arrow[r, "c"'] & Y
\end{tikzcd}$$

in which the composite through the lower left corner is $c(a + b)$. We have

$$\begin{aligned}
p_1(c \amalg c)(a\perp b) &= (\mathrm{Id}_Y + 0)(c \amalg c)(a\perp b) = (c + 0)(a\perp b) \\
&= c(\mathrm{Id}_X + 0)(a\perp b) = cp_1(a\perp b) = ca = p_1(ca\perp cb)
\end{aligned}$$

and similarly for p_2 instead p_1. So $(c \amalg c)(a\perp b) = ca\perp cb$ since both sides have the same 'projections' to the two summands of $Y \amalg Y$. Since the composite through the upper right corner is $ca + cb$, we have shown $c(a + b) = ca + cb$.

Naturality in A is even easier. For a morphism $d : E \longrightarrow A$ we have $(a\perp b)d = ad\perp bd : E \longrightarrow X \amalg X$ since both sides have the same 'projections' ad and bd, respectively, to the two summands of $X \amalg X$. Thus $(a + b)d = ad + bd$ by the definition of '+'.

(ii) An arbitrary abelian monoid M has additive inverses if and only if the map

$$M^2 \longrightarrow M^2 , \quad (x, y) \longmapsto (x, x + y)$$

is bijective. Indeed, the inverse of $x \in M$ is the second component of the preimage of $(x, 0)$.

Every morphism $f : A \longrightarrow X \amalg X$ satisfies $f = (p_1 f)\perp(p_2 f)$, and hence

$$(p_1 f) + (p_2 f) = \nabla((p_1 f)\perp(p_2 f)) = \nabla f .$$

So for the abelian monoid $\mathcal{C}(A,X)$ the square

$$\begin{tikzcd}
\mathcal{C}(A, X \amalg X) \arrow[r, "(p_1\perp\nabla)\circ-"] \arrow[d, "(p_1\circ-{,}p_2\circ-)"', "\cong"] & \mathcal{C}(A, X \amalg X) \arrow[d, "\cong"', "(p_1\circ-{,}p_2\circ-)"] \\
\mathcal{C}(A,X)^2 \arrow[r, "(a{,}b)\mapsto(a{,}a+b)"'] & \mathcal{C}(A,X)^2
\end{tikzcd}$$

commutes. Moreover, both vertical maps are bijective. Since $\nabla\perp p_1$ is an isomorphism, the upper map is bijective, hence so is the lower map, and so the monoid $\mathcal{C}(A,X)$ has inverses.

(iii) Let $\tau : F \Longrightarrow G$ be any natural transformation between the underlying set-valued functors, i.e., the maps $\tau_X : F(X) \longrightarrow G(X)$ are not assumed to be additive. We write $i_1, i_2 : X \longrightarrow X \amalg X$ for the two morphisms that exhibit $X \amalg X$ as a coproduct of two instances of X. Because F and G preserve zero objects, they also preserve zero morphisms. We consider two classes x and y in $F(X)$; we claim that

$$\tau_{X\amalg X}(F(i_1)(x) + F(i_2)(y)) = G(i_1)(\tau_X(x)) + G(i_2)(\tau_X(y)) \tag{6.15}$$

in the abelian monoid $G(X \amalg X)$. To show this we observe that

$$\begin{aligned} G(p_1)(\tau_{X\amalg X}(F(i_1)(x) + F(i_2)(y))) &= \tau_X(F(p_1)(F(i_1)(x) + F(i_2)(y))) \\ &= \tau_X(F(p_1 i_1)(x) + F(p_1 i_2)(y))) \\ &= \tau_X(F(\mathrm{Id}_X)(x) + F(0)(y)) \ = \ \tau_X(x) \\ &= G(\mathrm{Id}_X)(\tau_X(x)) + G(0)(\tau_X(x)) \\ &= G(p_1 i_1)(\tau_X(x)) + G(p_1 i_2)(\tau_X(y)) \\ &= G(p_1)(G(i_1)(\tau_X(x)) + G(i_2)(\tau_X(y))) \end{aligned}$$

in $G(X)$. Similarly,

$$G(p_2)(\tau_{X\amalg X}(F(i_1)(x) + F(i_2)(y))) \ = \ G(p_2)(G(i_1)(\tau_X(x)) + G(i_2)(\tau_X(y))) \ .$$

Since the functor G preserves coproducts, and because abelian monoids form a preadditive category, the morphism

$$(G(p_1), G(p_2)) \ : \ G(X \amalg X) \ \longrightarrow \ G(X) \times G(X)$$

is bijective; so this shows the relation (6.15). The fold morphism $\nabla : X \amalg X \longrightarrow X$ satisfies $\nabla \circ i_1 = \nabla \circ i_2 = \mathrm{Id}_X$. So

$$F(\nabla)(F(i_1)(x) + F(i_2)(y)) \ = \ F(\nabla \circ i_1)(x) + F(\nabla \circ i_2)(y) \ = \ x + y \ .$$

So we can finally conclude with the desired relation:

$$\begin{aligned} \tau_X(x+y) &= \tau_X(F(\nabla)(F(i_1)(x) + F(i_2)(y))) \\ &= G(\nabla)(\tau_{X\amalg X}(F(i_1)(x) + F(i_2)(y))) \\ \overset{(6.15)}{} &= G(\nabla)(G(i_2)(\tau_X(x)) + G(i_2)(\tau_X(y))) \\ &= G(\nabla \circ i_1)(\tau_X(x)) + G(\nabla \circ i_2)(\tau_X(y)) \ = \ \tau_X(x) + \tau_X(y) \ . \end{aligned}$$

Part (iv) is a special case of (iii), with $F = \mathcal{C}(A, -)$ a represented functor. □

Definition 6.16. A category $\mathcal{C}$ is *additive* if it is preadditive and for every object X of $\mathcal{C}$, the shearing morphism $\nabla \perp p_1 : X \amalg X \longrightarrow X \amalg X$ is an isomorphism.

We let R be an orthogonal ring spectrum. We recall that $\gamma : R\text{-mod} \longrightarrow \mathcal{D}(R)$ denotes a localization of the category of R-modules at the class of stable equivalences. Since the stable equivalences participate in the cofibration structure of Theorem 6.8, the calculus of fractions is available to describe morphisms in $\mathcal{D}(R)$.

For $k \in \mathbb{Z}$, the functor $\pi_k : R\text{-mod} \longrightarrow \mathcal{A}b$ takes stable equivalences to isomorphisms. We abuse notation and also write $\pi_k : \mathcal{D}(R) \longrightarrow \mathcal{A}b$ for the unique factorization of π_k through the localization functor $\gamma : R\text{-mod} \longrightarrow \mathcal{D}(R)$.

Theorem 6.17. *Let R be an orthogonal ring spectrum.*

(i) *The localization functor $\gamma : R\text{-mod} \longrightarrow \mathcal{D}(R)$ preserves arbitrary coproducts and finite products. In particular, the derived category $\mathcal{D}(R)$ has arbitrary coproducts and finite products.*
(ii) *The derived category $\mathcal{D}(R)$ is additive.*
(iii) *A morphism $f : M \longrightarrow N$ of R-modules is a stable equivalence if and only if $\gamma(f)$ is an isomorphism in $\mathcal{D}(R)$.*

Proof. (i) The category of R-modules has arbitrary coproducts and products, and both are created on the underlying orthogonal spectra. Moreover, stable equivalences are preserved under arbitrary coproducts and finite products by parts (i) and (ii) of Proposition 1.27. So the localization functor preserves arbitrary coproducts by Proposition 5.15, and it preserves finite products by Proposition 5.17.

(ii) The trivial orthogonal spectrum is a zero object in R-mod. Since the localization functor preserves initial and terminal objects, the trivial orthogonal spectrum is also a zero object in $\mathcal{D}(R)$. For any pair of R-modules M and N, the canonical morphism

$$M \vee N \ \longrightarrow \ M \times N$$

is a stable equivalence by Proposition 1.24 (iii). The localization functor $\gamma : R\text{-mod} \longrightarrow \mathcal{D}(R)$ takes this stable equivalence to an isomorphism in $\mathcal{D}(R)$. Because γ preserves finite coproducts and products, the derived category $\mathcal{D}(R)$ is preadditive.

It remains to show that for every R-modules M, the morphism $\nabla\perp p_1 : M \oplus M \longrightarrow M \oplus M$ is an isomorphism in $\mathcal{D}(R)$. For every integer k the composite map

$$\pi_k(M) \oplus \pi_k(M) \xrightarrow[\cong]{(i_1)_* + (i_2)_*} \pi_k(M \oplus M) \xrightarrow{\pi_k(\nabla\perp p_2)} \pi_k(M \oplus M) \xrightarrow[\cong]{((p_1)_*, (p_2)_*)} \pi_k(M) \times \pi_k(M)$$

sends (x, y) to $(x + y, y)$ where the first and last maps are the canonical ones. The composite map is an isomorphism since $\pi_k(M)$ is a group, i.e., has additive inverses. Since canonical maps are isomorphisms, so is the middle map; thus $\nabla\perp p_1$ induces isomorphisms of homotopy groups. and is thus a stable equivalence. Hence the shearing morphism $\nabla\perp p_1$ is an isomorphism in $\mathcal{D}(R)$.

(iii) Since the functor $\pi_k : R\text{-mod} \longrightarrow \mathcal{A}b$ factors through the localization $\mathcal{D}(R)$, and since $\gamma(f)$ is an isomorphism in $\mathcal{D}(R)$, the morphism f induces isomorphisms on all homotopy groups. So f is a stable equivalence. □

Now we record that suspension functor for R-modules becomes invertible at the level of the derived category. In more detail, we recall from Proposition 2.16 that the functor $-\wedge S^1$ preserves stable equivalences of orthogonal spectra, and hence also of R-modules. So the composite functor $\gamma\circ(-\wedge S^1) : R\text{-mod} \longrightarrow \mathcal{D}(R)$ takes stable equivalences to isomorphisms. The universal property of the localization functor $\gamma : R\text{-mod} \longrightarrow \mathcal{D}(R)$ provides a unique functor

$$\Sigma \; : \; \mathcal{D}(R) \; \longrightarrow \; \mathcal{D}(R)$$

that satisfies $\Sigma \circ \gamma = \gamma \circ (- \wedge S^1)$. In particular, Σ is given on objects by $\Sigma X = X \wedge S^1$. Moreover, the behavior on morphisms is as follows. Every morphism $M \longrightarrow N$ in $\mathcal{D}(R)$ is of the form $\gamma(\tau)^{-1} \circ \gamma(f)$ for two morphisms of R-modules $f : M \longrightarrow Z$ and $\tau : N \longrightarrow Z$ such that τ is a stable equivalence. Then $\tau \wedge S^1 : N \wedge S^1 \longrightarrow Z \wedge S^1$ is a weak equivalence, too, and

$$\Sigma(\gamma(\tau)^{-1} \circ \gamma(f)) \; = \; \gamma(\tau \wedge S^1)^{-1} \circ \gamma(f \wedge S^1) \; .$$

Proposition 6.18. *For every orthogonal ring spectrum R, the suspension functor $\Sigma : \mathcal{D}(R) \longrightarrow \mathcal{D}(R)$ is a self-equivalence of the derived category of R.*

Proof. The loop functor $\Omega : R\text{-mod} \longrightarrow R\text{-mod}$ preserves stable equivalences by Proposition 2.16. The universal property of the localization functor $\gamma : R\text{-mod} \longrightarrow \mathcal{D}(R)$ thus provides a unique functor

$$\Sigma^{-1} \; : \; \mathcal{D}(R) \; \longrightarrow \; \mathcal{D}(R)$$

that satisfies $\Sigma^{-1} \circ \gamma = \gamma \circ \Omega$. The two composite constructions $\Omega(M \wedge S^1)$ and $(\Omega M) \wedge S^1$ come with natural stable equivalences

$$\eta \; : \; M \; \longrightarrow \; \Omega(M \wedge S^1) \qquad \text{and} \qquad \epsilon \; : \; (\Omega M) \wedge S^1 \; \longrightarrow \; M,$$

the unit and counit of the adjunction of $(- \wedge S^1, \Omega)$, see Proposition 1.13. Since all functors in sight descend to the derived category, these natural stable equivalences descend to natural isomorphisms

$$\mathrm{Id}_{\mathcal{D}(R)} \; \cong \; \Sigma^{-1} \circ \Sigma \qquad \text{and} \qquad \Sigma \circ \Sigma^{-1} \; \cong \; \mathrm{Id}_{\mathcal{D}(R)}$$

of endofunctors on $\mathcal{D}(R)$. So Σ^{-1} is a quasi-inverse to the suspension functor Σ, which is thus an equivalence of categories. □

7. Triangulated categories

We have seen that the stable homotopy category, and more generally the derived category of an orthogonal ring spectrum, is an additive category with coproducts and finite products. In this section we make $\mathcal{D}(R)$ into a triangulated category. The arguments apply more generally to certain classes of cofibration categories, and we work in that generality. First we recall the definition.

Let $\mathcal{T}$ be a category equipped with an endofunctor $\Sigma : \mathcal{T} \longrightarrow \mathcal{T}$. A *triangle* in $\mathcal{T}$ (with respect to the functor Σ) is a triple (f, g, h) of composable morphisms in $\mathcal{T}$ such that the target of h is equal to Σ applied to the source of f. We will often display a triangle in the form

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A \ .$$

A *morphism* from a triangle (f, g, h) to a triangle (f', g', h') is a triple of morphisms $a : A \longrightarrow A'$, $b : B \longrightarrow B'$ and $c : C \longrightarrow C'$ in $\mathcal{T}$ such that the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[r, "g"] \arrow[d, "b"'] & C \arrow[r, "h"] \arrow[d, "c"] & \Sigma A \arrow[d, "\Sigma a"] \\ A' \arrow[r, "f'"'] & B' \arrow[r, "g'"'] & C' \arrow[r, "h'"'] & \Sigma A' \end{tikzcd}$$

commutes. A morphism of triangles is an isomorphism (i.e., has an inverse morphism) if and only all three components are isomorphisms in $\mathcal{T}$.

Definition 7.1. A *triangulated category* is an additive category $\mathcal{T}$ equipped with a self-equivalence $\Sigma : \mathcal{T} \longrightarrow \mathcal{T}$ and a collection of triangles, called *distinguished triangles*, which satisfy the following axioms (T0) – (T5).

We refer to the equivalence Σ of a triangulated category as the *suspension*, since that is what it will be in our main example. In algebraic contexts, this equivalence is often denoted $X \mapsto X[1]$ and called the 'shift'.

- **(T0)** The class of distinguished triangles is closed under isomorphism.
- **(T1)** Every morphism f is part of a distinguished triangle (f, g, h).
- **(T2)** For every object X the triangle $0 \longrightarrow X \xrightarrow{\mathrm{Id}} X \longrightarrow 0$ is distinguished.
- **(T3)** [Rotation] If a triangle (f, g, h) is distinguished, then so is the triangle $(g, h, -\Sigma f)$.
- **(T4)** [Completion of triangles] Given distinguished triangles (f, g, h) and (f', g', h') morphisms (a, b) satisfying $bf = f'a$, there exists a morphism c making the following diagram commute:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[r, "g"] \arrow[d, "b"'] & C \arrow[r, "h"] \arrow[d, dotted, "c"] & \Sigma A \arrow[d, "\Sigma a"] \\ A' \arrow[r, "f'"'] & B' \arrow[r, "g'"'] & C' \arrow[r, "h'"'] & \Sigma A' \end{tikzcd}$$

- **(T5)** [Octahedral axiom] For every pair of composable morphisms $f : A \longrightarrow B$ and $f' : B \longrightarrow D$ there is a commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, equal] & B \arrow[r, "g"] \arrow[d, "f'"'] & C \arrow[r, "h"] \arrow[d, "x"] & \Sigma A \arrow[d, equal] \\ A \arrow[r, "f'f"'] & D \arrow[r, "g''"'] \arrow[d, "g'"'] & E \arrow[r, "h''"'] \arrow[d, "y"] & \Sigma A \arrow[d, "\Sigma f"] \\ & F \arrow[r, equal] \arrow[d, "h'"'] & F \arrow[r, "h'"'] \arrow[d, "(\Sigma g)\circ h'"] & \Sigma B \\ & \Sigma B \arrow[r, "\Sigma g"'] & \Sigma C & \end{tikzcd}$$

such that the triangles (f, g, h), (f', g', h'), $(f'f, g'', h'')$ and $(x, y, (\Sigma g) \circ h')$ are distinguished.

The above formulation of the axioms appears to be weaker, at first sight, than the original axioms of Verdier [50, II.1]; however, we show in Proposition 7.17 below that the weaker axioms imply the stronger properties: part (iii) establishes an 'if and only if' in the rotation axiom (T3), and part (iv) is the octahedral axiom in its original form.

We recall that a category $\mathcal{C}$ is *pointed* if it has a *zero object*, i.e., an object that is simultaneously initial and terminal; we will denote zero objects by '$*$'. In pointed categories, we will denote a coproduct of two objects X and Y by $X \vee Y$.

Definition 7.2. Let $\mathcal{C}$ be a pointed cofibration category. A *functorial cone* consists of a functor $C : \mathcal{C} \longrightarrow \mathcal{C}$ and a natural transformation $\iota : \mathrm{Id} \longrightarrow C$ such that for all $\mathcal{C}$-objects X, the morphism $\iota_X : X \longrightarrow CX$ is a cofibration, and the unique morphism $CX \longrightarrow *$ is a weak equivalence.

The main example for our purposes is the category of modules over an orthogonal ring spectrum. There, the standard cone $M \wedge [0,1]$ together with the 'end point inclusion' $-\wedge 1 : M \longrightarrow M \wedge [0,1]$ form a functorial cone for the cofibration structure of Theorem 6.8, compare also Example 6.10. However, there are many other examples, see Example 7.12 or Example 7.14 below.

Construction 7.3. Let $\mathcal{C}$ be a pointed cofibration category with a functorial cone. Then the *suspension functor*

$$\Sigma \ : \ \mathcal{C} \ \longrightarrow \ \mathcal{C}$$

is defined by

$$\Sigma X \ = \ (CX)/\iota_X \ ,$$

i.e., ΣX is defined as a pushout

$$\begin{tikzcd} X \arrow[r] \arrow[d, tail, "\iota_X"'] & * \arrow[d] \\ CX \arrow[r, "q_X"'] & \Sigma X \end{tikzcd}$$

Any morphism $CX \longrightarrow CY$ is a weak equivalence by the 2-out-of-3 property, because the unique morphisms $CX \longrightarrow *$ and $CY \longrightarrow *$ are weak equivalences. So if $f : X \longrightarrow Y$ is a weak equivalence, then the gluing lemma (Proposition 5.3) shows that the morphism $\Sigma f : \Sigma X \longrightarrow \Sigma Y$ is a weak equivalence.

Because the suspension functor preserves weak equivalences, it descends to a functor on the homotopy category $\Sigma : \mathrm{Ho}(\mathcal{C}) \longrightarrow \mathrm{Ho}(\mathcal{C})$, for which we use the same name. Indeed,the composite functor $\gamma \circ \Sigma : \mathcal{C} \longrightarrow \mathrm{Ho}(\mathcal{C})$ takes weak equivalences to isomorphisms. The universal property of the localization functor $\gamma : \mathcal{C} \longrightarrow \mathrm{Ho}(\mathcal{C})$ provides a unique functor

$$\Sigma \ : \ \mathrm{Ho}(\mathcal{C}) \ \longrightarrow \ \mathrm{Ho}(\mathcal{C})$$

that satisfies $\Sigma \circ \gamma = \gamma \circ \Sigma$. In particular, Σ is given on objects by the previous suspension functor, and the behavior on morphisms is as follows. Every morphism $X \longrightarrow Y$ in $\mathrm{Ho}(\mathcal{C})$ is of the form $\gamma(\tau)^{-1} \circ \gamma(f)$ for two $\mathcal{C}$-morphisms $f : X \longrightarrow Z$ and $\tau : Y \longrightarrow Z$ such that τ is a weak equivalence. Then $\Sigma\tau : \Sigma X \longrightarrow \Sigma Z$ is a weak equivalence, too, and

$$\Sigma(\gamma(\tau)^{-1} \circ \gamma(f)) \ = \ \gamma(\Sigma\tau)^{-1} \circ \gamma(\Sigma f) \ .$$

Construction 7.4. We introduce the distinguished triangles in the homotopy category of a pointed cofibration category with functorial cones. The *elementary distinguished triangle* associated to a $\mathcal{C}$-morphism $\psi : X \longrightarrow Y$ is the sequence

$$X \ \xrightarrow{\gamma(\psi)} \ Y \ \xrightarrow{\gamma(i)} \ C\psi \ \xrightarrow{\gamma(p)} \ \Sigma X \ .$$

Here $C\psi$ is the mapping cone of ψ, defined by a pushout square

$$\begin{tikzcd} X \arrow[r, "\psi"] \arrow[d, tail, "\iota_X"'] & Y \arrow[d, tail, "i"] \\ CX \arrow[r] & C\psi \end{tikzcd}$$

The third morphism is $p = q_X \cup 0 : C\psi = CX \cup_\psi Y \longrightarrow \Sigma X$.

A *distinguished triangle* is any triangle (f, g, h) in $\mathrm{Ho}(\mathcal{C})$ which is isomorphic to an elementary distinguished triangle, i.e., such that there is a $\mathcal{C}$-morphism $\psi : X \longrightarrow Y$ and isomorphisms $a : X \longrightarrow A$, $b : Y \longrightarrow B$ and $c : C\psi \longrightarrow C$ in $\mathrm{Ho}(\mathcal{C})$ that make the diagram

$$\begin{tikzcd}
X \arrow[r, "\gamma(\psi)"] \arrow[d, "a"', "\cong"] & Y \arrow[r, "\gamma(i)"] \arrow[d, "b"', "\cong"] & C\psi \arrow[r, "\gamma(p)"] \arrow[d, "\cong"', "c"] & \Sigma X \arrow[d, "\cong"', "\Sigma a"] \\
A \arrow[r, "f"'] & B \arrow[r, "g"'] & C \arrow[r, "h"'] & \Sigma A
\end{tikzcd}$$

commute.

Construction 7.5 (Distinguished triangles from cofibrations)**.** As we shall now explain, cofibrations are another source of distinguished triangles. Given any cofibration $j : A \longrightarrow B$ in $\mathcal{C}$, we write B/A for any cokernel of j, and $q : B \longrightarrow B/A$ for the projection. This is a slight abuse of notation, because B/A depends on the morphism j (and not just on its source and target). Applying the gluing lemma to the commutative diagram

$$\begin{tikzcd}
CA \arrow[d, "\sim"'] & A \arrow[l, tail, "\iota_A"'] \arrow[r, tail, "j"] \arrow[d, equal] & B \arrow[d, equal] \\
* & A \arrow[l] \arrow[r, tail, "j"'] & B
\end{tikzcd}$$

shows that the morphism

$$0 \cup q \ : \ Cj \ = \ CA \cup_j B \ \longrightarrow \ B/A$$

is a weak equivalence. We define the *connecting morphism* $\delta(j) : B/A \longrightarrow \Sigma A$ in $\mathrm{Ho}(\mathcal{C})$ as

$$\delta(j) \ = \ \gamma(p) \circ \gamma(0 \cup q)^{-1} \ : B/A \ \longrightarrow \ \Sigma A \ .$$

Here $p = (q_X \cup 0) : Cj \longrightarrow \Sigma A$ is the 'projection' that was already considered above. The following diagram commutes by definition:

$$\begin{tikzcd}
A \arrow[r, "\gamma(j)"] \arrow[d, equal] & B \arrow[r, "\gamma(i)"] \arrow[d, equal] & Cj \arrow[r, "\gamma(p)"] \arrow[d, "\cong"', "\gamma(0\cup q)"] & \Sigma A \arrow[d, equal] \\
A \arrow[r, "\gamma(j)"'] & B \arrow[r, "\gamma(q)"'] & B/A \arrow[r, "\delta(j)"'] & \Sigma A
\end{tikzcd}$$

The upper row is an elementary distinguished triangle and all vertical morphisms are isomorphisms. So the lower triangle is distinguished.

Proposition 7.6. *Let $\mathcal{C}$ be a pointed cofibration category with functorial cones. A triangle if $\mathrm{Ho}(\mathcal{C})$ is distinguished if and only if it is isomorphic to a triangle of the form $(\gamma(j), \gamma(q), \delta(j))$ for some cofibration $j : A \longrightarrow B$.*

Proof. We already showed that triangles of the form $(\gamma(j), \gamma(q), \delta(j))$ for cofibrations j are distinguished. For the reverse implication we consider any $\mathcal{C}$-morphism $\psi : X \longrightarrow Y$. We choose a factorization $\psi = \pi j$ for some cofibration $j : X \longrightarrow Z$ and some weak equivalence $\pi : Z \longrightarrow Y$. All vertical morphisms in the following commutative diagram are weak equivalences, either by definition of by the gluing lemma:

$$\begin{tikzcd}
X \arrow[r, "j"] \arrow[d, equal] & Z \arrow[r, "q"] \arrow[d, equal] & Z/X & \\
X \arrow[r, "j"] \arrow[d, equal] & Z \arrow[r, "i_j"] \arrow[d, "\pi", "\sim"'] & Cj \arrow[r, "p_j"] \arrow[u, "0\cup q"', "\sim"] \arrow[d, "\mathrm{Id}_{CX}\cup\pi", "\sim"'] & \Sigma X \arrow[d, equal] \\
X \arrow[r, "\psi"'] & Y \arrow[r, "i_\psi"'] & C\psi \arrow[r, "p_\psi"'] & \Sigma X
\end{tikzcd}$$

After applying the localization functor, we thus obtain a commutative diagram in $\mathrm{Ho}(\mathcal{C})$ in which all vertical morphisms are isomorphisms:

$$\begin{tikzcd}
X \arrow[r, "\gamma(j)"] \arrow[d, equal] & Z \arrow[r, "\gamma(q)"] \arrow[d, "\gamma(\pi)", "\cong"'] & Z/X \arrow[r, "\delta(j)"] \arrow[d, "\gamma(\mathrm{Id}\cup\pi)\circ\gamma(0\cup q)^{-1}", "\cong"'] & \Sigma X \arrow[d, equal] \\
X \arrow[r, "\gamma(\psi)"'] & Y \arrow[r, "\gamma(i_\psi)"'] & C\psi \arrow[r, "\gamma(p_\psi)"'] & \Sigma X
\end{tikzcd}$$

This shows that the elementary distinguished triangle of $\psi : X \longrightarrow Y$ is isomorphic to one of the form $(\gamma(j), \gamma(q), \delta(j))$. □

Proposition 7.7. *Let $\mathcal{C}$ be a pointed cofibration category with functorial cones, and suppose that $\mathrm{Ho}(\mathcal{C})$ is additive. Let $\psi : X \longrightarrow Y$ be a $\mathcal{C}$-morphism. Then the connecting homomorphism of the cofibration $i : Y \longrightarrow C\psi$ satisfies the relation*

$$\delta(i) \;=\; -\Sigma\gamma(\psi) \;:\; \Sigma X \;\longrightarrow\; \Sigma Y \;.$$

Proof. We start by showing that the two morphisms

$$q_Y \cup 0 \;,\; 0 \cup q_Y \;:\; CY \cup_Y CY \;\longrightarrow\; \Sigma Y$$

become additive inverses in the abelian monoid $\mathrm{Ho}(CY \cup_Y CY, \Sigma Y)$. To this end we consider the morphism $\xi : CY \cup_Y CY \longrightarrow \Sigma Y \vee \Sigma Y$ induced by taking horizontal pushouts of the commutative diagram

$$\begin{tikzcd}
CY \vee CY \arrow[d, equal] & Y \vee Y \arrow[l, tail, "\iota_Y \vee \iota_Y"'] \arrow[r, "\nabla"] \arrow[d, equal] & Y \arrow[d, "\sim"] \\
CY \vee CY & Y \vee Y \arrow[l, tail, "\iota_Y \vee \iota_Y"] \arrow[r] & *
\end{tikzcd}$$

Then

$$(\mathrm{Id}_{\Sigma Y} + 0) \circ \xi \;=\; q_Y \cup 0 \qquad \text{and} \qquad (0 + \mathrm{Id}_{\Sigma Y}) \circ \xi \;=\; 0 \cup q_Y \;.$$

This means that

$$\gamma(\xi) \;=\; \gamma(q_Y \cup 0) \perp \gamma(0 \cup q_Y)$$

as morphisms in $\mathrm{Ho}(\mathcal{C})$. The following square commutes in $\mathcal{C}$:

$$\begin{tikzcd}
CY \cup_Y CY \arrow[r, "\xi"] \arrow[d, "\mathrm{Id}\cup\mathrm{Id}"'] & \Sigma Y \vee \Sigma Y \arrow[d, "\nabla"] \\
CY \arrow[r, "q_Y"'] & \Sigma Y
\end{tikzcd}$$

Because the cone CY is weakly equivalent to the zero object, it becomes a zero object in $\mathrm{Ho}(\mathcal{C})$. We conclude that

$$\gamma(q_Y \cup 0) + \gamma(0 \cup q_Y) \;=\; \gamma(\nabla) \circ (\gamma(q_Y \cup 0) \perp \gamma(0 \cup q_Y)) \;=\; \gamma(\nabla) \circ \gamma(\xi) \;=\; 0$$

in the abelian monoid $\mathrm{Ho}(CY \cup_Y CY, \Sigma Y)$. This proves that $\gamma(q_Y \cup 0) = -\gamma(0 \cup q_Y)$

Now we consider the morphism $\zeta : CY \cup_i C\psi \longrightarrow CY \cup_Y CY$, induced by taking horizontal pushouts of the commutative diagram

$$\begin{tikzcd} CY \vee CX \arrow[d, "\mathrm{Id} \vee C(\psi)"'] & X \vee X \arrow[l, tail, "(\iota_Y \circ \psi) \vee \iota_X"'] \arrow[r, "\nabla"] \arrow[d, "\psi \vee \psi"'] & X \arrow[d, "\psi"] \\ CY \vee CY & Y \vee Y \arrow[l, tail, "\iota_Y \vee \iota_Y"] \arrow[r, "\nabla"'] & Y \end{tikzcd}$$

Then

$$(q_Y \cup 0) \circ \zeta \ = \ q_Y \cup 0 \qquad \text{and} \qquad (0 \cup q_Y) \circ \zeta \ = \ (\Sigma\psi) \circ (0 \cup p)$$

as $\mathcal{C}$-morphisms $CY \cup_i C\psi \longrightarrow \Sigma Y$ Thus

$$\begin{aligned} \delta(i) &= \gamma(q_Y \cup 0) \circ \gamma(0 \cup p)^{-1} \\ &= \gamma(q_Y \cup 0) \circ \gamma(\zeta) \circ \gamma(0 \cup p)^{-1} \\ &= -\gamma(0 \cup q_Y) \circ \gamma(\zeta) \circ \gamma(0 \cup p)^{-1} \\ &= -\gamma(\Sigma\psi) \circ \gamma(0 \cup p) \circ \gamma(0 \cup p)^{-1} \ = \ -\gamma(\Sigma\psi) \ . \end{aligned}$$

□

Now we can state and prove the main result of this section.

Theorem 7.8. *Let $\mathcal{C}$ be a pointed cofibration category with functorial cones. Suppose moreover that* $\mathrm{Ho}(\mathcal{C})$ *is additive, and that the suspension functor* $\Sigma : \mathrm{Ho}(\mathcal{C}) \longrightarrow \mathrm{Ho}(\mathcal{C})$ *is an autoequivalence of categories. Then the suspension functor and the class of distinguished triangles make the derived category* $\mathrm{Ho}(\mathcal{C})$ *into a triangulated category.*

Proof. It remains to prove the axioms (T0) – (T5).

(T0) By definition, the class of distinguished triangles is closed under isomorphism.

(T1) We let $f : A \longrightarrow B$ be a morphism in $\mathrm{Ho}(\mathcal{C})$. We appeal to the calculus of fractions (Theorem 5.13) to write $f = \gamma(s)^{-1} \circ \gamma(\psi)$ for two $\mathcal{C}$-morphisms $\psi : A \longrightarrow D$ and $s : B \longrightarrow D$ such that s is a weak equivalence. Then the following diagram of triangles commutes in $\mathrm{Ho}(\mathcal{C})$:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, equal] & B \arrow[r, "\gamma(is)"] \arrow[d, "\gamma(s)"', "\cong"] & C\psi \arrow[r, "\gamma(p)"] \arrow[d, equal] & \Sigma A \arrow[d, equal] \\ A \arrow[r, "\gamma(\psi)"'] & D \arrow[r, "\gamma(i)"'] & C\psi \arrow[r, "\gamma(p)"'] & \Sigma A \end{tikzcd}$$

All vertical morphisms in the diagram are isomorphisms, and the lower row is an elementary distinguished triangle. So the upper row is the desired distinguished triangle starting with f.

(T2) The identity $\mathrm{Id}_X : X \longrightarrow X$ is a cokernel of the unique morphism $0 : * \longrightarrow X$. So the triangle $(0, \mathrm{Id}_X, 0)$ is distinguished by Construction 7.5, applied to the cofibration $0 : * \longrightarrow X$.

(T3 – Rotation) We let (f, g, h) be a distinguished triangle; we need to show that the triangle $(g, h, -\Sigma f)$ is also distinguished. Since the class of distinguished triangles is closed under isomorphisms, it suffices to consider the elementary distinguished triangle $(\gamma(\psi), \gamma(i), \gamma(p))$ associated to a $\mathcal{C}$-morphism $\psi : X \longrightarrow Y$. The morphism $i : Y \longrightarrow C\psi$ is a cofibration, and the morphism $p = q_X \cup 0 : C\psi = CX \cup_\psi Y \longrightarrow \Sigma X$ exhibits the suspension ΣX as a cokernel of i. So the triangle

$$Y \xrightarrow{\gamma(i)} C\psi \xrightarrow{\gamma(p)} \Sigma X \xrightarrow{\delta(i)} \Sigma Y$$

is distinguished, as explained in Construction 7.5. By Proposition 7.7, the connecting morphism $\delta(i)$ of the cofibration $i : Y \longrightarrow C\psi$ is the *additive inverse* of the morphism $\Sigma\gamma(\psi) = \gamma(\Sigma\psi) : \Sigma X \longrightarrow \Sigma Y$. So the rotated triangle $(\gamma(i), \gamma(p), -\Sigma\gamma(\psi))$ is distinguished.

(T4 – Completion of triangles) We are given two distinguished triangles (f,g,h) and (f',g',h') and two morphisms a and b in $\mathrm{Ho}(\mathcal{C})$ satisfying $bf = f'a$ as in the diagram:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[r, "g"] \arrow[d, "b"] & C \arrow[r, "h"] \arrow[d, dashed, "c"] & \Sigma A \arrow[d, "\Sigma a"] \\
A' \arrow[r, "f'"'] & B' \arrow[r, "g'"'] & C' \arrow[r, "h'"'] & \Sigma A'
\end{tikzcd}$$

We have to extend this data to a morphism of triangles, i.e., to find a morphism c making the entire diagram commute. If we can solve the problem for isomorphic triangles, then we can also solve it for the original triangles. By Proposition 7.6 we can thus assume that the triangles (f,g,h) and (f',g',h') are the distinguished triangle arising from two cofibrations $j : A \longrightarrow B$ and $j' : A' \longrightarrow B'$ via Construction 7.5.

We start with the special case where $a = \gamma(\alpha)$ and $b = \gamma(\beta)$ for $\mathcal{C}$-morphisms $\alpha : A \longrightarrow A'$ and $\beta : B \longrightarrow B'$. Then $\gamma(j'\alpha) = \gamma(\beta j)$, so the calculus of fractions (Theorem 5.13 (ii)) provides an acyclic cofibration $s : B' \longrightarrow \bar{B}$, a cylinder object (I, i_0, i_1, p) for A and a homotopy $H : I \longrightarrow \bar{B}$ from $Hi_0 = sj'\alpha$ to $Hi_1 = s\beta j$. The following diagram of cofibrations on the left commutes in $\mathcal{C}$, so the diagram of distinguished triangles on the right commutes in $\mathrm{Ho}(\mathcal{C})$ by the naturality of the connecting morphisms:

$$\begin{tikzcd}
A \arrow[r, tail, "j"] \arrow[d, equal] & B \\
A \arrow[r, tail, "i_0"] \arrow[d, "\alpha"'] & I \cup_{i_1} B \arrow[u, "\sim"', "jp\cup B"] \arrow[d, "H\cup s\beta"] \\
A' \arrow[r, tail, "sj'"] \arrow[d, equal] & \bar{B} \\
A' \arrow[r, tail, "j'"'] & B' \arrow[u, "\sim"', "s"]
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "\gamma(j)"] \arrow[d, equal] & B \arrow[r, "\gamma(q)"] & B/A \arrow[r, "\delta(j)"] & \Sigma A \arrow[d, equal] \\
A \arrow[r, "\gamma(i_0)"] \arrow[d, "\gamma(\alpha)"'] & I \cup_{i_1} B \arrow[r, "\gamma(q)"] \arrow[u, "\gamma(jp\cup B)", "\cong"'] \arrow[d, "\gamma(H\cup s\beta)"'] & (I \cup_{i_1} B)/A \arrow[r, "\delta(i_0)"] \arrow[u, "\cong", "\gamma((jp\cup B)/A)"'] \arrow[d, "\gamma((H\cup s\beta)/\alpha)"] & \Sigma A \arrow[d, "\Sigma\gamma(\alpha)"] \\
A' \arrow[r, "\gamma(sj')"] \arrow[d, equal] & \bar{B} \arrow[r, "\gamma(\bar{q})"] & \bar{B}/A' \arrow[r, "\delta(sj')"] & \Sigma A' \arrow[d, equal] \\
A' \arrow[r, "\gamma(j')"'] & B' \arrow[r, "\gamma(q')"'] \arrow[u, "\gamma(s)", "\cong"'] & B'/A' \arrow[r, "\delta(j')"'] \arrow[u, "\cong", "\gamma(s/A')"'] & \Sigma A'
\end{tikzcd}$$

The canonical morphism $\kappa : B \longrightarrow I \cup_{i_1} B$ is right inverse to $jp \cup B : I \cup_{i_1} B \longrightarrow B$, so

$$\begin{aligned}
\gamma(s)^{-1} \circ \gamma(H \cup s\beta) \circ \gamma(jp \cup B)^{-1} \;&=\; \gamma(s)^{-1} \circ \gamma(H \cup s\beta) \circ \gamma(\kappa) \\
&=\; \gamma(s)^{-1} \circ \gamma(s\beta) \;=\; \gamma(\beta) \;=\; b\ .
\end{aligned}$$

So the morphism

$$c \;=\; \gamma(s/A')^{-1} \circ \gamma((H \cup s\beta)/\alpha) \circ \gamma((jp \cup B)/A)^{-1} \;:\; B/A \longrightarrow B'/A'$$

is the desired filler.

In the general case we write $a = \gamma(s)^{-1}\gamma(\alpha)$ where $\alpha : A \longrightarrow \bar{A}$ and $s : A' \longrightarrow \bar{A}$ are $\mathcal{C}$-morphisms and s is an acyclic cofibration. We choose a pushout

$$\begin{tikzcd}
\bar{A} \arrow[r, tail, "k"] & \bar{A} \cup_{A'} B' \\
A' \arrow[u, tail, "s", "\sim"'] \arrow[r, tail, "j'"'] & B' \arrow[u, tail, "\sim", "s'"']
\end{tikzcd}$$

Another application of the calculus of fractions lets us write $\gamma(s')b = \gamma(t)^{-1}\gamma(\beta) : B \longrightarrow \bar{A} \cup_{A'} B'$ where $\beta : B \longrightarrow \bar{B}$ and $t : \bar{A} \cup_{A'} B' \longrightarrow \bar{B}$ are $\mathcal{C}$-morphisms, and t is an acyclic cofibration. We then have

$$\gamma(tk)\gamma(\alpha) \;=\; \gamma(tk)\gamma(s)a \;=\; \gamma(ts')\gamma(j')a \;=\; \gamma(ts')b\gamma(j) \;=\; \gamma(\beta)\gamma(j)\ ,$$

so by the special case, applied to the cofibrations $j : A \longrightarrow B$ and $tk : \bar{A} \longrightarrow \bar{B}$ and the morphisms $\alpha : A \longrightarrow \bar{A}$ and $\beta : B \longrightarrow \bar{B}$, there exists a morphism $c : B/A \longrightarrow \bar{B}/\bar{A}$ in $\mathrm{Ho}(\mathcal{C})$ making the diagram

$$\begin{tikzcd}
A \arrow[r, "\gamma(j)"] \arrow[d, "\gamma(\alpha)"'] & B \arrow[r, "\gamma(q)"] \arrow[d, "\gamma(\beta)"'] & B/A \arrow[r, "\delta(j)"] \arrow[d, dotted, "c"'] & \Sigma A \arrow[d, "\Sigma\gamma(\alpha)"] \\
\bar{A} \arrow[r, "\gamma(tk)"] & \bar{B} \arrow[r, "\gamma(\bar{q})"] & \bar{B}/\bar{A} \arrow[r, "\delta(tk)"] & \Sigma\bar{A} \\
A' \arrow[r, "\gamma(j')"'] \arrow[u, "\gamma(s)", "\cong"'] & B' \arrow[r, "\gamma(q')"'] \arrow[u, "\gamma(ts')", "\cong"'] & B'/A' \arrow[r, "\delta(j')"'] \arrow[u, "\cong", "\gamma(ts'/s)"'] & \Sigma A' \arrow[u, "\cong", "\Sigma\gamma(s)"']
\end{tikzcd}$$

commute (the lower part commutes by naturality of connecting morphisms). Since s is an acyclic cofibration, so is its cobase change s'. By the gluing lemma the weak equivalences $s : A' \longrightarrow \bar{A}$ and $ts' : B' \longrightarrow \bar{B}$ induce a weak equivalence $ts'/s : B'/A' \longrightarrow \bar{B}/\bar{A}$ on quotients and the composite

$$B/A \xrightarrow{\ c\ } \bar{B}/\bar{A} \xrightarrow{\gamma(ts'/s)^{-1}} B'/A'$$

in $\mathrm{Ho}(\mathcal{C})$ thus solves the original problem.

(T5 - Octahedral axiom) We start with the special case where $f = \gamma(j)$ and $f' = \gamma(j')$ for cofibrations $j : A \longrightarrow B$ and $j' : B \longrightarrow D$. Then the composite $j'j : A \longrightarrow D$ is a cofibration with $\gamma(j'j) = f'f$. The diagram

$$\begin{tikzcd}
A \arrow[r, "\gamma(j)"] \arrow[d, equal] & B \arrow[r, "\gamma(q_j)"] \arrow[d, "\gamma(j')"'] & B/A \arrow[r, "\delta(j)"] \arrow[d, "\gamma(j'/A)"] & \Sigma A \arrow[d, equal] \\
A \arrow[r, "\gamma(j'j)"'] & D \arrow[r, "\gamma(q_{j'j})"'] \arrow[d, "\gamma(q_{j'})"'] & D/A \arrow[r, "\delta(j'j)"'] \arrow[d, "\gamma(D/j)"] & \Sigma A \arrow[d, "\Sigma\gamma(j)"] \\
 & D/B \arrow[r, equal] \arrow[d, "\delta(j')"'] & D/B \arrow[r, "\delta(j')"'] \arrow[d, "\delta(j'/A)=(\Sigma\gamma(q_j))\delta(j')"] & \Sigma B \\
 & \Sigma B \arrow[r, "\Sigma\gamma(q_j)"'] & \Sigma(B/A) &
\end{tikzcd}$$

then commutes by naturality of connecting morphisms. Moreover, the four triangles in question are the distinguished triangles of the cofibrations j, j', $j'j$ and $j'/A : B/A \longrightarrow D/A$.

In the general case we write $f = \gamma(s)^{-1}\gamma(a)$ for $\mathcal{C}$-morphisms $a : A \longrightarrow B'$ and $s : B \longrightarrow B'$, such that s is a weak equivalence. Then a can be factored as $a = pj$ for a cofibration $j : A \longrightarrow \bar{B}$ and a weak equivalence $p : \bar{B} \longrightarrow B'$. Altogether we then have $f = \varphi \circ \gamma(j)$ where $\varphi = \gamma(s)^{-1} \circ \gamma(p) : \bar{B} \longrightarrow B$ is an isomorphism in $\mathrm{Ho}(\mathcal{C})$. We can apply the same reasoning to the morphism $f'\varphi : \bar{B} \longrightarrow D$ and write it as $f' \circ \varphi = \psi \circ \gamma(j')$ for a cofibration $j' : \bar{B} \longrightarrow \bar{D}$ and an isomorphism $\psi : \bar{D} \longrightarrow D$ in $\mathrm{Ho}(\mathcal{C})$. The special case can then be applied to the cofibrations $j : A \longrightarrow \bar{B}$ and $j' : \bar{B} \longrightarrow \bar{D}$. The resulting commutative diagram that solves (T5) for $(\gamma(j), \gamma(j'))$ can then be translated back into a commutative diagram that solves (T5) for (f, f') by conjugating with the isomorphisms $\varphi : \bar{B} \longrightarrow B$ and $\psi : \bar{D} \longrightarrow D$. This completes the proof of the octahedral axiom (T5), and hence the proof of the theorem. □

Remark 7.9. Theorem 7.8 is not best possible in the sense that the functoriality of the cone is unnecessary and the additivity requirement for $\mathrm{Ho}(\mathcal{C})$ is redundant. Indeed, in a pointed cofibration category, we can always choose a cone, i.e., a factorization of the unique morphism $X \longrightarrow *$ as a cofibration $\iota : X \longrightarrow C$ followed by a weak equivalence $C \longrightarrow *$. The suspension is then again the cokernel $\Sigma X = C/X$ of the cofibration ι. While such cones might not be functorial at the level of the cofibration category, one can show that the suspension construction descends to an endofunctor of the homotopy category, compare [39, Proposition A.4]. In this sense, functoriality of cones is an unnecessary assumption. Moreover, the assumption that the suspension functor is an autoequivalence of $\mathrm{Ho}(\mathcal{C})$ already implies that $\mathrm{Ho}(\mathcal{C})$ is additive, see [39, Proposition A.8]. In this sense, the additivity hypothesis is redundant. The reason why we require the unnecessary assumptions in Theorem 7.8 is that they simplify the proof substantially. The reader is invite to consult [39, Theorem A.12] for a proof that does not require functoriality of cones additivity.

Example 7.10 (Triangulated structure on $\mathcal{D}(R)$)**.** Now we specialize to the most important case for our purposes, the pointed category of modules over an orthogonal ring spectrum R, with the cofibration structure from Theorem 6.8. A functorial cone is given by smashing with the interval $[0, 1]$, based at 0. We recall that $\mathbf{t} : [0, 1] \longrightarrow S^1$ is the quotient map defined by $\mathbf{t}(x) = \frac{2x-1}{x(1-x)}$. The morphism

$$A \wedge t \ : \ A \wedge [0, 1] \ \longrightarrow \ A \wedge S^1$$

witness the suspension $A \wedge S^1$ as a cokernel of the 'cone inclusion' $- \wedge 1 : A \longrightarrow A \wedge [0, 1]$. So the suspension functor associated to this functorial cone is the usual suspension.

The *elementary distinguished triangle* in $\mathcal{D}(R)$ associated to a morphism $\psi : A \longrightarrow B$ of R-modules is thus the sequence

$$A \xrightarrow{\gamma(\psi)} B \xrightarrow{\gamma(i)} C\psi \xrightarrow{\gamma(p)} \Sigma A \ .$$

Here $C\psi = A \wedge [0, 1] \cup_\psi B$ is the usual mapping cone of ψ, $i : B \longrightarrow C\psi$ the inclusion, and $p = (A \wedge \mathbf{t}) \cup 0 : C\psi \longrightarrow A \wedge S^1$.

A *distinguished triangle* is any triangle (f, g, h) in $\mathcal{D}(R)$ which is isomorphic to an elementary distinguished triangle, i.e., such that there is a morphism $\psi : A \longrightarrow B$ of R-modules and isomorphisms $a : A \longrightarrow X$, $b : B \longrightarrow Y$ and $c : C\psi \longrightarrow Z$ in $\mathcal{D}(R)$ that make the diagram

$$\begin{tikzcd}
A \arrow[r, "\gamma(\psi)"] \arrow[d, "a"'] & B \arrow[r, "\gamma(i)"] \arrow[d, "b"'] & C\psi \arrow[r, "\gamma(p)"] \arrow[d, "c"] & \Sigma A \arrow[d, "\Sigma a"] \\
X \arrow[r, "f"'] & Y \arrow[r, "g"'] & Z \arrow[r, "h"'] & \Sigma X
\end{tikzcd}$$

commute.

We showed in Theorem 6.17 (ii) that the derived category $\mathcal{D}(R)$ is additive, and in Proposition 6.18 that the suspension functor $\Sigma : \mathcal{D}(R) \longrightarrow \mathcal{D}(R)$ is a self-equivalence. So Theorem 7.8 applied, and yields:

Corollary 7.11. *For every orthogonal ring spectrum R, the suspension functor and the class of distinguished triangles make the derived category $\mathcal{D}(R)$ into a triangulated category.*

Example 7.12 (Triangulated structure on $\mathcal{K}(\mathcal{A})$)**.** We let $\mathcal{A}$ be an additive category, and we write $\mathrm{Ch}(\mathcal{A})$ for the category of $\mathbb{Z}$-graded chain complexes in $\mathcal{A}$. We write $\mathcal{K}(A)$ for the algebraic homotopy category, with the same objects as $\mathrm{Ch}(\mathcal{A})$, and with chain homotopy classes of chain maps as morphisms. We call a chain map $f : A \longrightarrow B$ in $\mathrm{Ch}(\mathcal{A})$ a *cofibration* if it is dimensionwise a split monomorphism, i.e., for every $n \in \mathbb{Z}$ there is an $\mathcal{A}$-object C and an isomorphism $A_n \oplus C \cong B_n$ that restricts to $f_n : A_n \longrightarrow B_n$ on the first summand. Exercise E.10 (c) is devoted to showing that these cofibrations and the chain homotopy equivalences form a cofibration structure on the category $\mathrm{Ch}(\mathcal{A})$. And the content of Exercise E.11 is to check that the quotient functor $\pi : \mathrm{Ch}(\mathcal{A}) \longrightarrow \mathcal{K}(\mathcal{A})$ to the algebraic homotopy category is a localization at the class of chain homotopy equivalence. So in this particular case, the abstract notion of homotopy category coincides with the concrete notion.

The category $\mathrm{Ch}(\mathcal{A})$ is clearly pointed: any complex consisting only of zero objects is a zero object in $\mathrm{Ch}(\mathcal{A})$. The usual algebraic cone provides a functorial cone in the sense of Definition 7.2: the cone CA of a complex A is defined by

$$(CA)_n \ = \ A_n \oplus A_{n-1}$$

with differential

$$d_n^{CA}(a, a') \ = \ (d_n^A(a) + (-1)^n \cdot a', \ d_{n-1}^A(a')) \ .$$

The cone CA is chain contractible, and the inclusions of the first summand define a cofibration $\iota_A : A \longrightarrow CA$, so we have exhibited a functorial cone. The projections to the second summands witness the shift $A[1]$ as a cokernel of the 'cone inclusion' $\iota_A : A \longrightarrow CA$. So the suspension functor associated to this functorial cone is the shift functor.

The *elementary distinguished triangle* in $\mathcal{K}(A)$ associated to a chain morphism $\psi : A \longrightarrow B$ is thus the sequence

$$A \xrightarrow{\pi(\psi)} B \xrightarrow{\pi(i)} C\psi \xrightarrow{\pi(p)} A[1] \ .$$

Here $C\psi = CA \cup_\psi B$ is the algebraic mapping cone of ψ; in more explicit terms, this complex is given by

$$(CA)_n \ = \ B_n \oplus A_{n-1}$$

with differential

$$d_n^{C\psi}(b,a) = (d_n^B(b) + (-1)^n \cdot \psi_{n-1}(a), d_{n-1}^A(a)) \ .$$

The morphism $i : B \longrightarrow C\psi$ the inclusion as the first summands, and $p : C\psi \longrightarrow A[1]$ is the projection to the second summands. A *distinguished triangle* is any triangle (f,g,h) in $\mathcal{K}(\mathcal{A})$ which is isomorphic to an elementary distinguished triangle.

In the previous example, already the cofibration category $\mathrm{Ch}(\mathcal{A})$ is additive, and so is the homotopy category $\mathcal{K}(\mathcal{A})$. Moreover, since the suspension functor is the shift functor, it is already invertible at the level of complexes, and hence also at the level of the homotopy category $\mathcal{K}(\mathcal{A})$. So Theorem 7.8 applies, and yields:

Corollary 7.13. *For every additive category $\mathcal{A}$, the shift functor and the class of distinguished triangles make the homotopy category of complexes $\mathcal{K}(\mathcal{A})$ into a triangulated category.*

Example 7.14 (Triangulated structure on $\mathcal{D}(S)$)**.** We let S be an associative and unital ring. We write $\mathrm{Ch}(S)$ for the category of $\mathbb{Z}$-graded chain complexes of left S-modules. We recall that a chain map in $\mathrm{Ch}(S)$ is a *quasi-isomorphism* if it induces isomorphisms of all homology groups. Exercise E.10 (d) is devoted to showing that the cofibrations from the previous Example 7.12 (i.e., dimensionwise split monomorphisms) and the quasi-isomorphism form another cofibration structure on the category $\mathrm{Ch}(S)$. We let $\gamma : \mathrm{Ch}(S) \longrightarrow \mathcal{D}(S)$ be a localization at the class of quasi-isomorphisms; the target $\mathcal{D}(S)$ is called the *derived category* of the ring S. Since every chain homotopy equivalence is in particular a quasi-isomorphism, this localization factors through the quotient $\pi : \mathrm{Ch}(S) \longrightarrow \mathcal{K}(S\text{-mod})$ discussed in the previous example, and the resulting functor $\bar{\gamma} : \mathcal{K}(S\text{-mod}) \longrightarrow \mathcal{D}(S)$ is also a localization at the quasi-isomorphisms (or rather their images in $\mathcal{K}(S\text{-mod})$).

The algebraic cone discussed in the previous example is also a functorial cone with respect to the quasi-isomorphisms; the shift functor preserves quasi-isomorphisms and thus descends to an invertible functor on $\mathcal{D}(S)$. The resulting distinguished triangles in the derived category $\mathcal{D}(S)$ can thus be described in two equivalent ways:

- as those triangles that are isomorphic to

$$A \xrightarrow{\gamma(\psi)} B \xrightarrow{\gamma(i)} C\psi \xrightarrow{\gamma(p)} A[1]$$

  form some morphism $\psi : A \longrightarrow B$ in $\mathrm{Ch}(S)$;
- and as those triangles that are isomorphic to distinguished triangles in $\mathcal{K}(S\text{-mod})$ under the localization functor $\bar{\gamma} : \mathcal{K}(S\text{-mod}) \longrightarrow \mathcal{D}(S)$.

Corollary 7.15. *For every ring S, the shift functor and the class of distinguished triangles make the derived category $\mathcal{D}(S)$ into a triangulated category.*

In fact, the arguments leading up to Corollary 7.15 work just as well for any abelian category instead of the special case S-mod of the category of left S-modules; however, as we have not discussed abelian categories, we concentrated on the special case of modules above.

Remark 7.16 ($\mathcal{D}(S)$ versus $\mathcal{D}(HS)$)**.** In Example 4.14 we associated to a ring S an Eilenberg–MacLane ring spectrum HS. The derived category $\mathcal{D}(S)$ of the ring S and the derived category of the orthogonal ring spectrum HS are in fact equivalent as triangulated categories, in such a way that the composite functor

$$\mathcal{D}(S) \ \cong\ \mathcal{D}(HS) \xrightarrow{\ \pi_n\ } S\text{-mod}$$

is naturally isomorphic to the homology functor H_n. In this sense, homology algebra of modules over rings is subsumed in stable homotopy theory.

We will not construct an equivalence $\mathcal{D}(S) \cong \mathcal{D}(HS)$ in this document. It can be obtained by combining the following results in the literature. A competing framework for ring and module spectra is given by the category of symmetric spectra of simplicial sets as developed by Hovey, Shipley and Smith [19]. In this context, there is also an Eilenberg–MacLane symmetric ring spectrum $(HS)^{\Sigma}_{\text{sset}}$, see [19, Example 1.2.5] (but the ring $\mathbb{Z}$ replaced by the ring S). Theorem 5.1.6 of [42] provides a chain of Quillen equivalence of model categories between the stable model structure on $(HS)^{\Sigma}_{\text{sset}}$-modules and a certain model structure on $\mathrm{Ch}(S)$ with the quasi-isomorphism as weak equivalences. By taking geometric realization levelwise we obtain an Eilenberg–MacLane symmetric ring spectrum $(HS)^{\Sigma}_{\text{top}}$ of topological spaces (as opposed to simplicial sets); the Quillen equivalence of geometric realization and singular complex extends to a Quillen equivalence between symmetric $(HS)^{\Sigma}_{\text{sset}}$-modules in simplicial sets and symmetric $(HS)^{\Sigma}_{\text{top}}$-modules in spaces, although I do not have an explicit reference for this fact. The symmetric ring spectrum $(HS)^{\Sigma}_{\text{top}}$ is also isomorphic to the underlying symmetric spectrum of the orthogonal ring spectrum HS. So [30, Corollary 0.6] shows that the forgetful functor is a right Quillen equivalence between orthogonal HS-modules and symmetric modules over $(HS)^{\Sigma}_{\text{top}}$. Quillen equivalences of model cateories in particular induce equivalence of the localizations at the respective classes of weak equivalences. So the chain of Quillen equivalences sketched above yields the desired equivalence between $\mathcal{D}(S)$ and $\mathcal{D}(HS)$.

Proposition 7.17. *Let $\mathcal{T}$ be a triangulated category. Then the following properties hold.*

(i) *For every distinguished triangle (f, g, h) and every object X of $\mathcal{T}$, the two sequences of abelian groups*

$$\mathcal{T}(\Sigma A, X) \xrightarrow{\mathcal{T}(h,X)} \mathcal{T}(C, X) \xrightarrow{\mathcal{T}(g,X)} \mathcal{T}(B, X) \xrightarrow{\mathcal{T}(f,X)} \mathcal{T}(A, X)$$

and

$$\mathcal{T}(X, A) \xrightarrow{\mathcal{T}(X,f)} \mathcal{T}(X, B) \xrightarrow{\mathcal{T}(X,g)} \mathcal{T}(X, C) \xrightarrow{\mathcal{T}(X,h)} \mathcal{T}(X, \Sigma A)$$

are exact.

(ii) *Let (a, b, c) be a morphism of distinguished triangles. If two out of the three morphisms are isomorphisms, then so is the third.*

(iii) *Let (f, g, h) be a triangle such that the triangle $(g, h, -\Sigma f)$ is distinguished. Then the triangle (f, g, h) is distinguished.*

(iv) *Let (f_1, g_1, h_1), (f_2, g_2, h_2) and (f_3, g_3, h_3) be three distinguished triangles such that f_1 and f_2 are composable and $f_3 = f_2 f_1$. Then there exist morphisms $\bar{x}$ and $\bar{y}$ such that $(\bar{x}, \bar{y}, (\Sigma g_1) \circ h_2)$ is a distinguished triangle and the following diagram commutes:*

$$\begin{tikzcd}
A \arrow[r, "f_1"] \arrow[d, equal] & B \arrow[r, "g_1"] \arrow[d, "f_2"'] & \bar{C} \arrow[r, "h_1"] \arrow[d, dashed, "\bar{x}"] & \Sigma A \arrow[d, equal] \\
A \arrow[r, "f_3"'] & D \arrow[r, "g_3"'] \arrow[d, "g_2"'] & \bar{E} \arrow[r, "h_3"'] \arrow[d, dashed, "\bar{y}"] & \Sigma A \arrow[d, "\Sigma f_1"] \\
 & \bar{F} \arrow[r, equal] \arrow[d, "h_2"'] & \bar{F} \arrow[r, "h_2"'] \arrow[d, "(\Sigma g_1)\circ h_2"] & \Sigma B \\
 & \Sigma B \arrow[r, "\Sigma g_1"'] & \Sigma C &
\end{tikzcd}$$

(v) *For every distinguished triangle (f, g, h) the following three conditions are equivalent:*

- *The morphism $f : A \longrightarrow B$ has a retraction, i.e., there is a morphism r such that $rf = \mathrm{Id}_A$.*
- *The morphism $g : B \longrightarrow C$ has a section, i.e., there is a morphism s such that $gs = \mathrm{Id}_C$.*
- *The morphism $h : C \longrightarrow \Sigma A$ is zero.*

(vi) *Let (f, g, h) be a distinguished triangle and $s : C \longrightarrow B$ a morphism such that $gs = \mathrm{Id}_C$. Then the morphisms $f : A \longrightarrow B$ and $s : C \longrightarrow B$ make B into a coproduct of A and C.*

(vii) *Let I be a set and let (f_i, g_i, h_i) be a distinguished triangle for every $i \in I$. Then the triangles*

$$\bigoplus_I A_i \xrightarrow{\oplus f_i} \bigoplus_I B_i \xrightarrow{\oplus g_i} \bigoplus_I C_i \xrightarrow{\kappa\circ(\oplus h_i)} \Sigma(\bigoplus_I A_i)$$

and

$$\prod_I A_i \xrightarrow{\prod f_i} \prod_I B_i \xrightarrow{\prod g_i} \prod_I C_i \xrightarrow{\kappa^{-1}\circ(\prod h_i)} \Sigma(\prod_I A_i)$$

are distinguished, whenever the respective coproducts and products exist. Here $\kappa : \bigoplus_I \Sigma A_i \longrightarrow \Sigma(\bigoplus_I A_i)$ and $\kappa : \Sigma(\prod_I A_i) \longrightarrow \prod_I \Sigma A_i$ are the canonical isomorphisms.

(viii) *Let $A \oplus B$ be a coproduct of two objects A and B of $\mathcal{T}$ with respect to the morphisms $i_A : A \longrightarrow A \oplus B$ and $i_B : B \longrightarrow A \oplus B$. Then the triangle*

$$A \xrightarrow{i_A} A \oplus B \xrightarrow{p_B} B \xrightarrow{0} \Sigma A$$

is distinguished, where p_B is the morphism determined by $p_B i_A = 0$ and $p_B i_B = \mathrm{Id}_B$.

Proof. We start by showing that for every distinguished triangle (f, g, h) the composite gf is zero. Indeed, by (T4) applied to the pair (Id, f) there is a (necessarily unique) morphism from any zero object to C such that the diagram

$$\begin{tikzcd} A \arrow[r, "\mathrm{Id}"] \arrow[d, equal] & A \arrow[r] \arrow[d, "f"'] & 0 \arrow[r] \arrow[d, dotted] & \Sigma A \arrow[d, equal] \\ A \arrow[r, "f"'] & B \arrow[r, "g"'] & C \arrow[r, "h"'] & \Sigma A \end{tikzcd}$$

commutes, so $gf = 0$ (the upper row is distinguished by (T2) and (T3)).

(i) Since $gf = 0$ the image of $\mathcal{T}(g, X)$ is contained in the kernel of $\mathcal{T}(f, X)$. Conversely, let $\psi : B \longrightarrow X$ be a morphism in the kernel of $\mathcal{T}(f, X)$, i.e., such that $\psi f = 0$. Applying (T4) to the pair $(0, \psi)$ gives a morphism $\varphi : C \longrightarrow X$ such that the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[r, "g"] \arrow[d, "\psi"'] & C \arrow[r, "h"] \arrow[d, dotted, "\varphi"] & \Sigma A \arrow[d] \\ 0 \arrow[r] & X \arrow[r, "\mathrm{Id}"'] & X \arrow[r] & 0 \end{tikzcd}$$

commutes (the lower row is distinguished by (T1)). So the first sequence is exact at $\mathcal{T}(B, X)$. Applying this to the triangle $(g, h, -\Sigma f)$ (which is distinguished by (T3)), we deduce that the first sequence is also exact at $\mathcal{T}(X, C)$.

The argument for the other sequence is similar, but slightly more involved and depends on the assumption that the functor Σ is fully faithful. Since $gf = 0$, the image of $\mathcal{T}(X, f)$ is contained in the kernel of $\mathcal{T}(X, g)$. Conversely, let $\psi : X \longrightarrow B$ be a morphism in the kernel of $\mathcal{T}(X, g)$, i.e., such that $g\psi = 0$. Applying (T4) to the pair $(\psi, 0)$ gives a morphism $\bar{\varphi} : \Sigma X \longrightarrow \Sigma A$ such that the diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "\psi"'] & 0 \arrow[r] \arrow[d] & \Sigma X \arrow[r, "-\mathrm{Id}"] \arrow[d, dotted, "\bar{\varphi}"] & \Sigma X \arrow[d, "\Sigma\psi"] \\ B \arrow[r, "g"'] & C \arrow[r, "h"'] & \Sigma A \arrow[r, "-\Sigma f"'] & \Sigma B \end{tikzcd}$$

commutes (both rows are distinguished by (T1) and (T2)). Since shifting is full, there exists a morphism $\varphi : X \longrightarrow A$ such that $\bar{\varphi} = \Sigma\varphi$, and hence $\Sigma(f\varphi) = (\Sigma f)(\Sigma\varphi) = \Sigma\psi$. Since shifting is faithful we have

$f\varphi = \psi$, so ψ is in the image of $\mathcal{T}(X, f)$. Altogether, the first sequence is exact at $\mathcal{T}(X, B)$. If we apply this to the triangle $(g, h, -\Sigma f)$ (which is distinguished by (T3)), we deduce that the first sequence is also exact at $\mathcal{T}(X, C)$.

(ii) We first treat the case where a and b are isomorphisms. If X is any object of $\mathcal{T}$ we have a commutative diagram

$$\begin{tikzcd}
\mathcal{T}(X,A) \arrow[r, "f_*"] \arrow[d, "a_*"'] & \mathcal{T}(X,B) \arrow[r, "g_*"] \arrow[d, "b_*"'] & \mathcal{T}(X,C) \arrow[r, "h_*"] \arrow[d, "c_*"'] & \mathcal{T}(X,\Sigma A) \arrow[r, "(-\Sigma f)_*"] \arrow[d, "(\Sigma a)_*"] & \mathcal{T}(X,\Sigma B) \arrow[d, "(\Sigma b)_*"] \\
\mathcal{T}(X,A') \arrow[r, "f'_*"'] & \mathcal{T}(X,B') \arrow[r, "g'_*"'] & \mathcal{T}(X,C') \arrow[r, "h'_*"'] & \mathcal{T}(X,\Sigma A') \arrow[r, "(-\Sigma(f'))_*"'] & \mathcal{T}(X,\Sigma B')
\end{tikzcd}$$

where we write f_* for $\mathcal{T}(X, f)$, etc. The top row is exact by part (i) applied to the distinguished triangles (f, g, h) and $(g, h, -\Sigma f)$. Similarly, the bottom row is exact. Since a and b (and hence Σa and Σb) are isomorphisms, all vertical maps except possibly the middle one are isomorphisms of abelian groups. So the five lemma says that c_* is an isomorphism. Since this holds for all objects X, the morphism $c : C \longrightarrow C'$ is an isomorphism.

If b and c are isomorphisms, we apply the previous argument to the triple $(b, c, \Sigma a)$. This is a morphism from the distinguished (by (T3)) triangle $(g, h, -\Sigma f)$ to the distinguished triangle $(g', h', -\Sigma f')$. By the above, Σa is an isomorphism, hence so is a since shifting is an equivalence of categories. The third case is similar.

(iii) If the triangle $(g, h, -\Sigma f)$ is distinguished, then so is $(-\Sigma f, -\Sigma g, -\Sigma h)$ by two applications of (T3). Axiom (T1) provides a distinguished triangle

$$A \xrightarrow{f} B \xrightarrow{\bar{g}} \bar{C} \xrightarrow{\bar{h}} \Sigma A$$

and by three applications of (T3), the triangle $(-\Sigma f, -\Sigma\bar{g}, -\Sigma\bar{h})$ is distinguished. By (T4) there is a morphism $\bar{c} : \Sigma C \longrightarrow \Sigma\bar{C}$ such that the diagram

$$\begin{tikzcd}
\Sigma A \arrow[r, "-\Sigma f"] \arrow[d, equal] & \Sigma B \arrow[r, "-\Sigma g"] \arrow[d, equal] & \Sigma C \arrow[r, "-\Sigma h"] \arrow[d, "\bar{c}"] & \Sigma^2 A \arrow[d, equal] \\
\Sigma A \arrow[r, "-\Sigma f"'] & \Sigma B \arrow[r, "-\Sigma\bar{g}"'] & \Sigma\bar{C} \arrow[r, "-\Sigma\bar{h}"'] & \Sigma^2 A
\end{tikzcd}$$

commutes. By part (ii), c is an isomorphism. Since suspension is an equivalence of categories, we have $\bar{c} = \Sigma c$ for a unique isomorphism $c : C \longrightarrow \bar{C}$. Then $(\mathrm{Id}_A, \mathrm{Id}_B, c)$ is an isomorphism from the triangle (f, g, h) to the distinguished triangle $(f, \bar{g}, \bar{h})$. So the triangle (f, g, h) is itself distinguished.

(iv) Axiom (T5) provides a commutative diagram

$$\begin{tikzcd}
A \arrow[r, "f_1"] \arrow[d, equal] & B \arrow[r, "g"] \arrow[d, "f_2"'] & C \arrow[r, "h"] \arrow[d, "x"] & \Sigma A \arrow[d, equal] \\
A \arrow[r, "f_3"'] & D \arrow[r, "g''"'] \arrow[d, "g'"'] & E \arrow[r, "h''"'] \arrow[d, "y"] & \Sigma A \arrow[d, "\Sigma f_1"] \\
 & F \arrow[r, equal] \arrow[d, "h'"'] & F \arrow[r, "h'"'] \arrow[d, "(\Sigma g)\circ h'"] & \Sigma B \\
 & \Sigma B \arrow[r, "\Sigma g"'] & \Sigma C &
\end{tikzcd}$$

such that the triangles (f_1, g, h), (f_2, g', h'), (f_3, g'', h'') and $(x, y, (\Sigma g) \circ h')$ are distinguished. By (T4) there is a morphism $\varphi : \bar{C} \longrightarrow C$ that makes $(\mathrm{Id}_A, \mathrm{Id}_B, \varphi)$ a morphism of triangles from (f_1, g_1, h_1) to (f_1, g, h); this morphism is an isomorphism by part (ii). Similarly, there is an morphism $\psi : \bar{F} \longrightarrow F$ such that $(\mathrm{Id}_B, \mathrm{Id}_D, \psi)$ an isomorphism of triangles from (f_2, g_2, h_2) to (f_2, g', h'). Finally, there is an morphism $\nu : \bar{E} \longrightarrow E$ such that $(\mathrm{Id}_A, \mathrm{Id}_D, \nu)$ an isomorphism of triangles from (f_3, g_3, h_3) to (f_3, g'', h''). If we set

$$\bar{x} = \nu^{-1} x \varphi : \bar{C} \longrightarrow \bar{E} \qquad \text{and} \qquad \bar{y} = \psi^{-1} y \nu : \bar{E} \longrightarrow \bar{F} ,$$

then the desired diagram commutes. Moreover, the triple (φ, ν, ψ) is an isomorphism from the triangle $(\bar{x}, \bar{y}, (\Sigma g_1)h_2)$ to the triangle $(x, y, (\Sigma g)h')$. Since the latter triangle is distinguished, so is the former.

(v) By part (i), the composite of two adjacent morphism in any distinguished triangle is zero. So if s is a section to g, then $h = hgs = 0$. Similarly, if r is a retraction to f, then $h = (-\Sigma r)(-\Sigma f)h = 0$ because the triangle $(g, h, -\Sigma f)$ is distinguished. Conversely, if $h = 0$, then the sequence

$$\mathcal{T}(C, B) \xrightarrow{\mathcal{T}(C,g)} \mathcal{T}(C, C) \longrightarrow 0$$

is exact by part (i), and any preimage of the identity of C is a section to g. Similarly, the sequence

$$\mathcal{T}(\Sigma B, \Sigma A) \xrightarrow{\mathcal{T}(-\Sigma f, \Sigma A)} \mathcal{T}(\Sigma A, \Sigma A) \longrightarrow 0$$

is exact because the triangle $(g, h, -\Sigma f)$ is distinguished. So there is a morphism $\bar{r} : \Sigma B \longrightarrow \Sigma A$ such that $-\bar{r} \circ \Sigma f = \mathrm{Id}_{\Sigma A}$. Since Σ is full, there is a morphism $r : B \longrightarrow A$ such that $\Sigma r = -\bar{r}$, hence $\Sigma(rf) = (\Sigma r)(\Sigma f) = \mathrm{Id}_{\Sigma A}$. Since Σ is faithful, r is a retraction to f.

(vi) Since s is a section to g, the morphism $\mathcal{T}(g, X)$ is injective. By part (v) the morphism f has a retraction, so $\mathcal{T}(f, X)$ is surjective. The first exact sequence of part (i) thus becomes a short exact sequence of abelian groups

$$0 \longrightarrow \mathcal{T}(C, X) \xrightarrow{\mathcal{T}(g,X)} \mathcal{T}(B, X) \xrightarrow{\mathcal{T}(f,X)} \mathcal{T}(A, X) \longrightarrow 0 .$$

Because $\mathcal{T}(s, X)$ is a section to the first map, the map $(\mathcal{T}(f, X), \mathcal{T}(s, X)) : \mathcal{T}(B, X) \longrightarrow \mathcal{T}(A, X) \times \mathcal{T}(C, X)$ is bijective, i.e., the morphisms f and s make B a coproduct of A and B.

(vii) We choose a distinguished triangle:

$$\bigoplus_I A_i \xrightarrow{\oplus f_i} \bigoplus_I B_i \xrightarrow{g} C \xrightarrow{h} \Sigma(\bigoplus_I A_i) .$$

We apply axiom (T3) to the canonical morphisms $\kappa_j : A_j \longrightarrow \bigoplus_I A_i$ and $\kappa'_j : B_j \longrightarrow \bigoplus_I B_i$ and obtain morphisms $\varphi_j : C_j \longrightarrow C$ such that the diagrams

$$\begin{tikzcd}
A_j \arrow[r, "f_j"] \arrow[d, "\kappa_j"'] & B_j \arrow[r, "g_j"] \arrow[d, "\kappa'_j"'] & C_j \arrow[r, "h_j"] \arrow[d, dotted, "\varphi_j"'] & \Sigma A_j \arrow[d, "\Sigma\kappa_j"] \\
\bigoplus_I A_i \arrow[r, "\oplus f_i"'] & \bigoplus_I B_i \arrow[r, "g"'] & C \arrow[r, "h"'] & \Sigma(\bigoplus_I A_i)
\end{tikzcd}$$

commute. We claim that then the morphisms $\varphi_i : C_i \longrightarrow C$ make C into a coproduct of the objects C_i. For this we observe that the diagram

$$\begin{tikzcd}
\mathcal{T}(\Sigma(\bigoplus_I B_i), X) \arrow[r, "-(\Sigma\oplus f_i)_*"] \arrow[d, "((\Sigma\kappa_i)_*)"'] & \mathcal{T}(\Sigma(\bigoplus_I A_i), X) \arrow[r, "h_*"] \arrow[d, "((\Sigma\kappa'_i)_*)"'] & \mathcal{T}(C, X) \arrow[r, "g_*"] \arrow[d, "((\varphi_i)_*)"'] & \mathcal{T}(\bigoplus_I B_i, X) \arrow[r, "(\oplus f_i)_*"] \arrow[d, "((\kappa'_i)_*)"] & \mathcal{T}(\bigoplus_I A_i, X) \arrow[d, "((\kappa_i)_*)"] \\
\prod_I \mathcal{T}(\Sigma B_i, X) \arrow[r, "-\prod \Sigma(f_i)_*"'] & \prod_I \mathcal{T}(\Sigma A_i, X) \arrow[r, "\prod(h_i)_*"'] & \prod_I \mathcal{T}(C_i, X) \arrow[r, "\prod(g_i)_*"'] & \prod_I \mathcal{T}(B_i, X) \arrow[r, "\prod(f_i)_*"'] & \prod_I \mathcal{T}(A_i, X)
\end{tikzcd}$$

commutes by construction of the morphisms φ_i. The top row is exact by part (i), the bottom row is exact as a product of exact sequences. The four outer vertical maps are isomorphisms by the universal property of coproducts, so the middle vertical map is an isomorphism by the 5-lemma. This shows that C is a coproduct of the C_i's in a way that makes $g = \oplus g_i : \bigoplus_I B_i \longrightarrow C$ and $h = \kappa \circ (\oplus h_i) : C \longrightarrow \Sigma(\bigoplus_I A_i)$.

The statement about products of triangles can be proved in an analogous fashion. Alternatively, one can reduce to the first case by exploiting that products in $\mathcal{T}$ are coproducts in the opposite category $\mathcal{T}^{\mathrm{op}}$, which is triangulated with respect to the opposite triangulation (compare Exercise E.14).

(viii) This the special case of part (vii) for the two exact triangles

$$A \xrightarrow{\mathrm{Id}_A} A \longrightarrow 0 \longrightarrow \Sigma A \qquad \text{and} \qquad 0 \longrightarrow B \xrightarrow{\mathrm{Id}_B} B \longrightarrow 0$$

whose sum is the triangle in question, which is thus distinguished. □

Example 7.18 (Shift preserves distinguished triangles)**.** The shift functor $\mathrm{sh} : R\text{-mod} \longrightarrow R\text{-mod}$ of Example 2.13 and Construction 4.6 preserves stable equivalences, see Proposition 2.16. So the shift functor descends to a functor on the derived category

$$\mathrm{sh} \;:\; \mathcal{D}(R) \;\longrightarrow\; \mathcal{D}(R)$$

for which we use the same name. Moreover, shifting commutes with smashing with a based space *on the nose*, i.e., $(\mathrm{sh}\, X) \wedge A = \mathrm{sh}(X \wedge A)$; so we can (and will) leave out parentheses in such expressions. Since the suspension functor on $\mathcal{D}(R)$ is induced by smashing with S^1, the shift functor commutes with the suspension functor, again on the nose, both on the point-set level and also on the level of the derived category. We will now argue that shifting also preserves distinguished triangles on the nose.

Proposition 7.19. *Let R be an orthogonal ring spectrum. For every distinguished triangle (f, g, h) in $\mathcal{D}(R)$, the triangle*

$$\mathrm{sh}\, A \;\xrightarrow{\;\mathrm{sh}\, f\;}\; \mathrm{sh}\, B \;\xrightarrow{\;\mathrm{sh}\, g\;}\; \mathrm{sh}\, C \;\xrightarrow{\;\mathrm{sh}\, h\;}\; \mathrm{sh}\, A \wedge S^1$$

is also distinguished.

Proof. We may assume without loss of generality that (f, g, h) is the elementary distinguished triangle $(\gamma(\psi), \gamma(i), \gamma(p))$ of some morphism of orthogonal spectra $\psi : A \longrightarrow B$. The shift functor $\mathrm{sh} : R\text{-mod} \longrightarrow R\text{-mod}$ on the level of orthogonal spectra commutes on the nose with suspension and mapping cones, i.e., $\mathrm{sh}(A \wedge S^1) = (\mathrm{sh}\, A) \wedge S^1$, $\mathrm{sh}(C\psi) = C(\mathrm{sh}\, \psi)$, and similarly for the morphisms that participate in the mapping cone sequences. So the elementary distinguished triangle

$$\mathrm{sh}\, A \;\xrightarrow{\;\gamma(\mathrm{sh}\, \psi)\;}\; \mathrm{sh}\, B \;\xrightarrow{\;\gamma(\mathrm{sh}\, i)\;}\; \mathrm{sh}\, C \;\xrightarrow{\;\gamma(\mathrm{sh}\, p)\;}\; \mathrm{sh}\, A \wedge S^1$$

associated to the shifted morphism $\mathrm{sh}\, \psi : \mathrm{sh}\, A \longrightarrow \mathrm{sh}\, B$ is *equal* to the triangle $(\mathrm{sh}\, \gamma(f), \mathrm{sh}\, \gamma(g), \mathrm{sh}\, \gamma(h))$. So the latter triangle is distinguished, which proves the claim. □

The previous proposition can be rephrased as saying that the shift functor $\mathrm{sh} : \mathcal{SH} \longrightarrow \mathcal{SH}$ is an exact functor of triangulated categories if we equip it with the identity isomorphism $\mathrm{sh} \circ (- \wedge S^1) = (- \wedge S^1) \circ \mathrm{sh}$.

8. Thom spectra and bordism

In Example 4.17 we introduced the commutative orthogonal ring spectrum MO; as the terms of this spectrum are Thom spaces of universal vector bundles, it is referred to as a *Thom spectrum*. In this section we sketch a proof of Thom's celebrated theorem [47] that the so-called *Thom-Pontryagin construction* defines an isomorphism of homology theories from the geometrically defined bordism theory to the homology theory represented by MO is an isomorphism.

8.1. **Bordism.** Now we recall bordism groups and their relationship to the homology theory defined by the Thom spectrum MO. In Construction 8.19 we introduce the Thom-Pontryagin map, which is an isomorphism from bordism to MO-homology.

Definition 8.1. A *singular manifold* over a space X is a pair (M, h) consisting of a closed smooth manifold M and a continuous map $h : M \longrightarrow X$. Two singular manifolds (M, h) and (M', h') are *bordant* if there is a triple (B, H, ψ) consisting of a compact smooth manifold B, a continuous map $H : B \longrightarrow X$ and an equivariant diffeomorphism

$$\psi \;:\; M \cup M' \;\cong\; \partial B$$

such that $(H \circ \psi)|_M = h$ and to $(H \circ \psi)|_{M'} = h'$.

Bordism of singular manifolds over X is an equivalence relation. Reflexivity and symmetry are straightforward; transitivity is established by gluing two bordisms along a common piece of the boundary. To get a smooth structure on the glued bordism that is compatible with the action one needs smooth collars.

We denote by $\mathcal{N}_n(X)$ the set of bordism classes of n-dimensional singular manifolds over X. This set becomes an abelian group under disjoint union. Every element x of $\mathcal{N}_n(X)$ satisfies $2x = 0$: for every closed smooth manifold M, the manifold $M \times [0,1]$ bounds a disjoint union of two copies of M. The groups $\mathcal{N}_n(X)$ are covariantly functorial in continuous maps, by post-composition.

Proposition 8.2. (i) *Let $\varphi, \varphi' : X \longrightarrow Y$ be homotopic continuous maps. Then $\varphi_* = \varphi'_*$ as homomorphisms from $\mathcal{N}_n(X)$ to $\mathcal{N}_n(Y)$.*
(ii) *For every weak equivalence $\varphi : X \longrightarrow Y$ the induced homomorphism $\varphi_* : \mathcal{N}_n(X) \longrightarrow \mathcal{N}_n(Y)$ is an isomorphism.*
(iii) *Let $\{X_i\}_{i\in I}$ be a family of spaces. Then the canonical map*

$$\bigoplus\nolimits_{i\in I} \mathcal{N}_n(X_i) \ \longrightarrow\ \mathcal{N}_n\left(\coprod\nolimits_{i\in I} X_i\right)$$

is an isomorphism.

Proof. (i) We let $H : X \times [0,1] \longrightarrow Y$ be a homotopy from φ to φ' and (M,h) a singular manifold over X. Then $(M\times[0,1], H\circ(h\times[0,1]), \psi)$ is a bordism from $(M,\varphi h)$ to $(M,\varphi' h)$, where $\psi : M\cup M \longrightarrow \partial(M\times[0,1])$ identifies one copy of M with $M \times \{0\}$ and the other copy with $M \times \{1\}$. So $\varphi_*[M,h] = [M,\varphi\circ h] = [M,\varphi'\circ h] = \varphi'_*[M,h]$.

(ii) For surjectivity of φ_* we consider any singular manifold (M,g) over Y. Every smooth manifold M admits a triangulation, and hence also the structure of a CW-complex, compare [53]. Since φ is a weak equivalence there exists a continuous map $h : M \longrightarrow X$ such that φh is equivariantly homotopic to g. Part (i) then shows that $\varphi_*[M,h] = (\varphi h)_*[M,\mathrm{Id}_M] = g_*[M,\mathrm{Id}_M] = [M,g]$.

The argument for injectivity is similar. We consider a singular manifold (M,h) over X that represents an element in the kernel of φ_*. There is then a null-bordism (B,H,ψ) of $(M,\varphi h)$. Again there is a CW-structure on B for which the boundary is a subcomplex. Since φ is a weak equivalence, there exists a continuous map $H' : B \longrightarrow X$ such that $H'\circ\psi = h$. The triple (B,H',ψ) thus witnesses that $[M,h] = 0$. Since φ_* is a group homomorphism, it is injective.

Property (iii) holds because compact manifolds only have finitely many connected components, so all continuous reference maps from singular manifolds or bordisms have image in a finite union. □

Now we state the key exactness property of bordism in the form of a Mayer-Vietoris sequence. The definition of the boundary map needs the existence of separating functions as provided by the following lemma.

Lemma 8.3. *Let M be a compact smooth manifold, C and C' two disjoint, closed subsets of M, and*

$$s \ :\ \partial M \ \longrightarrow\ \mathbb{R}$$

a smooth map such that

$$C\cap\partial M \subseteq s^{-1}(0) \qquad \text{and} \qquad C'\cap\partial M \subseteq s^{-1}(1)\ .$$

Then there exists a smooth extension $r : M \longrightarrow \mathbb{R}$ of s such that $C \subseteq r^{-1}(0)$ and $C' \subseteq r^{-1}(1)$.

Proof. Since M is compact, hence normal, the Tietze extension theorem provides a continuous map $r_0 : M \longrightarrow \mathbb{R}$ that extends s and satisfies $C \subseteq r_0^{-1}(0)$ and $C' \subseteq r_0^{-1}(1)$. By smooth approximation, we can then find a smooth map $r : M \longrightarrow \mathbb{R}$ that coincides with r_1 on $C\cup C'\cup\partial M$; this is the desired separating function. □

Construction 8.4 (Bordism boundary map)**.** We define a boundary homomorphism for a Mayer-Vietoris sequence. We let X be a space and $A, B \subset X$ open subsets with $X = A\cup B$. Then a homomorphism

$$\partial \ :\ \mathcal{N}_n(X) \ \longrightarrow\ \mathcal{N}_{n-1}(A\cap B)$$

is defined as follows.

We let (M, h) be a singular manifold that represents a class in $\mathcal{N}_n(X)$. The sets $h^{-1}(X - A)$ and $h^{-1}(X - B)$ are disjoint closed subsets of M; we let $r : M \longrightarrow \mathbb{R}$ be a separating function as provided by Lemma 8.3, i.e., such that $h^{-1}(X - A) \subseteq r^{-1}(0)$ and $h^{-1}(X - B) \subseteq r^{-1}(1)$. We let $t \in (0, 1)$ be any regular value of r. Then

$$M_t \; = \; r^{-1}(t)$$

is a smooth closed submanifold of M of dimension $n - 1$ (possibly empty), and $h_t = h|_{M_t}$ lands in $A \cap B$; so (M_t, h_t) is a singular manifold over $A \cap B$.

Proposition 8.5. *In the situation above, the bordism class $[M_t, h_t]$ is independent of the choice of regular value t, of the choice of separating function and of the representative for the given class in $\mathcal{N}_n(X)$. The resulting map*

$$\partial \; : \; \mathcal{N}_n(X) \; \longrightarrow \; \mathcal{N}_{n-1}(A \cap B) \; , \quad [M, h] \; \longmapsto \; [M_t, h_t]$$

is a group homomorphism.

Proof. We let $t < t'$ be two regular values in $(0, 1)$ of the separating function r. Then

$$(r^{-1}[t, t'], \, h|_{r^{-1}[t,t']}, \, \mathrm{incl})$$

is a bordism from $(r^{-1}(t), h|_{r^{-1}(t)})$ to $(r^{-1}(t'), h|_{r^{-1}(t')})$, so the bordism class does not depend on the regular value.

Now we let (M, h) and (N, g) be two singular manifolds over X in the same bordism class, and we let (B, H, ψ) be a bordism from (M, h) to (N, g). We let $r : M \longrightarrow \mathbb{R}$ and $\bar{r} : N \longrightarrow \mathbb{R}$ be two separating functions. Lemma 8.3 lets us extend this data to a smooth separating function

$$\Psi \; : \; B \; \longrightarrow \; \mathbb{R}$$

such that $\Psi \circ \psi|_M = r$, $\Psi \circ \psi|_N = \bar{r}$,

$$H^{-1}(X - A) \subseteq \Psi^{-1}(0) \quad \text{and} \quad H^{-1}(X - B) \subseteq \Psi^{-1}(1) \; .$$

We choose a simultaneous regular value $t \in (0, 1)$ for Ψ, r and $\bar{r}$. Then

$$(\Psi^{-1}(t), \, H|_{\Psi^{-1}(t)}, \, \psi|_{r^{-1}(t) \cup \bar{r}^{-1}(t)})$$

is a bordism from $(r^{-1}(t), h|_{r^{-1}(t)})$ to $(\bar{r}^{-1}(t), g|_{\bar{r}^{-1}(t)})$. This shows at the same time that the bordism class is independent of the choice of separating function and of the choice of representing singular manifold. Additivity of the resulting boundary map is then clear: a separating function for a disjoint union can be taken as the union of separating functions for each summand. $\square$

Now we formulate the property that makes bordism a homology theory. A proof of the following proposition can for example be found in [49, Proposition 21.1.7].

Proposition 8.6. *Let X be a space and $A, B \subset X$ open subsets with $X = A \cup B$. Let $i^A : A \cap B \longrightarrow A$, $i^B : A \cap B \longrightarrow B$, $j^A : A \longrightarrow X$ and $j^B : B \longrightarrow X$ denote the inclusions. Then the following sequence of abelian groups is exact:*

$$\ldots \longrightarrow \mathcal{N}_n(A \cap B) \xrightarrow{(i^A_*, i^B_*)} \mathcal{N}_n(A) \oplus \mathcal{N}_n(B) \xrightarrow{\binom{j^A_*}{-j^B_*}} \mathcal{N}_n(X) \xrightarrow{\;\partial\;} \mathcal{N}_{n-1}(A \cap B) \longrightarrow \ldots$$

We define the *reduced bordism group* of a based space X as

$$\widetilde{\mathcal{N}}_n(X) \; = \; \mathrm{coker}\,(\mathcal{N}_n(*) \longrightarrow \mathcal{N}_n(X)) \; ,$$

the cokernel of the homomorphism induced by the basepoint inclusion. If (M, h) is a singular manifold over X, then we use the notation $[[M, h]]$ for the class it represents in the reduced bordism group $\widetilde{\mathcal{N}}_n(X)$. The unique map $u : X \longrightarrow *$ is retraction to the basepoint inclusion, so the map

$$(\mathrm{proj}, u_*) \; : \; \mathcal{N}_n(X) \; \longrightarrow \; \widetilde{\mathcal{N}}_n(X) \oplus \mathcal{N}_n(*)$$

is an isomorphism. On the other hand, if we add a disjoint basepoint to an unbased space Y, then the composite

$$\mathcal{N}_n(Y) \xrightarrow{\text{incl}_*} \mathcal{N}_n(Y_+) \xrightarrow{\text{proj}} \widetilde{\mathcal{N}}_n(Y_+)$$

is an isomorphism.

Remark 8.7. In much of the classical literature, the reduced bordism group $\widetilde{\mathcal{N}}_n(X)$ of a based space (X,x_0) is defined differently, namely as the group $\mathcal{N}_n(X,x_0)$ of bordism classes of pairs (M,h) where M is a compact smooth manifold with boundary, and $h : M \longrightarrow X$ is a continuous map with $h(\partial M) = \{x_0\}$. In this context, a bordism from (M,h) to (M',h') is a triple (B,H,ψ) consisting of a compact smooth $(n+1)$-manifold B, a continuous map $H : B \longrightarrow X$, a decomposition $\psi : M \cup M' \cup V \cong \partial B$ as regularly embedded submanifolds such that

$$V \cap M = \partial M \ , \quad V \cap M' = \partial M' \ , \quad \partial V = \partial M \amalg \partial M' \ ,$$

$(H \circ \psi)|_M = h$ and $(H \circ \psi)|_{M'} = h'$ and $H(V) = \{x_0\}$.

The comparison map

$$\widetilde{\mathcal{N}}_n(X) \longrightarrow \mathcal{N}_n(X,x_0)$$

sends the class of $[[M,h]]$ to the class of (M,h); this is well defined because pairs (M,h) with closed M and h constant to x_0 represent the trivial class in $\mathcal{N}_n(X,x_0)$ of bordism classes of pairs (M,h). Indeed, as a witness we may take $B = M \times [0,1]$, define $H : M \times [0,1]$ by $H(x,t) = h(x)$ and use the decomposition $\psi : M \amalg V \cong \partial(B \times [0,1])$ with $V = M \times \{1\}$. Then $H(\psi(V)) = \{x_0\}$ because we assumed that h is constant at x_0.

For injectivity we consider a singular manifold (M,h) with M closed that represents the trivial class in $\mathcal{N}_n(X,x_0)$. We let (B,H,ψ) be a bordism that witnesses this. Because M is are closed, ψ identifies the boundary ∂B with the *disjoint* union $M \amalg V$ So we can also interpret (B,H,ψ) as bordism between the singular manifolds (M,h) and $(V,(H \circ \psi)|_V)$, and hence $[[M,h]] = [[V,(H \circ \psi)|_V]]$ in $\widetilde{\mathcal{N}}_n(X)$. Because $H(\psi(V)) = \{x_0\}$, the class $[[V,(H \circ \psi)|_V]]$ in trivial in the cokernel $\widetilde{\mathcal{N}}_n(X)$.

For surjectivity we consider any pair (M,h) representing a class in $\mathcal{N}_n(X,x_0)$. We let $DM = M \cup_{\partial M} M$ denote the *double* of M, and we define $\bar{h} : DM \longrightarrow X$ as the continuous map that restricts to h on the first copy of M, and that sends the second copy of M to the basepoint x_0. Then (M,h) is bordant to $(DM,\bar{h})$, so the class $[M,h]$ is the image of $[[DM,\bar{h}]]$.

Construction 8.8. We consider a continuous map $f : X \longrightarrow Y$ and let

$$Cf \ = \ CX \cup_f Y \ = \ (X \times [0,1] \cup_f Y)/X \times \{0\}$$

denote its unreduced mapping cone. The two open sets

$$A \ = \ X \times [0,1)/X \times \{0\} \qquad \text{and} \qquad B \ = \ X \times (0,1] \cup_f Y$$

are G-invariant and together cover the mapping cone. The intersection $A \cap B$ is homeomorphic to $X \times (0,1)$, so the open covering has an associated boundary homomorphism

$$\partial \ : \ \mathcal{N}_n(Cf) \longrightarrow \mathcal{N}_{n-1}(X \times (0,1))$$

as in Construction 8.4. We take the cone point as the basepoint of Cf; this is contained in the subset A, so the map $\iota : \mathcal{N}_*(*) \longrightarrow \mathcal{N}_*(Cf)$ induced by the basepoint inclusion factors through $j^A_* : \mathcal{N}_n(A) \longrightarrow \mathcal{N}_n(Cf)$, and the composite $\partial \circ \iota$ is trivial by exactness of the excision sequence. The boundary map thus factors over the reduced bordism group. We define a 'reduced boundary map' $\bar{\partial}$ as the composite

$$\widetilde{\mathcal{N}}_n(Cf) \xrightarrow{\partial} \mathcal{N}_{n-1}(X \times (0,1)) \xrightarrow{\text{proj}_*} \mathcal{N}_{n-1}(X) \ .$$

Proposition 8.9. *For every continuous map $f : X \longrightarrow Y$, the following sequence of abelian groups is exact:*

$$\ldots \longrightarrow \mathcal{N}_n(X) \xrightarrow{f_*} \mathcal{N}_n(Y) \xrightarrow{i_*} \widetilde{\mathcal{N}}_n(Cf) \xrightarrow{\bar{\partial}} \mathcal{N}_{n-1}(X) \longrightarrow \ldots$$

Proof. We use the open covering of the mapping cone Cf as in the definition of the boundary map. In the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "x\mapsto(x,1/2)"'] & Y \arrow[r, "i"] \arrow[d, "i"'] & Cf \arrow[d, equal] \\ A\cap B \arrow[r, "\mathrm{incl}"'] & B \arrow[r, "\mathrm{incl}"'] & Cf \end{tikzcd}$$

the right square commutes and the left square commutes up to homotopy. Moreover, all vertical maps are homotopy equivalences, so they induce isomorphisms in bordism, by Proposition 8.2. So the resulting diagram of bordism groups commutes:

$$\begin{tikzcd} \mathcal{N}_n(X) \arrow[r, "f_*"] \arrow[d, "\cong"'] & \mathcal{N}_n(Y) \arrow[r, "i_*"] \arrow[d, "\cong"', "i_*"] & \mathcal{N}_n(Cf) \arrow[d, equal] \\ \mathcal{N}_n(A\cap B) \arrow[r] & \mathcal{N}_n(B) \arrow[r] & \mathcal{N}_n(Cf) \end{tikzcd}$$

Moreover, all vertical maps in this diagram are isomorphisms, so we can substitute $\mathcal{N}_n(X)$ and $\mathcal{N}_n(Y)$ into the long exact excision sequence of Proposition 8.6. Since A is contractible to the cone point, we can also replace the corresponding summand by the coefficient group, and the result is an exact sequence

$$\dots \longrightarrow \mathcal{N}_n(X) \xrightarrow{(f_*,u_*)} \mathcal{N}_n(Y)\oplus\mathcal{N}_n(*) \longrightarrow \mathcal{N}_n(Cf) \xrightarrow{\partial} \mathcal{N}_{n-1}(X) \longrightarrow \dots$$

The sequence then remains exact if we divide out the summand $\mathcal{N}_n(*)$ and replace the absolute bordism group of Cf by the reduced group $\widetilde{\mathcal{N}}_n(Cf)$. □

If $f : A \longrightarrow B$ is an h-cofibration of spaces, then the projection $q : Cf \longrightarrow B/A$ from the mapping cone to the quotient is a based homotopy equivalence. So we can substitute $\widetilde{\mathcal{N}}_*(B/A)$ into the long exact mapping cone sequence of Proposition 8.9 and obtain a long exact sequence of abelian groups:

$$\dots \longrightarrow \mathcal{N}_n(A) \xrightarrow{f_*} \mathcal{N}_n(B) \xrightarrow{q_*} \widetilde{\mathcal{N}}_n(B/A) \longrightarrow \mathcal{N}_{n-1}(A) \longrightarrow \dots$$

The bordism groups come with natural products, given by the biadditive maps

$$\times : \mathcal{N}_m(X) \times \mathcal{N}_n(Y) \longrightarrow \mathcal{N}_{m+n}(X\times Y), \quad [M,h]\times[N,g] = [M\times N, h\times g] .$$

These products are suitably associative, commutative and unital. The product pairing descends to a pairing on reduced bordism groups if the spaces X and Y are based. Indeed, the composite

$$\mathcal{N}_m(X)\otimes\mathcal{N}_n(Y) \xrightarrow{\times} \mathcal{N}_{m+n}(X\times Y) \xrightarrow{q_*} \mathcal{N}_{m+n}(X\wedge Y) \xrightarrow{\mathrm{proj}} \widetilde{\mathcal{N}}_{m+n}(X\wedge Y)$$

annihilates the image of $\mathcal{N}_m(*)\otimes\mathcal{N}_n(Y)$ and the image of $\mathcal{N}_m(X)\otimes\mathcal{N}_n(*)$, where $q : X\times Y \longrightarrow X\wedge Y$ is the quotient map; so the composite factors uniquely over a homomorphism

$$\wedge : \widetilde{\mathcal{N}}_m(X) \otimes \widetilde{\mathcal{N}}_n(Y) \longrightarrow \widetilde{\mathcal{N}}_{m+n}(X\wedge Y) .$$

We shall now recall the suspension isomorphism in bordism. We define

$$d = [[S^1, \mathrm{Id}_{S^1}]] \in \widetilde{\mathcal{N}}_1(S^1) ,$$

the reduced bordism class of the identity of S^1; here S^1 is given the standard smooth structure, with atlas given by the inclusion $\mathbb{R} \longrightarrow S^1 = \mathbb{R}\cup\{\infty\}$ and the open embedding $\mathbb{R}\longrightarrow S^1$ sending x to $1/x$.

Proposition 8.10. *For every based CW-complex X, the exterior product map*

$$-\wedge d : \widetilde{\mathcal{N}}_n(X) \longrightarrow \widetilde{\mathcal{N}}_{n+1}(X\wedge S^1)$$

is an isomorphism. For every continuous map $f : X \longrightarrow Y$ between based spaces, the homomorphism $p_* : \widetilde{\mathcal{N}}_{n+1}(Cf) \longrightarrow \widetilde{\mathcal{N}}_{n+1}(X\wedge S^1)$ agrees with the composite

$$\widetilde{\mathcal{N}}_{n+1}(Cf) \xrightarrow{\bar{\partial}} \widetilde{\mathcal{N}}_n(X) \xrightarrow{-\wedge d} \widetilde{\mathcal{N}}_{n+1}(X\wedge S^1) .$$

Proof. We apply Proposition 8.9 to the map $f : X \longrightarrow *$ to a one-point space. The cone of this map is

$$X^\diamond \;=\; X \times [0,1]/\sim \;,$$

the unreduced suspension of X, where $X \times \{0\}$ and $X \times \{1\}$ are collapsed to one point each. The map $f_* : \mathcal{N}_*(X) \longrightarrow \mathcal{N}_*(*)$ is a split epimorphism. So the long exact sequence provided by Proposition 8.9 reduces to a short exact sequence:

$$0 \;\longrightarrow\; \widetilde{\mathcal{N}}_{n+1}(X^\diamond) \;\xrightarrow{\bar\partial}\; \mathcal{N}_n(X) \;\xrightarrow{f_*}\; \mathcal{N}_n(*) \;\longrightarrow\; 0$$

Since X is cofibrant in the based sense, the projection

$$\psi \;:\; X^\diamond \;\longrightarrow\; X \wedge S^1 \;, \quad q[x,s] \;=\; x \wedge \frac{2s-1}{s(1-s)}$$

that collapses $\{x_0\} \times [0,1]$ is a homotopy equivalence.

Then the composite

$$\widetilde{\mathcal{N}}_{n+1}(X \wedge S^1) \;\xrightarrow[\cong]{\psi_*^{-1}}\; \widetilde{\mathcal{N}}_{n+1}(X^\diamond) \;\xrightarrow{\bar\partial}\; \mathcal{N}_n(X) \;\xrightarrow{\text{proj}}\; \widetilde{\mathcal{N}}_n(X)$$

is an isomorphism. We claim that the relation

$$\text{proj}(\bar\partial(\psi_*^{-1}(x \wedge d))) \;=\; x$$

holds for all classes $x \in \widetilde{\mathcal{N}}_n(X)$. Since $\text{proj} \circ \bar\partial \circ \psi_*^{-1}$ is an isomorphism, so is smash product with the class d.

This relation, in turn, is a consequence of the geometric origin of the class d, the product in bordism and the boundary map. In more detail, we suppose that $x = [[M,h]]$ for a singular manifold (M,h) over X. We define a continuous map $H : M \times S^1 \longrightarrow X^\diamond$ by

$$H(m,z) \;=\; \begin{cases} [h(m),(z+1)/2] & \text{for } z \in [-1,1], \text{ and} \\ [x_0,(z^{-1}+1)/2] & \text{for } z \in S^1 \setminus (-1,1). \end{cases}$$

Then the following square commutes up to homotopy:

$$\begin{tikzcd} M \times S^1 \arrow[r, "H"] \arrow[d, "h \times S^1"'] & X^\diamond \arrow[d, "\psi"] \\ X \times S^1 \arrow[r, "q"'] & X \wedge S^1 \end{tikzcd}$$

Hence

$$\psi_*[[M \times S^1, H]] \;=\; [[M \times S^1, q \circ (h \times S^1)]] \;=\; [[M,h]] \wedge d \;,$$

and thus

$$\bar\partial(\psi_*^{-1}([[M,h]] \wedge d)) \;=\; \bar\partial[[M \times S^1, H]] \;.$$

To calculate this geometric boundary we use the smooth separating function

$$r \;:\; M \times S^1 \;\longrightarrow\; [0,1] \;, \quad r(m,z) \;=\; \begin{cases} (z+1)/2 & \text{for } z \in [-1,1], \text{ and} \\ (z^{-1}+1)/2 & \text{for } z \in S^1 \setminus (-1,1). \end{cases} .$$

Then $1/2$ is a regular value of this separating function, and the preimage over this regular value is $r^{-1}(1/2) = M \times \{0,\infty\}$, two disjoint copies of M. The function H takes the copy $M \times \{\infty\}$ to the basepoint of X, so this copy does not contribute to the reduced bordism group. The restriction of H to the other copy $M \times \{0\}$ is the original map h, so we obtain

$$\bar\partial[M \times S^1, H] \;=\; [[M \times \{0,\infty\}, H|_{M \times \{0,\infty\}}]] \;\equiv\; [[M,h]]$$

in the reduced bordism group of X. $\square$

Now we work our way towards the Thom-Pontryagin construction that assigns to every bordism class over a space X a homology class in $MO_*(X_+)$. We break the construction up into two steps, and we first discuss the *fundamental class*, a basic invariant associated with a closed smooth manifold.

Construction 8.11. It will be convenient to slightly modify the definition of MO, given in Example 3.6. We introduce a sequential spectrum MO' by $MO'_0 = *$, and

$$MO'_n \;=\; \mathrm{Th}(Gr_n(\mathbb{R}^\infty)) \ ,$$

the Thom space of the tautological n-plane bundle over the Grassmannian of n-planes in $\mathbb{R}^\infty$, for $n \geq 1$. The structure map

$$\sigma_n \;:\; S^1 \wedge MO'_n \;\longrightarrow\; MO'_{1+n} \quad \text{is given by} \quad \sigma_n(x \wedge (v, L)) \;=\; ((x, v), \mathbb{R} \oplus L) \ ;$$

here $x \in \mathbb{R}$, $v \in L \subset \mathbb{R}^\infty$.

We observe that MO' is stably equivalent to the underlying sequential spectrum of the commutative orthogonal ring spectrum MO defined in Example 4.17. The reader should beware that MO' does not extend to an orthogonal ring spectrum in any natural way, so the multiplicative structure gets lost on the way from MO to MO'.

Any linear monomorphism $\alpha : V \longrightarrow W$ between euclidean inner product spaces, possibly infinite dimensional, induces a continuous based map of Thom spaces

$$\alpha_* \;:\; \mathrm{Th}(Gr_n(V)) \;\longrightarrow\; \mathrm{Th}(Gr_n(W)) \ , \quad \alpha_*(v, L) \;=\; (\alpha(v), \alpha(L)) \ .$$

For $n \geq 1$, we define a linear isometric embedding

(8.12) $$\psi^n \;:\; \mathbb{R}^\infty \;\longrightarrow\; (\mathbb{R}^n)^\infty \quad \text{by}$$

$$\psi^n(x_1, x_2, x_3, \dots) \;=\; ((x_1, x_2, \dots, x_n), (0, \dots, 0, x_{n+1}), (0, \dots, 0, x_{n+2}), (0, \dots, 0, x_{n+3}), \dots) \ .$$

The main point about this definition is that the following diagram of linear isometric embeddings commutes:

$$\begin{tikzcd}
\mathbb{R} \oplus \mathbb{R}^\infty \arrow[r, "{(x,(y_1,y_2,\dots)) \mapsto (x,y_1,y_2,\dots)}"] \arrow[d, "\mathbb{R}\oplus\psi^n"'] & \mathbb{R}^\infty \arrow[d, "\psi^{1+n}"] \\
\mathbb{R} \oplus (\mathbb{R}^n)^\infty \arrow[r, "{(x,(v_1,v_2,v_3,\dots)) \mapsto ((x,v_1),(0,v_2),(0,v_3),\dots)}"'] & (\mathbb{R}^{1+n})^\infty
\end{tikzcd}$$

The upper horizontal isometry is implicitly used in the definition of the structure map of MO'; and the lower horizontal isometry is implicitly used in the definition of the structure map of MO. So the following diagram of Thom spaces commutes:

$$\begin{tikzcd}
S^1 \wedge MO'_n \arrow[r, equal] \arrow[d, "S^1\wedge(\psi_n)_*"'] & S^1 \wedge \mathrm{Th}(Gr_n(\mathbb{R}^\infty)) \arrow[r, "\sigma_n"] \arrow[d, "S^1\wedge\psi^n_*"'] & S^1 \wedge \mathrm{Th}(Gr_{1+n}(\mathbb{R}^\infty)) \arrow[r, equal] \arrow[d, "\psi^{1+n}_*"] & MO'_{1+n} \arrow[d, "\psi^{1+n}_*"] \\
S^1 \wedge MO(\mathbb{R}^n) \arrow[r, equal] & S^1 \wedge \mathrm{Th}(Gr_n((\mathbb{R}^n)^\infty)) \arrow[r, "\sigma_n"'] & S^1 \wedge \mathrm{Th}(Gr_{1+n}((\mathbb{R}^{1+n})^\infty)) \arrow[r, equal] & MO(\mathbb{R}^{1+n})
\end{tikzcd}$$

The upshot is that the based continuous maps $\psi^n_* : MO'_n \longrightarrow MO(\mathbb{R}^n)$ for a morphism of sequential spectra $\psi : MO' \longrightarrow u(MO)$.

We claim that moreover, ψ^n_* is a based homotopy equivalence for every $n \geq 1$. Indeed, the space $\mathbf{L}(\mathbb{R}^\infty, (\mathbb{R}^n)^\infty)$ of linear isometric embeddings is contractible; so there is a path through linear isometric embeddings from ψ^n to a linear isometric isomorphism $\kappa : \mathbb{R}^\infty \cong (\mathbb{R}^n)^\infty$. Any such path induces a homotopy between ψ^n_* and the homeomorphism κ_*. So ψ^n_* is based homotopic to a homeomorphism, it is a based homotopy equivalence. So for every based space A, the map $\psi^n_* : MO'_n \wedge A \longrightarrow MO(\mathbb{R}^n) \wedge A$ is a based homotopy equivalence, and the morphism $\psi \wedge A : MO' \wedge A \longrightarrow u(MO) \wedge A = u(MO \wedge A)$ induces isomorphism of homotopy groups

(8.13) $$MO'_k(A) \;\cong\; MO_k(A) \ .$$

Construction 8.14 (Thom-Pontryagin construction)**.** To every smooth compact manifold M of dimension k, possibly with boundary, we associate a *fundamental class*

$$\langle M\rangle \ \in \ MO'_k(M/\partial M) \ = \ \pi_k(MO'\wedge(M/\partial M)) \ .$$

This class records the homotopical information in the stable normal bundle of M, and it is the geometric input for the Thom-Pontryagin map to MO-homology.

For the construction we choose a smooth embedding $i: M \longrightarrow \mathbb{R}^{n+k}$, for some $n\geq 0$. We use the standard inner product on $\mathbb{R}^{n+k}$ to define the normal bundle ν of the embedding by

$$\nu \ = \ \{(v,m)\in\mathbb{R}^{n+k}\times M\colon v\perp(Di)(T_mM)\} \ ;$$

in other words: the fiber of this normal bundle over a point $m\in M$ is the orthogonal complement in $\mathbb{R}^{n+k}$ of the image of the tangent space T_mM. By multiplying with a suitably large scalar, if necessary, we can assume that the embedding is *wide* in the sense that the exponential map

$$D(\nu) \ \longrightarrow \ \mathbb{R}^{n+k} \ , \quad (v,m) \ \longmapsto \ v \,+\, i(m)$$

is injective on the unit disc bundle of the normal bundle, and hence a closed embedding. The image of this map is a tubular neighborhood of radius 1 around $i(M)$. The boundary of $D(\nu)$ is the union of the unit sphere bundle $S(\nu)$ and the restriction $D(\nu|_{\partial M})$ of the unit disc bundle to the boundary of M. So the exponential map restricts to an open embedding on the open unit disc bundle $\mathring{D}(\nu|_{M\setminus\partial M})$ over the complement $M\setminus\partial M$ of the boundary. It thus determines a Thom-Pontryagin collapse map

$$c(i) \ : \ S^{n+k} \ \longrightarrow \ \mathrm{Th}(Gr_n(\mathbb{R}^{n+k}))\wedge(M/\partial M)$$

as follows: every point outside of $\mathring{D}(\nu|_{M\setminus\partial M})$ is sent to the basepoint, and a point $v+i(m)$, for $(v,m)\in\mathring{D}(\nu|_{M\setminus\partial M})$, is sent to

$$c(i)(v+i(m)) \ = \ \left(\frac{v}{1-|v|}, ((Di)(T_mM))^{\perp}\right)\wedge m \ .$$

Here and below, the Thom space is always taken for the tautological vector bundle over the respective Grassmannian. Now we let $\langle M\rangle$ denote the class of the composite

$$\begin{aligned} S^{n+k} \ &\xrightarrow{c(i)} \ \mathrm{Th}(Gr_n(\mathbb{R}^{n+k}))\wedge(M/\partial M) \\ &\xrightarrow{\kappa_*^{n+k}\wedge(M/\partial M)} \ \mathrm{Th}(Gr_n(\mathbb{R}^{\infty}))\wedge(M/\partial M) \ = \ MO'_n\wedge(M/\partial M) \ , \end{aligned}$$

where the second map is induced by the 'standard' linear isometric embedding

$$\kappa^{n+k} \ : \ \mathbb{R}^{n+k} \ \longrightarrow \ \mathbb{R}^{\infty} \ , \quad (x_1,\dots,x_{n+k}) \ \longmapsto \ (x_1,\dots,x_{n+k},0,0,\dots) \ .$$

Proposition 8.15. *The fundamental class $\langle M\rangle$ in $MO'_k(M/\partial M)$ associated to the smooth compact k-manifold M is independent of the choice of wide embedding $i: M\longrightarrow\mathbb{R}^{n+k}$.*

Proof. If we enlarge the embedding $i: M\longrightarrow\mathbb{R}^{n+k}$ by the linear isometric embedding $\alpha:\mathbb{R}^{n+k}\longrightarrow\mathbb{R}^{1+n+k}$ with $\alpha(v)=(0,v)$, then the collapse map $c(\alpha i)$ associated with the composite embedding $\alpha\circ i: M\longrightarrow\mathbb{R}^{1+n+k}$ is homotopic to the composite

$$S^{1+n+k} \ \xrightarrow{S^1\wedge c(i)} \ S^1\wedge\mathrm{Th}(Gr_n(\mathbb{R}^{n+k}))\wedge(M/\partial M) \ \xrightarrow{\sigma_n^k\wedge(M/\partial)} \ \mathrm{Th}(Gr_{1+n}(\mathbb{R}^{1+n+k}))\wedge(M/\partial M) \ ,$$

where

$$\sigma_n^k \ : \ S^1\wedge\mathrm{Th}(Gr_n(\mathbb{R}^{n+k})) \ \longrightarrow \ \mathrm{Th}(Gr_{1+n}((\mathbb{R}^{1+n+k})) \quad \text{is defined by} \quad x\wedge(v,L) \ = \ ((x,v),\mathbb{R}\oplus L) \ .$$

Moreover, the following square commutes:

$$\begin{tikzcd}
S^1 \wedge \mathrm{Th}(Gr_n(\mathbb{R}^{n+k})) \arrow[r, "S^1\wedge\kappa_*^{n+k}"] \arrow[d, "\sigma_n^k"'] & S^1 \wedge \mathrm{Th}(Gr_n(\mathbb{R}^\infty)) \arrow[r, equal] & S^1 \wedge MO'_n \arrow[d, "\sigma_n"] \\
\mathrm{Th}(Gr_{1+n}(\mathbb{R}^{1+n+k})) \arrow[r, "\kappa_*^{1+n+k}"'] & \mathrm{Th}(Gr_{1+n}(\mathbb{R}^\infty)) \arrow[r, equal] & MO'_{1+n}
\end{tikzcd}$$

So the stabilization of $(\kappa_*^{n+k}\wedge(M/\partial))\circ c(i) : S^{n+k} \longrightarrow MO'_n\wedge(M/\partial)$ is homotopic to $(\kappa_*^{1+n+k}\wedge(M/\partial M))\circ c(\alpha i) : S^{1+n+k} \longrightarrow MO'_{1+n} \wedge (M/\partial)$; thus the classes in $MO'_k(M/\partial)$ arising from i and $\alpha \circ i$ coincide.

Now we consider two different wide embeddings $i : M \longrightarrow \mathbb{R}^{n+k}$ and $j : M \longrightarrow \mathbb{R}^{\bar{n}+k}$. By working with the maximum of n and $\bar{n}$ and appealing to the previous paragraph, we can assume without loss of generality that $\bar{n} = n$. The map

$$M \times [0,1] \ \longrightarrow\ \mathbb{R}^{n+k} \oplus \mathbb{R}^{n+k}\ , \quad (m,t) \ \longmapsto\ (t\cdot i(m),\ (1-t)\cdot i(m))$$

is a smooth isotopy through wide embeddings between i into the second summand and i into the first summand. This isotopy induces a homotopy between the two collapse maps. Similarly, the map

$$M \times [0,1] \ \longrightarrow\ \mathbb{R}^{n+k} \oplus \mathbb{R}^{n+k}\ , \quad (m,t) \ \longmapsto\ (t\cdot i(m),\ (1-t)\cdot j(m))$$

is a smooth isotopy through wide embeddings between j into the second summand and i into the first summand. This isotopy induces a homotopy between the two collapse maps. So altogether the collapse maps based on i and j become homotopic after composition with the embedding $\mathbb{R}^{n+d} \longrightarrow \mathbb{R}^{n+d} \oplus \mathbb{R}^{n+d}$ as the second summand. By the first paragraph, the class $\langle M\rangle$ does not depend on the smooth embedding i. $\square$

Example 8.16. We claim that the image of the fundamental class $\langle S^1\rangle \in MO_1(S^1_+)$ under the reduction map $q : S^1_+ \longrightarrow S^1$ that identifies the additional basepoint with the 'intrinsic' basepoint ∞ satisfies the relation

$$q_*\langle S^1\rangle \ =\ 1 \wedge S^1 \tag{8.17}$$

in the group $MO_1(S^1)$.

One possible wide smooth embedding of S^1 into an inner product space is the embedding

$$j\ :\ S^1 \ \longrightarrow\ \mathbb{C} = \mathbb{R}^2\ , \quad j(x) \ =\ \frac{ix-1}{ix+1}$$

into the complex numbers, where the symbol 'i' refers to the imaginary unit in $\mathbb{C}$; this map is sometimes called the 'Cayley transform', and it maps S^1 onto the unit sphere $S(\mathbb{C})$. The class $q_*\langle S^1\rangle$ is then represented by the composite

$$S^2 \ \xrightarrow{c(j)}\ \mathrm{Th}(Gr_1(\mathbb{R}^2)) \wedge S^1_+ \ \xrightarrow{\kappa_*^2\wedge q}\ \mathrm{Th}(Gr_1(\mathbb{R}^\infty)) \wedge S^1\ .$$

The Hurewicz theorem lets us calculate the homotopy group that is home to this composite:

$$\pi_2(\mathrm{Th}(Gr_1(\mathbb{R}^\infty))\wedge S^1) \ \cong\ \pi_2(\mathbb{R}P^\infty \wedge S^1) \ \cong\ H_2(\mathbb{R}P^\infty \wedge S^1;\mathbb{Z}) \ \cong\ H_1(\mathbb{R}P^\infty;\mathbb{Z}) \ \cong\ \mathbb{Z}/2\ .$$

To finish the argument, one would now have to argue that the previous composite represents the generator of this group.

Now we show that the fundamental class is compatible with disjoint unions and taking boundaries. For part (ii) of the following proposition we exploit that for every smooth compact manifold M, the inclusion of the boundary $\partial M \longrightarrow M$ has the homotopy extension property. One way to see this is exploit that M can be triangulated in such a way that ∂M is a subcomplex. In particular, the pair $(M, \partial M)$ admits a relative CW-structure. Alternatively, we can appeal to the existence of collars.

Proposition 8.18. *Let M and N be smooth compact k-manifolds.*

(i) *Let* $i^1 : M \longrightarrow M \cup N$ *and* $i^2 : N \longrightarrow M \cup N$ *denote the inclusions into a disjoint union. Then the relation*

$$\langle M \cup N \rangle \;=\; i^1_*\langle M \rangle \;+\; i^2_*\langle N \rangle$$

holds in the group $MO'_k((M \cup N)/(\partial M \cup \partial N))$.

(ii) *The relation*

$$\delta\langle M \rangle \;=\; \langle \partial M \rangle$$

holds, where

$$\delta \;:\; MO'_k(M/(\partial M)) \;\longrightarrow\; MO'_{k-1}((\partial M)_+)$$

is the connecting homomorphism (1.19) *of the inclusion* $\partial M \longrightarrow M$.

Proof. (i) We let $p^1 : (M \cup N)/(\partial M \cup \partial N) \longrightarrow M/\partial M$ and $p^2 : (M \cup N)/(\partial M \cup \partial N) \longrightarrow N/\partial N$ denote the two projections, i.e., p^1 takes N to the collapsed boundary, and p^2 takes M to the collapsed boundary. We choose any wide smooth embedding $i : M \cup N \longrightarrow \mathbb{R}^{n+k}$ and observe that the composite

$$S^{n+k} \xrightarrow{c(i)} \mathrm{Th}(Gr_n(\mathbb{R}^{n+k})) \wedge (M \cup N)/(\partial M \cup \partial N) \xrightarrow{\mathrm{Id} \wedge p^1} \mathrm{Th}(Gr_n(\mathbb{R}^{n+k})) \wedge (M/\partial M)$$

is on the nose the collapse map for M based on the restriction of the embedding i to M. We then obtain

$$p^1_*\langle M \cup N \rangle \;=\; \langle M \rangle \;=\; p^1_*(i^1_*\langle M \rangle + i^2_*\langle N \rangle) \;;$$

the second relation uses that $p^1 \circ i^1$ is the identity, $p^2 \circ i^1$ is the trivial map and p^1_* is additive. The analogous argument shows that $p^2_*\langle M \cup N \rangle \;=\; p^2_*(i^1_*\langle M \rangle + i^2_*\langle N \rangle)$. Since homotopy groups are additive on wedges, the map

$$(p^1_*, p^2_*) \;:\; \pi_k(MO' \wedge (M \cup N)/(\partial M \cup \partial N)) \;\longrightarrow\; \pi_k(MO' \wedge (M/\partial M)) \times \pi_k(MO' \wedge (N/\partial N))$$

is bijective, and this proves the claim.

(ii) By definition, the connecting homomorphism (1.19) is the composite of three maps

$$MO'_k(M/(\partial M)) \xleftarrow[\cong]{(0\cup \mathrm{Id}_M)_*} MO'_k(C(\partial M) \cup_{\partial M} M)) \xrightarrow{p_*} MO'_k((\partial M)_+ \wedge S^1) \xleftarrow[\cong]{-\wedge S^1} MO'_{k-1}((\partial M)_+) \;.$$

We choose a *collar*, i.e., a smooth embedding $c : \partial M \times [0,2] \longrightarrow M$ such that $c(-,0) : \partial M \longrightarrow M$ is the inclusion and the image of $\partial M \times [0,2)$ is an open neighborhood of the boundary inside M. The collar allows us to identify M with the mapping cylinder of the inclusion $\iota : \partial M \longrightarrow M$. More precisely, we define a continuous map

$$\zeta \;:\; M \;\longrightarrow\; \partial M \times [0,1] \cup_{\partial M \times 1} M$$

by

$$\zeta(m) \;=\; \begin{cases} (x,t) & \text{if } m = c(x,t) \text{ for } (x,t) \in \partial M \times [0,1], \\ c(x, 2(t-1)) & \text{if } m = c(x,t) \text{ for } (x,t) \in \partial M \times [1,2], \\ m & \text{if } m \in M \setminus c(M \times [0,2)). \end{cases}$$

The map ζ is a continuous bijection between compact Hausdorff spaces, and hence a homeomorphism. The map ζ identifies the boundary of M with the 'start' $\partial M \times \{0\}$ of the cylinder; it factors through a based homeomorphism

$$\bar{\zeta} \;:\; M/\partial M \xrightarrow{\cong} C(\partial M) \cup_{\partial M \times 1} M \;=\; C\iota$$

to the unreduced mapping cone of the inclusion. The composite $(0 \cup \mathrm{Id}_M) \circ \bar{\zeta} : C\iota \longrightarrow C\iota$ is based homotopic to the identity, by interpolating through the collar coordinate. The effects of the maps $(0 \cup \mathrm{Id}_M) \circ \bar{\zeta} : C\iota \longrightarrow M/\partial$ and $\bar{\zeta} : M/\partial M \longrightarrow C\iota$ on MO'-homology are thus inverse to each other. So the connecting homomorphism equals

$$MO'_k(M/(\partial M)) \xrightarrow{(p\bar{\zeta})_*} MO'_k((\partial M)_+ \wedge S^1) \xleftarrow[\cong]{-\wedge S^1} MO'_{k-1}((\partial M)_+) \;.$$

The map $p\bar{\zeta} : M/(\partial M) \longrightarrow (\partial M)_+ \wedge S^1$ is explicitly given by

$$(p\bar{\zeta})(m) \ = \ \begin{cases} x \wedge \frac{2t-1}{t(1-t)} & \text{if } m = c(x,t) \text{ for } (x,t) \in \partial M \times [0,1], \\ * & \text{if } m \in M \setminus c(M \times [0,1)). \end{cases}$$

Now we choose a wide smooth embedding

$$i \ : \ \partial M \ \longrightarrow \ \mathbb{R}^{n+k} \ .$$

Then we choose a wide smooth embedding

$$j \ : \ M \ \longrightarrow \ \mathbb{R}^{n+k+1}$$

that 'coincides with $i \times \mathbb{R}$ near the boundary', with respect to the chosen collar. More precisely, we arrange j so that

$$j(c(x,t)) \ = \ (j(x), t)$$

for all $(x,t) \in \partial M \times [0,1]$.

Because the embedding j is the product of i with the identity on the collar, the following square commutes up to based homotopy:

$$\begin{tikzcd} S^{n+k+1} \arrow[r, "c(j)"] \arrow[d, "c(i)\wedge S^1"'] & \mathrm{Th}(Gr_n(\mathbb{R}^{n+k+1})) \wedge (M/\partial M)_+ \arrow[d, "\mathrm{Id}\wedge(p\bar{\zeta})"] \\ \mathrm{Th}(Gr_n(\mathbb{R}^{n+k})) \wedge (\partial M)_+ \wedge S^1 \arrow[r, "\alpha_*\wedge\mathrm{Id}"'] & \mathrm{Th}(Gr_n(\mathbb{R}^{n+k+1})) \wedge (\partial M)_+ \wedge S^1 \end{tikzcd}$$

where $\alpha : \mathbb{R}^{n+k} \longrightarrow \mathbb{R}^{n+k+1}$ is $\alpha(x) = (x, 0)$. This witnesses the relation

$$(p\bar{\zeta})_* \langle M \rangle \ = \ \langle \partial M \rangle \wedge S^1$$

in the group $\pi_{k+1}(MO \wedge (\partial M)_+ \wedge S^1)$. The desired relation follows:

$$\langle \partial M \rangle \ = \ (p\bar{\zeta})_* \langle M \rangle \wedge S^{-1} \ = \ \delta \langle M \rangle \ . \qquad \square$$

Construction 8.19. The Thom-Pontryagin construction defines a natural transformation of homology theories

$$\Theta \ = \ \Theta(X) \ : \ \widetilde{\mathcal{N}}_*(X) \ \longrightarrow \ MO'_*(X) \ ,$$

as we now recall. We let (M, h) be a k-dimensional singular manifold over a based space X. The way we have set things up, all the geometry is already encoded in the fundamental class $\langle M \rangle \in MO_k(M_+)$; the rest is a formal procedure, by simply pushing the class forward and use the functoriality of $MO'_*(-)$:

$$\Theta[M,h] \ = \ h_* \langle M \rangle \ \in \ MO'_k(X) \ .$$

Proposition 8.20. *The class $\Theta[M,h]$ in $MO_k(X)$ only depends on the bordism class of the singular manifold (M,h).*

Proof. We let (M,h) be a singular manifold that is null-bordant. We choose a null-bordism $(B, H : B \longrightarrow X, \psi : M \cong \partial B)$, so that $H|_{\partial B} \circ \psi = h$. We write $\iota : \partial B \longrightarrow B$ for the inclusion. Then

$$h_* \langle M \rangle \ = \ (H \circ \iota \circ \psi)_* \langle M \rangle \ = \ H_*(\iota_* \langle \partial B \rangle) = \ H_*(\iota_*(\delta \langle B \rangle)) \ = \ 0 \ .$$

The third equation is Proposition 8.18 (ii); the last equation exploits that the connecting homomorphism $\delta : MO'_{k+1}(B/\partial B) \longrightarrow MO'_k((\partial B)_+)$ and the map $\iota_* a : MO'_k((\partial B)_+) \longrightarrow MO'_k(B_+)$ occur back-to-back in an exact sequence, so their composite is the zero homomorphism. Since the fundamental class is additive on disjoint unions (Proposition 8.18 (i)), $\Theta[M,h]$ only depends on the bordism class of (M,h). $\square$

Theorem 8.21. (i) *The Thom-Pontryagin map $\Theta : \widetilde{\mathcal{N}}_k(X) \longrightarrow MO_k(X)$ is additive.*

(ii) *The Thom-Pontryagin map is compatible with the boundary maps in the mapping cone sequences in bordism and MO-homology, i.e., Θ is a transformation of homology theories.*

Proof. Part (i) is a direct consequence of the additivity of the fundamental classes established in Proposition 8.18 (i). We let (M,h) and (N,g) be two singular k-manifolds over X. The sum $[M,h]+[N,g]$ is represented by the singular k-manifold $(M \amalg N, h+g)$. So

$$\begin{aligned}\Theta([M,h]+[N,g]) \ &= \ \Theta[M\amalg N, h+g] \ = \ (h+g)_*\langle M\amalg N\rangle \\ &= \ (h+g)_*(i^1_*\langle M\rangle + i^2_*\langle N\rangle) \ = \ ((h+g)i^1)_*\langle M\rangle + ((h+g)i^2)_*\langle N\rangle \\ &= \ h_*\langle M\rangle + g_*\langle N\rangle \ = \ \Theta[M,h]+\Theta[N,g] \ .\end{aligned}$$

(ii) We let $f : X \longrightarrow Y$ be a continuous map. Compatibility of the Thom-Pontryagin construction with the boundary homomorphism amounts to the commutativity of the following square:

$$\begin{tikzcd} \widetilde{\mathcal{N}}_{k+1}(Cf) \arrow[r, "p_*"] \arrow[d, "\Theta"'] & \widetilde{\mathcal{N}}_{k+1}(X_+\wedge S^1) \arrow[r, "(-\wedge d)^{-1}", "\cong"'] \arrow[d, "\Theta"'] & \mathcal{N}_k(X) \arrow[d, "\Theta"] \\ MO_{k+1}(Cf) \arrow[r, "p_*"'] & MO_{k+1}(X_+\wedge S^1) \arrow[r, "\cong", "(-\wedge S^1)^{-1}"'] & MO_k(X_+) \end{tikzcd}$$

Here $p : Cf \longrightarrow X_+ \wedge S^1$ is the projection. Indeed, the upper composite agrees with the boundary map in bordism by Proposition 8.10; the lower composite is the homotopy theoretic boundary map by the definition in (1.17) and the fact that the suspension isomorphism in MO-homology is exterior multiplication with the class $1\wedge S^1 \in MO_1(S^1)$.

The Thom-Pontryagin construction is natural for continuous maps, so it remains to show the commutativity of the right square above. Assuming multiplicativity of the Thom-Pontryagin construction – see Proposition 8.23 below – the relation (8.17) gives

$$\Theta(x\wedge d) \ = \ \Theta(x)\wedge\Theta(d) \ = \ \Theta(x)\wedge(1\wedge S^1) \ = \ \Theta(x)\wedge S^1$$

for all $x \in \widetilde{\mathcal{N}}_k(X)$. $\square$

The following is Thom's celebrated theorem [47, Theorem IV.8]:

Theorem 8.22. *For every space X and $k \geq 0$, the Thom-Pontryagin map*

$$\Theta(X) \ : \ \mathcal{N}_k(X) \ \longrightarrow \ MO'_k(X_+)$$

is an isomorphism.

Proof. We start with the special case where $X = *$ is a one-point space. Then the reference map to X is no information, so the group $\mathcal{N}_k(*)$ becomes the bordism group $\mathcal{N}_k$ of smooth closed k-manifolds. So we must show that the Thom-Pontryagin map

$$\Theta \ : \ \mathcal{N}_k \ \longrightarrow \ \pi_k(MO')$$

is an isomorphism. This is a classical argument using basic tool from differential topology; we review the argument for surjectivity.

We consider a based continuous map $f : S^{n+k} \longrightarrow MO'_n = \mathrm{Th}(Gr_n(\mathbb{R}^\infty))$ that represents a class in $\pi_k(MO')$. Since the sphere S^{n+k} is compact, the map f has image in $\mathrm{Th}(Gr_n(V))$ for some finite-dimensional linear subspace V of $\mathbb{R}^\infty$.

Because $Gr_n(V)$ is compact, the Thom space of the tautological n-plane bundle over $Gr_n(V)$ is the onepoint compactification of the total space. So $\mathrm{Th}(Gr_n(V))$ has a preferred smooth structure away from the basepoint at ∞, and the zero section of the tautological bundle defines a smooth embedding

$$s \ : \ Gr_n(V) \ \longrightarrow \ \mathrm{Th}(Gr_n(V)) \ , \quad L \ \longmapsto \ (0,L) \ .$$

We choose a based homotopy from f to another map $g : S^{n+k} \longrightarrow \mathrm{Th}(Gr_n(V))$ that is smooth (away from the basepoints) and transverse to the zero section. The zero section has codimension n, so

$$M \ = \ g^{-1}(s(Gr_n(V)))$$

is a smooth closed submanifold of S^{n+k} of codimension n that misses the basepoint at ∞. In other words, M is a smooth closed k-manifold inside $S^{n+k} \setminus \{\infty\} = \mathbb{R}^{n+k}$; if we use the inclusion $M \longrightarrow \mathbb{R}^{n+k}$ to calculate the class $\Theta[M]$, the representing map will be homotopic to g, and hence $\Theta[M] = [g] = [f]$ is the class in $\pi_k(MO)$ we started with. This proves surjectivity of the Thom-Pontryagin map.

Assuming now that the map $\Theta : \mathcal{N}_k \longrightarrow \pi_k(MO')$ is an isomorphism for all integers k, we can complete the argument by formal reasoning. Source and target of Θ are homology theories in X. So cell induction proves the claim when X is a CW-complex. Both functors $\mathcal{N}_k$ and $\pi_k(MO' \wedge (-)_+)$ take weak equivalences of spaces to isomorphisms; so by CW-approximation, the claim follows for all spaces X. □

The sequential spectrum MO' is stably equivalent to the commutative orthogonal ring spectrum MO discussed in Example 4.17. As we explained in (8.13), the homology theories represented by MO' and MO are naturally isomorphic. In the rest of the discussion, we use the isomorphism (8.13) to identify these two homology theories. In particular, we now view the Thom-Pontryagin construction as taking values in $MO_k(X)$ (as opposed to $MO'_k(X)$). Because MO underlies an orthogonal ring spectrum, the homology theory MO_*, which comes with exterior products as explained in Proposition **??**. We now explain that the Thom-Pontryagin construction is also multiplicative.

Proposition 8.23. *Let M and N be smooth closed manifolds of dimension k and l, respectively.*

(i) *The relation*

$$\langle M \times N \rangle \ = \ \langle M \rangle \times \langle N \rangle$$

holds in the group $MO_{k+l}(M \times N)$.

(ii) *The Thom-Pontryagin map is multiplicative, i.e., for all classes $x \in \widetilde{\mathcal{N}}_k(X)$ and $y \in \widetilde{\mathcal{N}}_l(Y)$, the relation*

$$\Theta(x \wedge y) \ = \ \Theta(x) \wedge \Theta(y)$$

holds in $MO_{k+l}(X \wedge Y)$.

Proof. (i) We choose wide smooth embeddings

$$i \ : \ M \ \longrightarrow \ \mathbb{R}^{m+k} \qquad \text{and} \qquad j \ : \ N \ \longrightarrow \ \mathbb{R}^{n+l} \ .$$

The product map

$$i \times j \ : \ M \times N \ \longrightarrow \ \mathbb{R}^{m+k+n+l}$$

is then another wide smooth embedding that we use for the Thom-Pontryagin construction of $M \times N$. The normal bundle of $i \times j$ is the exterior direct sum of the normal bundles of i and j. The unit disc $D(V \oplus W)$ of the direct sum is contained in the product $D(V) \times D(W)$ of the unit discs, so the exponential tubular neighborhood for $i \times j$ is contained in the product of the exponential tubular neighborhoods for i and j. The collapse map

$$S^{m+k+n+l} \ \xrightarrow{c(i \times j)} \ \mathrm{Th}(Gr_{m+n}(\mathbb{R}^{m+k+n+l})) \wedge (M \times N)_+$$

is homotopic to the composite

$$\begin{aligned} S^{m+k} \wedge S^{n+l} \ &\xrightarrow{c(i) \wedge c(j)} \ (\mathrm{Th}(Gr_m(\mathbb{R}^{m+k}) \wedge M_+) \wedge (\mathrm{Th}(Gr_n(\mathbb{R}^{n+l})) \wedge N_+) \\ &\xrightarrow{\text{shuffle}} \ \mathrm{Th}(Gr_m(\mathbb{R}^{m+k}) \wedge \mathrm{Th}(Gr_n(\mathbb{R}^{n+l})) \wedge (M \times N)_+ \\ &\xrightarrow{\mu^{k,l}_{m,n} \wedge (M \times N)_+} \ \mathrm{Th}(Gr_{m+n}(\mathbb{R}^{m+k+n+l})) \wedge (M \times N)_+ \ ; \end{aligned}$$

here

$$\begin{aligned} \mu^{k,l}_{m,n} : \ & \mathrm{Th}(Gr_m(\mathbb{R}^{m+k}) \wedge \mathrm{Th}(Gr_n(\mathbb{R}^{n+l})) \ \longrightarrow \ \mathrm{Th}(Gr_{m+n}(\mathbb{R}^{m+k+n+l})) \\ & \text{is defined by} \qquad (v, L) \wedge (w, K) \quad \longmapsto \quad ((v, w), L \oplus K) \ . \end{aligned}$$

The following diagram commutes up to homotopy through linear isometric embeddings, simply because the Stiefel manifold $\mathbf{L}(\mathbb{R}^{m+k} \oplus \mathbb{R}^{n+l}, (\mathbb{R}^{m+n})^\infty)$ is contractible:

$$\begin{tikzcd}
\mathbb{R}^{m+k} \oplus \mathbb{R}^{n+l} \arrow[r, "\mu^{k,l}_{m,n}"] \arrow[d, "\kappa^{m+k} \oplus \kappa^{n+l}"'] & \mathrm{Th}(Gr_{m+n}(\mathbb{R}^{m+k+n+l})) \arrow[d, "\kappa^{m+k+n+l}_*"] \\
\mathbb{R}^\infty \oplus \mathbb{R}^\infty \arrow[d, "\psi^m \oplus \psi^n"'] & \mathbb{R}^\infty \arrow[d, "\psi^{m+n}"] \\
(\mathbb{R}^m)^\infty \oplus (\mathbb{R}^n)^\infty \arrow[r, "\kappa_{\mathbb{R}^m, \mathbb{R}^n}"'] & (\mathbb{R}^{m+n})^\infty
\end{tikzcd}$$

Here $\psi^n : \mathbb{R}^\infty \longrightarrow (\mathbb{R}^n)^\infty$ was defined in (8.12) for the purposes of comparing MO' and MO. The multiplicativity now follows from the fact that the following diagram of induced maps of Thom spaces commutes up to based homotopy:

$$\begin{tikzcd}
\mathrm{Th}(Gr_m(\mathbb{R}^{m+k}) \wedge \mathrm{Th}(Gr_n(\mathbb{R}^{n+l})) \arrow[r, "\mu^{k,l}_{m,n}"] \arrow[d, "\kappa^{m+k}_* \wedge \kappa^{n+l}_*"'] & \mathrm{Th}(Gr_{m+n}(\mathbb{R}^{m+k+n+l})) \arrow[d, "\kappa^{m+k+n+l}_*"] \\
\mathrm{Th}(Gr_m(\mathbb{R}^\infty)) \wedge \mathrm{Th}(Gr_n(\mathbb{R}^\infty)) \arrow[d, "\psi^m_* \wedge \psi^n_*"'] & \mathrm{Th}(Gr_{m+n}(\mathbb{R}^\infty)) \arrow[d, "\psi^{m+n}_*"] \\
\mathrm{Th}(Gr_m((\mathbb{R}^m)^\infty)) \wedge \mathrm{Th}(Gr_n((\mathbb{R}^n)^\infty)) \arrow[d, equal] & \mathrm{Th}(Gr_{m+n}((\mathbb{R}^{m+n})^\infty)) \arrow[d, equal] \\
MO_m \wedge MO_n \arrow[r, "\mu_{m,n}"'] & MO_{m+n}
\end{tikzcd}$$

Part (ii) is a formal consequence of the multiplicativity of the fundamental classes. We consider singular manifolds $(M, h : M \longrightarrow X)$ and $(N, g : N \longrightarrow Y)$. The class $[[M, h]] \wedge [[N, g]]$ is then represented by the singular manifold $(M \times N, q \circ (h \times g))$, where $q : X \times Y \longrightarrow X \wedge Y$ is the quotient map. Then

$$\begin{aligned}
\Theta[[M \times N, q \circ (h \times g)]] &= (q \circ (h \times g))_* \langle M \times N \rangle \\
&= (q \circ (h \times g))_* (\langle M \rangle \times \langle N \rangle) \\
&= q_*(h_*\langle M \rangle \times g_*\langle N \rangle) \\
&= h_*\langle M \rangle \wedge g_*\langle N \rangle \;=\; \Theta[[M, h]] \wedge \Theta[[N, g]] \ .
\end{aligned}$$

in the group $MO_{k+l}(X \wedge Y)$. The second equation is Proposition 8.18 (ii). □

Construction 8.24 (The orientation)**.** Every closed k-manifold M has a *fundamental homology class*

$$[M] \ \in \ H_k(M; \mathbb{F}_2)$$

that is uniquely characterized by the property that for every point $x \in M$, the image of $[M]$ in the local homology groups $H_k(M, M \setminus \{x\}; \mathbb{F}_2)$ is non-zero. We obtain a natural transformation of homology theories from bordism to mod-2 singular homology by

$$\mathcal{N}_k(X) \ \longrightarrow \ H_k(X; \mathbb{F}_2) \ , \quad [M, h] \ \longmapsto \ h_*[M] \ .$$

This is well-defined on bordism classes because whenever M is the boundary of a compact $(k+1)$-manifold W, then the fundamental class $[M]$ is the image of the fundamental class $[W] \in H_{k+1}(W, M; \mathbb{F}_2)$ under the connecting homomorphism $\partial : H_{k+1}(W, M; \mathbb{F}_2) \longrightarrow H_k(M; \mathbb{F}_2)$ of the pair (W, M). Hence the class $[M]$ is in the kernel of the homomorphism $H_k(M; \mathbb{F}_2) \longrightarrow H_k(W; \mathbb{F}_2)$ induced by the inclusion $M \longrightarrow W$. Thom showed in [47, Théorème III.2] that the map from bordism to mod-2 homology is surjective.

We have seen in Theorem 8.22 that bordism is represented by the Thom spectrum MO. Moreover, singular homology with coefficients in an abelian group A is represented by the Eilenberg–MacLane spectrum HA. One can show that there is a unique non-zero morphism

$$\omega \ : \ MO \longrightarrow H\mathbb{F}_2$$

is the stable homotopy category $\mathcal{SH}$, sometimes called the *orientation* of MO; and this morphism represents the natural transformation that evaluates at the fundamental class.

Now we turn to explicit calculation of the unoriented bordism ring $\mathcal{N}_*$.

Example 8.25. A 0-dimensional closed manifold is a finite set. Bordism between 0-dimensional manifold are finite disjoint unions of closed intervals. Since an interval connects two points, two 0-dimensional closed manifolds are bordant if and only their cardinalities have the same parities. In other words, the map

$$\mathcal{N}_0 \xrightarrow{\cong} \mathbb{Z}/2 \ , \quad [M] \longmapsto |M| \mod 2$$

is an isomorphism.

A 1-dimensional smooth closed manifold is a finite disjoint union of circles. A circle is the boundary of a 2-disc, so every 1-dimensional smooth closed manifold is null-bordant. Hence the bordism group $\mathcal{N}_1$ is trivial.

The cardinality of a finite set is also its Euler characteristic. We observe that the Euler characteristic modulo 2 is in fact a bordism invariant in all even dimensions.

Proposition 8.26. *Let $k \in \mathbb{N}$ be even.*

(i) *The boundary of every compact $(k+1)$-manifold has an even Euler characteristic.*
(ii) *The Euler characteristic modulo 2 is a bordism invariant of smooth closed k-manifolds.*

Proof. (i) We let B be a compact $(k+1)$-dimensional manifold with boundary ∂B. Then the double $DB = B \cup_{\partial B} B$ is a closed $(k+1)$-manifold. Since $k+1$ is odd, Poincaré duality implies that $\chi(DB) = 0$. Additivity of Euler characteristic shows that

$$2 \cdot \chi(B) - \chi(\partial B) \ = \ \chi(DB) \ = \ 0 \ .$$

So the boundary ∂B has an even Euler characteristic, as claimed.

(ii) If M and N are closed k-manifolds and $M \amalg N \cong \partial B$ for some compact $(k+1)$-manifold B, then

$$\chi(M) + \chi(N) \ = \ \chi(\partial B)$$

is even by part (i). □

Example 8.27 (Real projective spaces)**.** For every *even* $k \geq 2$, the real projective k-space $\mathbb{R}P^k$ has Euler characteristic 1, so it represents a non-trivial element of the bordism group $\mathcal{N}_k$ by Proposition 8.26.

For every *odd* $k \geq 1$, the real projective k-space $\mathbb{R}P^k$ bounds a smooth closed $(k+1)$-manifold. Such a manifold can be described fairly explicit, based on the fact that $\mathbb{R}P^k$ admits a *free involution*, i.e., a diffeomorphism $\tau : \mathbb{R}P^k \longrightarrow \mathbb{R}P^k$ without fixed points such that $\tau^2 = \mathrm{Id}$. For this purpose we write $k = 2l-1$ and take $\mathbb{R}P^k$ as the quotient of the unit sphere $S(\mathbb{C}^l)$ in $\mathbb{C}^l$ by the antipodal action. The free involution is then given by

$$\tau[\lambda_1, \dots, \lambda_l] \ = \ [i\lambda_1, \dots, i\lambda_l] \ ,$$

where $i \in \mathbb{C}$ is the imaginary unit.

Now we let $\tau : M \longrightarrow M$ be any free involution of some smooth closed k-manifold M. Then $M \times [0,1]$ is a smooth $(k+1)$-manifold with boundary $M \times \{0,1\}$. Because τ is free, the smooth involution

$$M \times [0,1] \longrightarrow M \times [0,1] \ , \quad (x,t) \longmapsto (\tau x, 1-t)$$

is also free. So the quotient space

$$B \ = \ M \times [0,1]/(x,t) \sim (\tau x, 1-t)$$

inherits the structure of a smooth $(k+1)$-manifold, and the composite

$$M \xrightarrow{(-,0)} M \times [0,1] \xrightarrow{\text{proj}} B$$

is a diffeomorphism from M to the boundary of B.

The Euler characteristic module 2 in an example of a *Stiefel–Whitney number* in the sense of the following definition. We will show next that Stiefel–Whitney numbers provide more bordism invariants, and that these can be used to distinguish, for example, the bordism classes of $\mathbb{R}P^2 \times \mathbb{R}P^2$ and $\mathbb{R}P^4$.

Construction 8.28 (Stiefel–Whitney numbers)**.** Now we let M be a smooth closed k-manifold. We write

$$w_i(M) \; = \; w_i(\tau_M) \; \in \; H^i(M;\mathbb{F}_2)$$

for the i-th Stiefel–Whitney class of the tangent bundle of M. We write

$$[M] \; \in \; H_k(M;\mathbb{F}_2)$$

for the fundamental class of M.

We consider natural numbers $r_1, r_2 \dots, r_k \geq 0$ such that

$$r_1 + 2r_2 + \dots + kr_k \; = \; k \; .$$

The *Stiefel–Whitney number* of a smooth closed k-manifold M associated to $(r_1, \dots, r_k)$ is

$$(w_1(M)^{r_1} \cdot w_2(M)^{r_2} \cdot \dots \cdot w_k(M)^{r_k}) \cap [M] \; \in \; \mathbb{Z}/2 \; ,$$

the cap product of the characteristic class

$$w_1(M)^{r_1} \cdot w_2(M)^{r_2} \cdot \dots \cdot w_k(M)^{r_k} \; \in \; H^k(M;\mathbb{F}_2) \tag{8.29}$$

with the fundamental class of M.

If the k-manifold M is connected, then the top cohomology group $H^k(M;\mathbb{F}_2)$ is 1-dimensional, and then the Stiefel–Whitney number associated to $(r_1, \dots, r_k)$ is 1 if and only if the characteristic class (8.29) is non-zero.

Example 8.30. The sequence $(0, \dots, 0, 1)$ defines a Stiefel–Whitney number for smooth closed k-manifolds M. And in fact,

$$w_k(M) \cap [M] \; \equiv \; \chi(M) \mod 2 \; ,$$

the Euler characteristic modulo 2.

Proposition 8.31. *Then the Stiefel–Whitney number associated to any sequence of natural numbers is a bordism invariant of smooth closed k-manifolds.*

Proof. Homology and cohomology take finite disjoint unions to products, and the fundamental classes and characteristic classes of a disjoint union is the product of the fundamental and characteristic classes of the summands. So the Stiefel–Whitney number associated to $(r_1, \dots, r_k)$ is additive on disjoint unions. So it suffices to show that all Stiefel–Whitney numbers of boundaries of $(k+1)$-manifolds vanish.

We let B be a smooth compact $(k+1)$-manifold. The restriction of the tangent bundle τ_B to the boundary ∂B is isomorphic to the sum of $\tau_{\partial B}$ and a trivial line bundle. Stiefel–Whitney classes do not change upon addition of trivial bundles, so

$$w_i(\partial B) \; = \; w_i(\tau_B|_{\partial B}) \; = \; \iota^*(w_i(B))$$

in $H^i(\partial B;\mathbb{F}_2)$, where $\iota : \partial B \longrightarrow B$ is the inclusion. So

$$\begin{aligned} (w_1(\partial B)^{r_1} \cdot w_2(\partial B)^{r_2} \cdot \dots \cdot w_k(\partial B)^{r_k}) \cap [\partial B] \; &= \; \iota^*(w_1(B)^{r_1} \cdot w_2(B)^{r_2} \cdot \dots \cdot w_k(B)^{r_k}) \cap [\partial B] \\ &= \; (w_1(B)^{r_1} \cdot w_2(B)^{r_2} \cdot \dots \cdot w_k(B)^{r_k}) \cap \iota_*[\partial B] \; = \; 0 \; . \end{aligned}$$

The last relation uses that $\iota_*[\partial B] = 0$ in $H_k(B;\mathbb{F}_2)$ because ∂B is a boundary in B. $\square$

Thom [47, Corollaire IV.11] also showed a converse to the previous Proposition 8.31: two smooth closed k-manifolds all of whose Stiefel–Whitney numbers agree are already bordant.

Example 8.32 (Stiefel–Whitney classes of real projective spaces)**.** We revisit the Stiefel–Whitney numbers of the real projective spaces. To this end recall that tangent bundle of $\mathbb{R}P^k$ satisfies the relation

$$\tau_{\mathbb{R}P^k} \oplus \underline{\mathbb{R}} \ \cong\ \underbrace{\gamma_k \oplus \cdots \oplus \gamma_k}_{k+1} \ ,$$

see for example [34, Theorem 4.5]. Here γ_k is the tautological line bundle over $\mathbb{R}P^k$, with total space

$$\{(x, L) \ \in\ \mathbb{R}^{k+1} \times \mathbb{R}P^k \colon x \in L\} \ .$$

We also recall that the cohomology algebra of $\mathbb{R}P^k$ is given by

$$H^*(\mathbb{R}P^k; \mathbb{F}_2) \ =\ \mathbb{F}_2[a]/(a^{k+1}) \ ,$$

a truncated polynomial algebra generated by the non-zero class $a \in H^1(\mathbb{R}P^k; \mathbb{F}_2)$, which is also the first Stiefel–Whitney class of tautological line bundle, i.e.,

$$a \ =\ w_1(\gamma_k) \ .$$

We conclude that

$$w_{\text{total}}(\mathbb{R}P^k) \ =\ w_{\text{total}}(\tau_{\mathbb{R}P^k} \oplus \underline{\mathbb{R}}) \ =\ w_{\text{total}}(\gamma_k^{k+1}) \ =\ (w_{\text{total}}(\gamma_k))^{k+1} \ =\ (1 + a)^{k+1} \ .$$

Comparing the summand of degree i yields

$$w_i(\mathbb{R}P^k) \ =\ \binom{k+1}{i} \cdot a^i \ \in\ H^i(\mathbb{R}P^k; \mathbb{F}_2) \ .$$

For example,

$$\begin{aligned} w_{\text{total}}(\mathbb{R}P^1) \ &=\ 1 \ , \\ w_{\text{total}}(\mathbb{R}P^2) \ &=\ 1 + a + a^2 \ , \\ w_{\text{total}}(\mathbb{R}P^3) \ &=\ 1 \ , \\ w_{\text{total}}(\mathbb{R}P^4) \ &=\ 1 + a + a^4 \ . \end{aligned}$$

For *odd* numbers k, the projective space $\mathbb{R}P^k$ bounds a $(k+1)$-manifold, compare Example 8.27. Hence all Stiefel–Whitney numbers must vanish.

Let us use this calculate all Stiefel–Whitney numbers of $\mathbb{R}P^4$. The degree 4 Stiefel–Whitney numbers are indexed by the sequences $(4, 0, 0, 0)$, $(2, 1, 0, 0)$, $(0, 2, 0, 0)$, $(1, 0, 1, 0)$ and $(0, 0, 0, 1)$; or equivalently by the monomials in Stiefel–Whitney classes $w_1^4, w_1^2 w_2, w_2^2, w_1 w_3$ and w_4. Since $w_2(\mathbb{R}P^4)$ and $w_3(\mathbb{R}P^4)$ are trivial, the Stiefel–Whitney numbers of $\mathbb{R}P^4$ associated to the monomials $w_1^2 w_2, w_2^2$ and $w_1 w_3$ are zero. Since

$$w_1(\mathbb{R}P^4)^4 \ =\ w_4(\mathbb{R}P^4) \ =\ a^4 \ ,$$

the Stiefel–Whitney numbers of $\mathbb{R}P^4$ associated to w_1^4 and w_4 are 1.

Example 8.33. We can now show that the bordism classes of $\mathbb{R}P^2 \times \mathbb{R}P^2$ and $\mathbb{R}P^4$ are linearly independent in $\mathcal{N}_4$. We already identified the Stiefel–Whitney numbers of $\mathbb{R}P^4$ in the previous example. To calculate the Stiefel–Whitney numbers of $\mathbb{R}P^2 \times \mathbb{R}^2$, we recall that

$$H^*(\mathbb{R}P^2 \times \mathbb{R}P^2; \mathbb{F}_2) \ =\ \mathbb{F}_2[b, c]/(b^3, c^3)$$

by the Künneth theorem, where $b, c \in H^1(\mathbb{R}P^2 \times \mathbb{R}P^2; \mathbb{F}_2)$ are the restriction of the generator $a \in H^1(\mathbb{R}P^2; \mathbb{F}_2)$ along the projections to the two factors. The tangent bundle of a product is the product of the tangent bundles, and the total Stiefel–Whitney classes of an exterior product is the exterior product of total Stiefel–Whitney classes. So we obtain

$$w_{\text{total}}(\mathbb{R}P^2 \times \mathbb{R}P^2) \ =\ w_{\text{total}}(\tau_{\mathbb{R}P^2} \times \tau_{\mathbb{R}P^2}) \ =\ (1 + b + b^2) \cdot (1 + c + c^2) \ .$$

Comparing the summand of the same degree yields

$$\begin{aligned} w_1(\mathbb{R}P^2 \times \mathbb{R}P^2) &= b + c \\ w_2(\mathbb{R}P^2 \times \mathbb{R}P^2) &= b^2 + bc + c^2 \\ w_3(\mathbb{R}P^2 \times \mathbb{R}P^2) &= b^2c + bc^2 \\ w_4(\mathbb{R}P^2 \times \mathbb{R}P^2) &= b^2c^2 \ . \end{aligned}$$

This yields that the Stiefel–Whitney numbers associated to $w_1^4, w_1^2 w_2$ and $w_1 w_3$ are 0, and the Stiefel–Whitney numbers associated to w_2^2 and w_4 are 1. Comparing Stiefel–Whitney numbers shows that $\mathbb{R}P^2 \times \mathbb{R}P^2$ and $\mathbb{R}P^4$ are linearly independent in $\mathcal{N}_4$, as claimed.

Thom showed in [47, Théorème IV.12] that the ring $\mathcal{N}_*$ is a polynomial algebra over $\mathbb{F}_2$ on classes x_k for all $k \geq 1$ with $k \neq 2^j - 1$ for any $j \geq 0$; and he showed that for even k, the classes of the projective spaces $[\mathbb{R}P^k] \in \mathcal{N}_k$ can be taken as polynomial generators; in particular, these classes are algebraically independent in the bordism ring $\mathcal{N}_*$. Explicit manifolds that represent the odd dimensional polynomial generators were first constructed by Dold [11], and are therefore called the *Dold manifolds*.

Exercises

Exercise E.1 (Coordinatized orthogonal spectra)**.** A *coordinatized orthogonal spectrum* consists of the following data:

- a sequence of based spaces X_n for $n \geq 0$,
- a based continuous left $O(n)$-action on X_n for each $n \geq 0$,
- based maps $\sigma_n : S^1 \wedge X_n \longrightarrow X_{1+n}$ for $n \geq 0$.

This data is subject to the following condition: for all $m, n \geq 0$, the iterated structure map $S^m \wedge X_n \longrightarrow X_{m+n}$ defined as the composition

$$S^m \wedge X_n \xrightarrow{S^{m-1}\wedge\sigma_n} S^{m-1} \wedge X_{1+n} \xrightarrow{S^{m-2}\wedge\sigma_{1+n}} \cdots \xrightarrow{\sigma_{m-1+n}} X_{m+n}$$

is $(O(m)\times O(n))$-equivariant. Here the orthogonal group $O(m)$ acts on S^m as the one-point compactification of the tautological action on $\mathbb{R}^m$, and $O(m) \times O(n)$ acts on the target by restriction, along orthogonal sum, of the $O(m + n)$-action.

A *morphism* $f : X \longrightarrow Y$ of coordinatized orthogonal spectra consists of $O(n)$-equivariant based maps $f_n : X_n \longrightarrow Y_n$ for $n \geq 0$, which are compatible with the structure maps in the sense that $f_{n+1} \circ \sigma_n = \sigma_n \circ (S^1 \wedge f_n)$ for all $n \geq 0$.

Let X be a coordinatized orthogonal spectrum.

(a) Let W be an inner product space of dimension n. We define a based space

$$X^\flat(W) \ = \ (\mathbf{L}(\mathbb{R}^n, W)_+ \wedge X_n) / \sim \ ,$$

the quotient space of $\mathbf{L}(\mathbb{R}^n, W)_+ \wedge X_n$ by the equivalence relation

$$(\varphi \circ A) \wedge x \ \sim \ \varphi \wedge (A \cdot x)$$

for all linear isometries $\varphi : \mathbb{R}^n \cong W$ and all $A \in O(n)$ and $x \in X_n$. Show that $X^\flat(W)$ is homeomorphic to X_n.

(b) Let U be another inner product space of the same dimension as W. Show that the continuous map

$$\begin{aligned} \mathbf{L}(U, W)_+ \wedge \mathbf{L}(\mathbb{R}^n, U)_+ \wedge X_n &\longrightarrow \mathbf{L}(\mathbb{R}^n, W) \wedge X_n \\ \psi \wedge \varphi \wedge x &\longmapsto (\psi \circ \varphi) \wedge x \end{aligned}$$

factors through a continuous map

$$X^\flat(U, W) \ : \ \mathbf{L}(U, W)_+ \wedge X^\flat(U) \ \longrightarrow \ X^\flat(W) \ .$$

(c) Let V be another inner product space of dimension m, and let $\psi : \mathbb{R}^m \cong V$ be a linear isometry. Show that the continuous map

$$\begin{aligned} S^V \wedge \mathbf{L}(\mathbb{R}^n, W)_+ \wedge X_n &\longrightarrow \mathbf{L}(\mathbb{R}^{m+n}, V \oplus W)_+ \wedge X_{m+n} \\ v \wedge \varphi \wedge x &\longmapsto (\psi \oplus \varphi) \wedge \sigma^m(\psi^{-1}(v) \wedge x) \end{aligned}$$

factors through a continuous map

$$\sigma_{V,W} \ : \ S^V \wedge X^\flat(W) \longrightarrow X^\flat(V \oplus W)$$

that is independent of the choice of ψ.

(d) Show that there is a unique orthogonal spectrum $X^\flat$ whose values are the spaces $X^\flat(W)$ from (a), whose functoriality in isometries is as in (b), and whose structure maps are as in (c).

(e) Extend the assignment $X \mapsto X^\flat$ to functor

$$(-)^\flat \ : \ \mathcal{S}p^{\mathrm{coord}} \longrightarrow \mathcal{S}p \ .$$

(f) A *forgetful functor*

$$\mathcal{S}p \longrightarrow \mathcal{S}p^{\mathrm{coord}}$$

is defined on objects by $(UX)_n = X(\mathbb{R}^n)$, where $\mathbb{R}^n$ is endowed with the standard inner product $\langle x, y\rangle = \sum_{i=1}^n x_i y_i$. The structure map is

$$\sigma_n \ = \ \sigma_{\mathbb{R},\mathbb{R}^n} \ : \ S^1 \wedge X_n \ = \ S^{\mathbb{R}} \wedge X(\mathbb{R}^n) \longrightarrow X(\mathbb{R}^{1+n}) \ = \ X_{1+n} \ .$$

On morphisms, the forgetful functor evaluates at $\mathbb{R}^n$. Show that the forgetful functor is an equivalence of categories by exhibiting natural isomorphisms $U(X^\flat) \cong X$ and $Y \cong (UY)^\flat$.

Exercise E.2. Find a family $\{X^i\}_{i\in I}$ of orthogonal spectra for which the natural map

$$\pi_0\left(\prod\nolimits_{i\in I} X^i\right) \longrightarrow \prod\nolimits_{i\in I} \pi_0(X^i)$$

is not surjective.

Exercise E.3. We recall that the m-th stable homotopy group $\pi^{\mathrm{s}}_m(K)$ of a based space K is defined as the colimit of the sequence of abelian groups

$$\pi_m(K) \xrightarrow{S^1\wedge -} \pi_{1+m}(S^1 \wedge K) \xrightarrow{-\wedge S^1} \pi_{2+m}(S^2 \wedge K) \xrightarrow{S^1\wedge -} \cdots \ .$$

Smashing with the identity of S^1 from the right provides an isomorphism $-\wedge S^1 : \pi^{\mathrm{s}}_m(K) \longrightarrow \pi^{\mathrm{s}}_{m+1}(K\wedge S^1)$, a special case of the suspension isomorphism (see Proposition 1.13) for the suspension spectrum of K.

Show that the homotopy groups of an orthogonal spectrum X can also be calculated from the system of *stable* as opposed to *unstable* homotopy groups of the individual spaces X_n: exhibit $\pi_k(X)$ as a colimit of the sequence

$$\pi^{\mathrm{s}}_{k+n}(X_n) \xrightarrow{-\wedge S^1} \pi^{\mathrm{s}}_{k+n+1}(X_n \wedge S^1) \xrightarrow{(\sigma^{\mathrm{op}}_{\mathbb{R}^n,\mathbb{R}})_*} \pi^{\mathrm{s}}_{k+n}(X_{n+1}) \ .$$

Exercise E.4. Exhibit a left adjoint and a right adjoint to the shift functor $\mathrm{sh}^V : \mathcal{S}p \longrightarrow \mathcal{S}p$ for orthogonal spectra, introduced in Example 2.13

Exercise E.5. Define orthogonal ring spectra via $\iota_V : S^V \longrightarrow R(V)$ and multiplication maps $\mu_{V,W} : R(V) \wedge R(W) \longrightarrow R(V \oplus W)$, requiring associativity, unit and centrality.

Exercise E.6. Let M be a monoid and A a ring. The *monoid ring* $A[M]$ is the A-linearization of the underlying set of M, endowed with the multiplication by the A-bilinear extension of the multiplication of M:

$$(\sum\nolimits_i a_i \cdot m_i) \cdot (\sum\nolimits_j a'_j \cdot m'_j) \ = \ \sum\nolimits_{i,j} (a_i \cdot a'_j) \cdot (m_i \cdot m'_j) \ .$$

This construction extends degreewise from rings to graded rings.

Now we let R be an orthogonal ring spectrum, and we endow the M with the discrete topology. Exhibit an isomorphism of graded rings

$$\pi_*(R)[M] \ \cong \ \pi_*(RM) \ ,$$

where RM is the monoid ring spectrum introduced in Example 4.13.

Exercise E.7. Let $A = \{A_k\}_{k\in\mathbb{Z}}$ be a graded ring, and $m \geq 1$. We define the *graded matrix ring* $M_m(A)$ by taking $(m\times m)$-matrices degreewise: the abelian group $(M_m(A))_k$ is the group of $(m\times m)$-matrices with entries in A_k, and the multiplication maps $(M_m(A))_k \times (M_m(A))_l \longrightarrow (M_m(A))_{k+l}$ are defined by the usual matrix multiplication, using the multiplication of the graded ring A.

Let R be an orthogonal ring spectrum. Exhibit an isomorphism of graded rings

$$\pi_*(M_m(R)) \;\cong\; M_m(\pi_*(R)) \ ,$$

where $M_m(R)$ is the $(m\times m)$-matrix ring spectrum introduced in Example 4.15.

Exercise E.8. Let $A = \{A_k\}_{k\in\mathbb{Z}}$ be a graded ring. The *graded-opposite ring* A^{op} has the same underlying graded abelian group as A, but the multiplication in A^{op} is defined by

$$x \cdot_{\mathrm{op}} y \;=\; (-1)^{kl} \cdot y \cdot x \ ,$$

where $x \in A_k$, $y \in A_l$ and the right hand side is the multiplication in A.

Let R be an orthogonal ring spectrum. Show that $\pi_*(R^{\mathrm{op}}) = (\pi_*(R))^{\mathrm{op}}$.

Exercise E.9. We let $i : A \longrightarrow B$ and $p : X \longrightarrow Y$ be morphisms in a category $\mathcal{C}$. Then i has the *left lifting property* with respect to p if the following holds: for all morphisms $\alpha : A \longrightarrow X$ and $\beta : B \longrightarrow Y$ such that $p\alpha = \beta i$, there is a morphism $\lambda : B \longrightarrow X$ such that $\lambda i = \alpha$ and $p\lambda = \beta$:

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, "\beta"'] \arrow[ur, dotted, "\lambda"] & Y \end{tikzcd}$$

For a class $\mathcal{E}$ of $\mathcal{C}$-morphisms, we denote by $\mathcal{E}^\perp$ the class of all $\mathcal{C}$-morphisms that have the left lifting property with respect to all morphisms in $\mathcal{E}$. Show that the class $\mathcal{E}^\perp$ has the following closure properties:

(a) The class $\mathcal{E}^\perp$ is closed under composition: for all composable morphisms $i : A \longrightarrow B$ and $j : B \longrightarrow C$ in $\mathcal{E}^\perp$, the composite ji belongs to $\mathcal{E}^\perp$.

(b) The class $\mathcal{E}^\perp$ is closed under cobase change: for every pushout square in $\mathcal{C}$

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & C \arrow[d, "j"] \\ B \arrow[r] & D \end{tikzcd}$$

such that $i \in \mathcal{E}^\perp$, the morphism j belongs to $\mathcal{E}^\perp$.

(c) The class $\mathcal{E}^\perp$ is closed under retracts: for every commutative diagram in $\mathcal{C}$

$$\begin{tikzcd} C \arrow[r, "s"] \arrow[d, "j"'] & A \arrow[r, "r"] \arrow[d, "i"'] & C \arrow[d, "j"] \\ D \arrow[r, "t"'] & B \arrow[r, "u"'] & C \end{tikzcd}$$

such that $rs = \mathrm{Id}_C$ and $ut = \mathrm{Id}_D$, if i belongs to $\mathcal{E}^\perp$, then so does j.

(d) The class $\mathcal{E}^\perp$ is closed under sequential composition: for every sequence of morphisms in $\mathcal{E}^\perp$

$$A_0 \longrightarrow A_1 \longrightarrow A_2 \longrightarrow \dots$$

that has a colimit in $\mathcal{C}$, the canonical morphism $A_0 \longrightarrow \mathrm{colim}_n A_n$ belongs to $\mathcal{E}^\perp$.

Exercise E.10. Show that the following classes of cofibrations and weak equivalences define cofibration structures on the respective categories.

(a) The category of compactly generated spaces with respect to the h-cofibrations and the weak homotopy equivalences.

(b) The category of simplicial sets with respect to the monomorphisms (i.e., morphisms that are dimensionwise injective) and the weak equivalences.
(c) The category of $\mathbb{Z}$-graded chain complexes in an additive category $\mathcal{A}$, with respect to the chain maps that are dimensionwise split monomorphisms, and the chain homotopy equivalences.
(d) For a ring S, the category of $\mathbb{Z}$-graded chain complexes of S-modules with respect to the chain maps that are dimensionwise split monomorphisms, and the quasi-isomorphisms.
(e) For a ring S, the category of S-modules with respect to monomorphism of S-modules and the *op-stable equivalences*. Here a morphism $f : M \longrightarrow N$ of S-modules is an op-stable equivalence if there exists a morphism $g : N \longrightarrow M$ of S-modules such that the morphism $gf - \mathrm{Id}_M : M \longrightarrow M$ and $fg - \mathrm{Id}_N : N \longrightarrow N$ each factor through an injective S-module.

In each example, find a class of morphisms $\mathcal{E}$ such that the respective cofibrations are characterized by the left lifting property with respect to all morphisms in $\mathcal{E}$.

Exercise E.11. Let $\mathcal{A}$ be an additive category. We consider the cofibration structure on the category $\mathrm{Ch}(\mathcal{A})$ of $\mathbb{Z}$-graded chain complexes in $\mathcal{A}$ from Exercise E.10 (c).

(a) Show that two chains maps are homotopic in this cofibration structure if and only if they are chain homotopic.
(b) We define a category $\mathcal{K}(\mathcal{A})$ with objects all $\mathbb{Z}$-graded chain complexes in $\mathcal{A}$, and with morphisms the chain homotopy classes of chain maps. Show that the quotient functor from $\mathrm{Ch}(\mathcal{A}) \longrightarrow \mathcal{K}(\mathcal{A})$ is a localization at the class of chain homotopy equivalence.

Exercise E.12. Let E be an orthogonal Ω-*spectrum*, i.e., the adjoint $\tilde{\sigma}_n : E_n \longrightarrow \Omega(E_{1+n})$ of the structure map is a weak homotopy equivalence for every $n \geq 0$. Show that for every based CW-complex A (not necessarily finite), the map

$$[A, E_0] \longrightarrow \mathcal{SH}(\Sigma^\infty A, E)$$

that sends a continuous based map $A \longrightarrow E_0$ to the image of the adjoint $\Sigma^\infty A \longrightarrow E$ under the localization functor is an isomorphism of abelian groups. (Hint: induction over a CW-structure on A)

Exercise E.13. Let $\mathcal{T}$ be a triangulated category. We call a triangle (f, g, h) in $\mathcal{T}$ *anti-distinguished* if the triangle $(-f, -g, -h)$ is distinguished in the original triangulation of $\mathcal{T}$. Show that the class of anti-distinguished triangles is also a triangulation of $\mathcal{T}$ (with respect to the same suspension functor).

Exercise E.14. Let $\mathcal{T}$ be a triangulated category and $\Sigma^{-1} : \mathcal{T} \longrightarrow \mathcal{T}$ a quasi-inverse to the suspension functor, i.e., a functor endowed with a natural isomorphism $\psi_A : A \cong \Sigma(\Sigma^{-1}A)$. We call a triangle

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma^{-1}A$$

in the opposite category $\mathcal{T}^{\mathrm{op}}$ *op-distinguished* if the triangle

$$\Sigma^{-1}A \xrightarrow{h} C \xrightarrow{g} B \xrightarrow{\psi_A \circ f} \Sigma(\Sigma^{-1}A)$$

is distinguished in the original triangulation of $\mathcal{T}$. Show that the opposite category $\mathcal{T}^{\mathrm{op}}$ is a triangulated category with respect to the functor $\Sigma^{-1} : \mathcal{T}^{\mathrm{op}} \longrightarrow \mathcal{T}^{\mathrm{op}}$ as suspension functor and the class of op-distinguished triangles.

Exercise E.15. Let $\mathcal{T}$ be a triangulated category and $f_n : X_n \longrightarrow X_{n+1}$ a sequence of composable morphism for $n \geq 0$. Let $(\bar{X}, \varphi_n)$ and $(\bar{X}', \varphi'_n)$ be two homotopy colimits of the sequence (X_n, f_n). Construct an isomorphism $\psi : \bar{X} \longrightarrow \bar{X}'$ satisfying $\psi\varphi_n = \varphi'_n$ and commuting with the connecting morphisms to the suspension of $\bigoplus_{n\geq 0} X_n$. To what is extent it the isomorphism ψ unique?

Exercise E.16. Let $\mathcal{T}$ be a triangulated category with countable sums. Let X be any object of $\mathcal{T}$ and $e : X \longrightarrow X$ an idempotent endomorphism. Show that e splits in the following sense: there are objects eX and $(1-e)X$ and an isomorphism between X and the sum $eX \oplus (1-e)X$ under which $e : X \longrightarrow X$ corresponds to the endomorphism $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ of $eX \oplus (1-e)X$. (Hint: use that homotopy colimits exist in $\mathcal{T}$ and construct eX as the homotopy colimit of the sequence of e's).

Exercise E.17. Let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ be morphisms in a triangulated category $\mathcal{T}$ such that the composite $gf : X \longrightarrow Z$ zero and the group $[\Sigma X, Z]$ is trivial. Show that there is at most one morphism $h : Z \longrightarrow \Sigma X$ such that (f, g, h) is a distinguished triangle.

Exercise E.18. Let $\psi : X \longrightarrow Y$ be a morphism of orthogonal spectra that is levelwise a Serre fibration. Let $\iota : F \longrightarrow X$ denote the inclusion of the strict fiber of ψ.

(a) Show that the morphism $l : F \wedge S^1 \longrightarrow C\psi$ induced by the pullback square

$$\begin{tikzcd} F \arrow[r, "\iota"] \arrow[d] & X \arrow[d, "\psi"] \\ * \arrow[r] & Y \end{tikzcd}$$

on the vertical mapping cones is a stable equivalence.

(b) As before we let $i : Y \longrightarrow C\psi$ denote the inclusion into the mapping cone. Show the triangle

$$F \xrightarrow{\gamma(\iota)} X \xrightarrow{\gamma(\psi)} Y \xrightarrow{-\gamma(l)^{-1} \circ \gamma(i)} \Sigma F$$

is distinguished in the stable homotopy category.

Exercise E.19. Let X be any space. Show that the homomorphism

$$\mathcal{N}_k(X) \longrightarrow H_k(X; \mathbb{F}_2) , \quad [M, h] \longmapsto h_*[M]$$

that evaluates a singular manifold at the mod-2 fundamental class is an isomorphism for $k = 0$, and surjective for $k = 1$.

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