EQUIVARIANT K-THEORY AS A GLOBAL LOCALIZATION

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Abstract. We establish a global equivariant refinement of the theorem of Snaith that presents the complex K-theory spectrum as a localization of the suspension spectrum of $\mathbb{C}P^\infty$. We prove that the morphism of ultra-commutative global ring spectra from $\Sigma^\infty_+ B_{\mathrm{gl}}U(1)$ that classifies the tautological $U(1)$-representation exhibits the global K-theory spectrum as a specific localization, in both the ∞-categories of ultra-commutative ring spectra and of commutative global ring spectra. The classes to be inverted are certain representation-graded $U(n)$-equivariant homotopy classes of $\Sigma^\infty_+ B_{\mathrm{gl}}U(1)$, the *pre-Bott classes*. We also show that for every compact Lie group G, the G-equivariant K-theory spectrum is a telescopic localization of $\Sigma^\infty_+ B_G U(1)$ by the pre-Bott classes of all unitary G-representations.

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Introduction

A celebrated theorem of Snaith [43, 44] shows that complex K-theory can be obtained from the unreduced suspension spectrum of $\mathbb{C}P^\infty$ by inverting the Bott class. A contemporary formulation of this result is that a specific morphism of E_∞-ring spectra $\Sigma^\infty_+\mathbb{C}P^\infty \longrightarrow KU$ that classifies the tautological complex line bundle over $\mathbb{C}P^\infty$ is initial, in the higher categorical sense, among morphisms that take the Bott class in $\pi_2(\Sigma^\infty_+\mathbb{C}P^\infty)$ to an invertible homotopy class, see [31, Theorem 6.5.1]. Lurie reformulates this fact as the statement that complex K-theory classifies orientations of the multiplicative group in derived algebraic geometry, compare [29, Section 3.1] and [31, Corollary 4.3.27]. In this paper we prove a highly structured, globally-equivariant refinement of Snaith's

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localization result. Our protagonist is a specific morphism of ultra-commutative global ring spectra

$$\eta \ : \ \Sigma^\infty_+ \mathbf{P} \ \longrightarrow \ \mathbf{KU}$$

introduced in [38, Remark 6.4.20]. Here $\mathbf{P}$ is a multiplicative model of the global classifying space of $U(1)$, made from complex projective spaces, and $\Sigma^\infty_+\mathbf{P}$ is its unreduced global suspension spectrum. The target $\mathbf{KU}$ is Joachim's ultra-commutative ring spectrum model for global equivariant K-theory [23, Definition 4.3], see also [38, Construction 6.4.9].

In Construction 2.4 we introduce the *pre-Bott classes* b_n in $\pi^{U(n)}_{\nu_n}(\Sigma^\infty_+\mathbf{P})$, for $n \geq 1$, where ν_n is the tautological $U(n)$-representation on $\mathbb{C}^n$. The definition of these classes involves the global structure and power operations. The morphism $\eta\colon \Sigma^\infty_+\mathbf{P} \longrightarrow \mathbf{KU}$ takes the pre-Bott class b_n to the $U(n)$-equivariant Bott class of the representation ν_n, thence the name. We then extend these classes by naturality in group homomorphisms to *pre-Bott classes* $b_{G,V}$ in $\pi^G_V(\Sigma^\infty_+\mathbf{P})$, for all unitary G-representations V, see Construction 2.9. By Bott periodicity in equivariant K-theory, the morphism η in particular inverts all these pre-Bott classes. We show that the morphism $\eta\colon \Sigma^\infty_+\mathbf{P} \longrightarrow \mathbf{KU}$ is in fact a localization by the pre-Bott classes in several different senses: in the ∞-categories of ultra-commutative ring spectra and of commutative global ring spectra; for every compact Lie group G, in the ∞-category of commutative G-ring spectra; for every compact Lie group G, at the level of G-equivariant cohomology theories; and in the 'Real-global' context.

We expand on these localization results. The global formulation starts from the fact that $\eta\colon \Sigma^\infty_+\mathbf{P} \longrightarrow \mathbf{KU}$ inverts the pre-Bott class b_n for $n \geq 1$, see Theorem 4.4. We show in Theorem 4.8 that η is in fact initial with this property, even with two different levels of coherence in the multiplication:

Theorem A. *The morphism of ultra-commutative ring spectra $\eta\colon \Sigma^\infty_+\mathbf{P} \longrightarrow \mathbf{KU}$ is a localization by the pre-Bott classes b_n for $n \geq 1$, both in the ∞-category of ultra-commutative ring spectra, and in the ∞-category of commutative global ring spectra.*

In Theorem A, 'localization' is to be understood in the coherent and higher categorical sense, i.e., as an initial example of a morphism with source $\Sigma^\infty_+\mathbf{P}$ that sends all b_n to invertible classes, see Definitions 1.13 and 1.17. Readers so inclined may interpret Theorem A as the statement that global K-theory classifies ultra-commutative orientations of the global multiplicative group. The main ingredient for the proof of Theorem A is a characterization of the effect of $\eta\colon \Sigma^\infty_+\mathbf{P} \longrightarrow \mathbf{KU}$ on the geometric fixed points spectra as a localization, see Theorem 4.6.

We emphasize that Theorem A is a fully global result, encompassing all compact Lie groups; previous work on the subject [45, 47] had restricted attention to *abelian* compact Lie groups. Because $\mathbf{P}$ is a global classifying space of $U(1)$, the spectrum $\Sigma^\infty_+\mathbf{P}$ is *left* induced from the global family of abelian compact Lie groups; in contrast, $\mathbf{KU}$ is *right* induced from the global family of abelian compact Lie groups. Being left and right induced from some proper subclass of compact Lie groups are antithetical properties; so it might come as a surprise that inverting the pre-Bott classes mediates the gap between these.

In a forthcoming paper, we shall use the localization property of the morphism $\eta\colon \Sigma^\infty_+\mathbf{P} \longrightarrow \mathbf{KU}$ to promote the k-th power endomorphism of $\mathbf{P}$ to the *stable global Adams operation*, a morphism of ultra-commutative ring spectra

$$\Psi^k \ : \ \mathbf{KU} \ \longrightarrow \ \mathbf{KU}[t_k^{-1}]$$

that realizes the k-th Adams operation in equivariant K-theory. Here

$$t_k \ = \ \mathrm{tr}^{C_k}_{\{1\}}(1) \ \in \ \pi^{C_k}_0(\mathbf{KU})$$

is the transfer of the multiplicative unit to the cyclic group of order k, and $\mathbf{KU}[t_k^{-1}]$ denotes the ultra-commutative localization of $\mathbf{KU}$ by t_k.

In view of Theorem A, one might wonder about localizations that only invert some of the pre-Bott classes. The pre-Bott b_{n-1} is a factor of a restriction of b_n, see Theorem 2.10 (i). So inverting b_n also inverts all the previous pre-Bott classes $b_1, \dots, b_{n-1}$. We show in Theorem 6.1 that inverting b_{n-1} in the ∞-category of ultra-commutative ring spectra does *not* make b_n invertible; our witness is the group $SU(n)$. Hence the localization in the ∞-category of commutative global ring spectra does not make b_n invertible, either. So the localizations by the various pre-Bott classes are all distinct, and one needs to invert all b_n to obtain global K-theory from $\Sigma^\infty_+\mathbf{P}$ on all compact Lie groups.

On the family of *abelian* compact Lie groups, inverting only the first pre-Bott class b_1 already turns $\Sigma^\infty_+\mathbf{P}$ into global K-theory. However, inverting only b_1 leads to two different answers, depending on the level of coherence in the commutativity, i.e., global-commutative versus ultra-commutative:

- As we show in Theorem 6.2, the localization of $\Sigma^\infty_+\mathbf{P}$ by b_1 in the ∞-category of commutative global ring spectra is left induced from the global family $\mathcal{A}b$ of abelian compact Lie groups, and $\mathcal{A}b$-globally equivalent to $\mathbf{KU}$. In other words: the global localization by b_1 is equivalent to the result of first restricting $\mathbf{KU}$ to the global family $\mathcal{A}b$, and then left inducing to full global equivariance. The ultimate reason why inverting only b_1 already yields $\mathbf{KU}$ on abelian compact Lie groups is that inverting b_1 already inverts the pre-Bott classes of all 1-dimensional representations, and of all sums of such. For abelian compact Lie groups, this accounts for all unitary representations, so on this global family inverting b_1 makes all other pre-Bott classes invertible as well.
- As we show in Theorem 6.3, the localization of $\Sigma^\infty_+\mathbf{P}$ by b_1 in the ∞-category of ultra-commutative ring spectra is left induced from the larger global family $\mathcal{A}b^\circ$ of compact Lie groups whose identity component is abelian, and $\mathcal{A}b^\circ$-globally equivalent to $\mathbf{KU}$. In other words: the ultra-commutative localization by b_1 is equivalent to the result of first restricting $\mathbf{KU}$ to the global family $\mathcal{A}b^\circ$, and then left inducing to full global equivariance.

Clearly, the global family $\mathcal{A}b^\circ$ is substantially larger than $\mathcal{A}b$; for example, it also contains all finite groups.

In addition to our global results, we also prove localization properties for the morphism of genuine G-ring spectra underlying η, for every compact Lie group G. We write $\mathbf{P}_G = P(\mathcal{U}_G^{\mathbb{C}})$ for the G-space of complex lines in a complete complex G-universe $\mathcal{U}_G^{\mathbb{C}}$, which is also the underlying G-space of the global space $\mathbf{P}$, and a classifying space for G-equivariant complex line bundles. When restricting to a fixed compact Lie group, we can describe G-equivariant K-theory as an algebraic $RO(G)$-graded localization of G-equivariant $\Sigma^\infty_+\mathbf{P}_G$-cohomology by inverting the pre-Bott classes of all unitary G-representations. The key ingredient for this is Theorem 5.9, identifying $\mathbf{KU}_G$ as a telescopic localization by the set of these pre-Bott classes. The following result is a combination of Theorem 5.2 and Theorem 5.10.

Theorem B. *Let G be a compact Lie group.*

(i) *The morphism of commutative G-ring spectra $\eta_G\colon \Sigma^\infty_+\mathbf{P}_G \longrightarrow \mathbf{KU}_G$ is a localization by the pre-Bott classes $b_{G,V}$ for all unitary G-representations V, in the ∞-category of commutative G-ring spectra.*

(ii) *For every $\Sigma^\infty_+\mathbf{P}_G$-module M in $\mathcal{S}p_G^\wedge$, the morphism*

$$\pi_\star^G(\eta \wedge_{\Sigma^\infty_+\mathbf{P}_G} M) \;:\; \pi_\star^G(M) \;\longrightarrow\; \pi_\star^G(\mathbf{KU}_G \wedge_{\Sigma^\infty_+\mathbf{P}_G} M)$$

exhibits the target as a localization, in $RO(G)$-graded $\pi_\star^G(\Sigma^\infty_+\mathbf{P})$-modules, of the source by the set of pre-Bott classes of all unitary G-representations.

(iii) *The morphism of commutative $RO(G)$-graded rings*

$$\pi_\star^G(\eta) \;:\; \pi_\star^G(\Sigma^\infty_+\mathbf{P}) \;\longrightarrow\; \pi_\star^G(\mathbf{KU})$$

exhibits the target as an $RO(G)$-graded localization of the source by the set of pre-Bott classes of all unitary G-representations.

(iv) *For every finite based G-CW complex A, the morphism*

$$\eta_G^\star(A) \;:\; (\Sigma^\infty_+\mathbf{P})^\star_G(A) \;\longrightarrow\; \mathbf{KU}^\star_G(A)$$

exhibits the target as a localization, in $RO(G)$-graded $(\Sigma^\infty_+\mathbf{P})^\star_G$-modules, of the source by the set of pre-Bott classes of all unitary G-representations.

Theorem B should be contrasted with the incorrect claim in [47] that for abelian compact Lie groups G, the theory $\mathbf{KU}_G$ can be obtained as the telescopic localization of $\Sigma^\infty_+\mathbf{P}_G$ by inverting a single putative 'Bott class' in the integer-graded homotopy group $\pi_2^G(\Sigma^\infty_+\mathbf{P}_G)$. We explain in Proposition 5.16 that there does not exist any integer-graded class with this property. Instead, one needs to invert $RO(G)$-graded classes, and our pre-Bott classes serve the purpose. When the group G is finite, one can obtain $\mathbf{KU}_G$ by telescopically inverting a single pre-Bott class; any representation that contains all irreducibles does the job, for example the complex regular representation.

The global K-theory spectrum is underlying a 'Real-global' refinement $\mathbf{KR}$, which also knows about conjugate-linear phenomena, and represents Real-equivariant K-theory for compact Lie groups augmented to the Galois group of $\mathbb{C}$ over $\mathbb{R}$. In Section 7 we extend our localization results to this Real-global context. This might sound like another vast generalization, but the main work is already done in proving Theorem A, and then only one small extra step is needed. In Construction 7.3 we introduce the *Real-global pre-Bott class* $\tilde{b}_1$ in the group in $\pi^{\tilde{T}}_{\nu_1}(\Sigma^\infty_+\mathbf{P})$, where $\tilde{T} = T \rtimes C$ is the extended circle group, the semidirect product of $C = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ acting on $T = U(1)$ by complex conjugation. As the name suggests, the Real-global pre-Bott class restricts to the pre-Bott class b_1, and it maps to the Bott class for the tautological Real $\tilde{T}$-representation, see Proposition 7.4. As we explain in the proof of Theorem 7.5 (i), the key feature of the refined pre-Bott class $\tilde{b}_1$ is that inverting it kills the geometric fixed point spectra of all surjectively augmented Lie groups $G \longrightarrow C$. Consequently, as far as localization is concerned, all new 'Real' features are already taken care of by $\tilde{b}_1$, and the pre-Bott classes b_n for $n \geq 2$ do not have to be modified, or extended, in the 'Real' direction. The final result, to be proved as Theorem 7.5, is the following.

Theorem C. *The morphism of Real-global ultra-commutative ring spectra $\eta\colon \Sigma^\infty_+\mathbf{P} \longrightarrow \mathbf{KR}$ is a localization by the Real-global pre-Bott class $\tilde{b}_1$ and the pre-Bott classes b_n for $n \geq 2$, both in the ∞-category of ultra-commutative Real-global ring spectra, and in the ∞-category of commutative Real-global ring spectra.*

This paper includes several appendices. In Appendix A we collect the background material on the ∞-categories of genuine G-spectra that we need in the body of the paper; it does not contain any new mathematics. In Appendix B we review global spectra and global ring spectra form the higher categorical perspective. This appendix also contains two rectification results, and the bimonadity of ultra-commutative ring spectra. In more detail: we show in Theorem B.17 that for every orthogonal ring spectrum A, the 1-category of left A-modules in orthogonal spectra presents the stable ∞-category of modules over the global ring spectrum presented by A. Theorem B.20 is a symmetric monoidal refinement of this rectification: for every commutative orthogonal ring spectrum A, the symmetric monoidal 1-category of A-modules in orthogonal spectra presents the symmetric monoidal ∞-category of modules over the commutative global ring spectrum presented by A. And in Theorem B.28 we show that ∞-category of ultra-commutative ring spectra is both monadic and comonadic over the ∞-category of commutative global ring spectra.

Appendix C develops global stable homotopy theory relative to a *global class*, i.e., a non-empty class of compact Lie groups that is closed under isomorphism and passage to closed subgroups; the emphasis is on multiplicative and ultra-commutative features. Global classes are more general than global families in the sense of [38, Definition 1.4.1], in that they need not be closed under quotient groups. Two key results of this appendix are reflectivity statements for multiplicative structures. Theorem C.6 includes the statement that for every *multiplicative* global class $\mathcal{F}$, the $\mathcal{F}$-left induced commutative global ring spectra are coreflective inside all commutative global ring spectra. And Theorem C.10 shows, among other things, that for multiplicative global classes which are additionally closed under wreath products, the $\mathcal{F}$-left induced ultra-commutative ring spectra are coreflective inside all ultra-commutative ring spectra. In the relatively short Appendix D we generalize the construction and some properties of power operations and norms for ultra-commutative ring spectra to the representation-graded setting. Readers familiar with such operations on degree 0 classes as developed in [38, Section 5.1] will not find anything surprising in this appendix. Appendix E is a brief review on $RO(G)$-gradings for equivariant homotopy groups, following Lewis–Mandell [26, Appendix A] and Dugger [14].

History and related results. The first published version of Snaith's theorem [43, II Theorem 9.1.1] amounts to the equivalence of underlying infinite loop spaces obtained from the equivalence $(\Sigma^\infty_+ \mathbb{C}P^\infty)[\beta^{-1}] \sim KU$ of spectra. In [44, Theorem 2.12], Snaith states the result as a natural isomorphism of cohomology theories, with a different proof. A translation of Snaith's work into contemporary language as a localization of E_∞-ring spectra can be found in [31, Theorem 6.5.1]. This formulation says that a specific morphism of E_∞-ring spectra $\Sigma^\infty_+ \mathbb{C}P^\infty \longrightarrow KU$ that classifies the tautological complex line bundle over $\mathbb{C}P^\infty$ is initial, in the higher categorical sense, among morphisms that take the Bott class in $\pi_2(\Sigma^\infty_+ \mathbb{C}P^\infty)$ to an invertible homotopy class. Lurie interprets this fact as the statement that complex K-theory classifies orientations of the multiplicative group in derived algebraic geometry, compare [29, Section 3.1] and [31, Corollary 4.3.27], and he gives a proof of Snaith's theorem from this perspective in [31, Section 6.5]. This suggests interpreting our main result Theorem 4.8 as the statement that 'the global ring spectrum $\mathbf{KU}$ classifies ultra-commutative orientations of the global multiplicative group'.

Spitzweck–Østvær [46] and Gepner–Snaith [16] independently proved analogues of Snaith's theorem in the context of motivic homotopy theory. There are also two previous papers [45, 47] devoted to describing G-equivariant K-theory, for abelian compact Lie groups G, as the result of 'inverting the Bott class on $\Sigma^\infty_+ \mathbf{P}_G$'. In [45, Theorem 3.1], Snaith establishes an equivariant version of his non-equivariant theorem for *abelian* compact Lie groups. The result is different from our Theorem B, however, in that Snaith implicitly works with naive-equivariant (as opposed to genuine-equivariant) G-spectra. Indeed, the definition [45, (2.6)] of the cohomology theory $h_G(X)$ for G-spaces X is by taking equivariant maps into $\mathbf{P}_G$ while simultaneously stabilizing by spheres with trivial G-action and multiplication by the 'naive' equivariant Bott class, i.e., the non-equivariant Bott class inflated to G-spectra. In modern jargon, this amounts to restricting the genuine G-spectrum $\Sigma^\infty_+ \mathbf{P}_G$ to a trivial G-universe, and then naively inverting the Bott class. If we write $\{-,-\}_G^{\text{naive}}$ for the morphism group in the homotopy category of G-spectra indexed on a trivial universe, then Snaith's main result [45, Theorem 3.1] is a natural isomorphism

$$h_G(X) = \{\Sigma^\infty_+ X, \Sigma^\infty_+ \mathbf{P}_G\}_G^{\text{naive}}[\beta^{-1}] \ \cong \ K_G(X) \ ,$$

for finite G-CW-complexes X, and abelian compact Lie groups G, between a naive equivariant cohomology theory on the one side and a genuine equivariant cohomology theory on the other side.

In [47], Spitzweck and Østvær aim to describe the G-equivariant K-theory spectrum, again for abelian compact Lie groups, as a localization of the unreduced suspension spectrum of $\mathbf{P}_G$, by inverting a class in $\pi_2^G(\Sigma^\infty_+ \mathbf{P}_G)$ that they call the 'Bott class'. However, their main result [47, Theorem 1.1] is incorrect. As we explain in Proposition 5.16, already for a two-element group, there does not exist any G-equivariant stable homotopy class in integer degree 2 of $\mathbf{P}_G$ with this property.

Notation and conventions. We state all our main results in higher categorical terms in the ∞-categories of commutative or ultra-commutative global or G-equivariant ring spectra. However to get there, we use the 1-categorical models for unstable and stable global homotopy theory provided in [38]. In other words: we model global spaces by orthogonal spaces relative to the class of global equivalences; and we model global spectra by orthogonal spectra relative to the class of global equivalences. In fact, I would not know how to construct the morphism $\eta\colon \Sigma^\infty_+\mathbf{P} \longrightarrow \mathbf{KU}$ with the full coherence that it supports (i.e., as a morphism of Real-global ultra-commutative ring spectra) in model independent terms.

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1. Localizations of commutative global ring spectra

In this section we introduce localizations of commutative and ultra-commutative global ring spectra, see Definitions 1.13 and 1.17, respectively. In each case, a localization is an initial example of a morphism that inverts the given set of representation-graded equivariant homotopy classes. The *geometric fixed point criterion*, established in Theorem 1.16, is our main tool to recognize global localizations; as the name suggests, it provides a sufficient criterion at the level of geometric fixed point rings.

The pre-Bott classes do not live in integer gradings, but rather they are *representation-graded* equivariant homotopy classes. We briefly recall what we mean by this concept.

Definition 1.1 (Representation-graded homotopy groups)**.** We let G be a compact Lie group, E an orthogonal G-spectrum, and V an orthogonal G-representation. The V-graded G-equivariant homotopy group of E is

$$\pi_V^G(E) \;=\; \mathrm{colim}_{U\in s(\mathcal{U}_G)}[S^{U\oplus V}, E(U)]^G_* \ .$$

The colimit is taken over the poset $s(\mathcal{U}_G)$, under inclusion, of finite-dimensional G-subrepresentations of a complete G-universe $\mathcal{U}_G$, along stabilization maps much like in the integer graded case.

We emphasize that we grade by actual representations, as opposed to isomorphism classes of such, or elements in the representation ring. Invertibility in the representation-graded sense is nevertheless closely related to invertibility in the $RO(G)$-graded homotopy ring, see Remark E.9 for details. One can imagine grading more generally by pairs of representations that represent formal differences '$V - W$'. Since the pre-Bott classes live in 'honest' representation degrees, we will not need the additional generality in the present paper, and we stick to the simpler situation.

The group $\pi_V^G(E)$ is naturally isomorphic to $\pi_0^G(\Omega^V E)$, so the functor π_V^G takes $\underline{\pi}_*$-isomorphisms of orthogonal G-spectra to isomorphisms, see [38, Proposition 3.1.40]. So whenever convenient we can—and will—consider π_V^G as functor on the ∞-category $\mathcal{S}p_G$ of genuine G-spectra, or on the homotopy category of G-spectra $h(\mathcal{S}p_G)$. On the homotopy category, the functor π_V^G is represented by the suspension spectrum of S^V: a representation-graded homotopy class

$$e_{G,V} \;\in\; \pi_V^G(\Sigma^\infty S^V)$$

is represented by the identity of $S^V = (\Sigma^\infty S^V)(0)$; and the pair $(\Sigma^\infty S^V, e_{G,V})$ represents the functor $\pi_V^G \colon h(\mathcal{S}p_G) \longrightarrow \mathcal{A}b$.

If H is a closed subgroup of G, E an orthogonal G-spectrum, and V a G-representation, then the restriction and transfer homomorphisms easily generalize from integer-graded to representation-graded homotopy groups to homomorphisms

$$\mathrm{res}^G_H \ : \ \pi_V^G(E) \longrightarrow \pi_V^H(E) \qquad \text{and} \qquad \mathrm{tr}^G_H \ : \ \pi_V^H(E) \longrightarrow \pi_V^G(E) \ ;$$

here π_V^H refers to the homotopy group graded by the underlying H-representation of V. Indeed, we could simply define these representation-graded restriction and transfer operations as the degree 0 restrictions and transfers for the orthogonal G-spectrum $\Omega^V E$.

We let R be a *homotopy G-ring spectrum*, i.e., an algebra in the symmetric monoidal stable G-homotopy category $h(\mathcal{S}p_G^\wedge)$. Since the suspension G-spectrum $\Sigma^\infty S^V$ represents the functor π_V^G, a representation-graded homotopy class $v \in \pi_V^G(R)$ yields a unique morphism $v^\circ \colon \Sigma^\infty S^V \longrightarrow R$ in $h(\mathcal{S}p_G)$ such that $v^\circ_*(e_{G,V}) = v$. We denote by $v^\flat \colon R \wedge S^V \longrightarrow R$ the composite

$$R \wedge S^V \cong R \otimes (\Sigma^\infty S^V) \xrightarrow{R \otimes v^\circ} R \otimes R \xrightarrow{\mu} R \tag{1.2}$$

in $h(\mathcal{S}p_G)$, where $\mu \colon R \otimes R \longrightarrow R$ is the multiplication morphism of a homotopy G-ring spectrum R.

Definition 1.3 (Invertibility)**.** Let G be a compact Lie group, let R be a homotopy G-ring spectrum, and let V be a G-representation. A representation-graded homotopy class $v \in \pi_V^G(R)$ is *invertible* if the morphism $v^\flat \colon R \wedge S^V \longrightarrow R$ is an equivalence of G-spectra.

Invertibility in the sense of Definition 1.3 is essentially the same as invertibility in the $RO(G)$-graded homotopy ring $\pi_\star^G(R)$. The caveat is that while the representation-graded homotopy group $\pi_V^G(R)$ is isomorphic to the $RO(G)$-graded homotopy group $\pi_{[V]}^G(R)$, there is no preferred or 'canonical' way to identify them. We refer to Remark E.9 for details.

The next proposition characterizes invertibility of representation-graded homotopy classes in terms of geometric fixed points. We recall the geometric fixed point group $\Phi_k^G(X)$ of an orthogonal G-spectrum in (A.11). A biadditive pairing

$$\times \ : \ \Phi_k^G(X) \times \Phi_l^G(Y) \longrightarrow \Phi_{k+l}^G(X \wedge Y)$$

is defined in (A.12), providing a lax symmetric monoidal structure on the $\mathbb{Z}$-graded geometric fixed point groups $\Phi_*^G(X)$. Since the functor Φ_*^G inverts $\underline{\pi}_*$-isomorphisms of orthogonal G-spectra, it descends to a lax symmetric monoidal functor $\bar{\Phi}_*^G \colon \mathcal{S}p_G^\wedge \longrightarrow \mathcal{G}r\mathcal{A}b^\otimes$ on the ∞-category of G-spectra; see Construction A.13 for more details. For homotopy G-ring spectra, the geometric fixed point groups inherit a graded multiplication, compare (B.15).

Now we let V be a G-representation. We write $k = \dim(V^G)$ for the dimension of the G-fixed subspace of V. A choice of orientation of V^G determines a homotopy class of linear isometry between $\mathbb{R}^k$ and V^G, and thus a *geometric fixed point homomorphism*:

$$\begin{aligned} \Phi^G \ : \ \pi_V^G(X) \quad &\longrightarrow \quad \Phi_k^G(X) \\ [f \colon S^{U \oplus V} \longrightarrow X(U)] \ &\longmapsto \ [f^G \colon S^{U^G \oplus \mathbb{R}^k} \cong S^{(U \oplus V)^G} \longrightarrow X(U)^G] \end{aligned} \tag{1.4}$$

For $V = \mathbb{R}^k$ with trivial G-action this fixed point homomorphism specializes to the one defined in (A.14). The geometric fixed point homomorphism (1.4) is additive and multiplicative, provided we orient $(V \oplus W)^G$ by concatenating the chosen orientations of V^G and W^G. Choosing the opposite orientation changes this isomorphism into its negative, which does not influence whether or not a class is invertible.

Proposition 1.5. *Let V be a representation of a compact Lie group G, and let R be a commutative homotopy G-ring spectrum. Then for a representation-graded homotopy class $v \in \pi_V^G(R)$, the following conditions are equivalent.*

(a) *The class v is invertible.*
(b) *There exists an element $u \in \pi_0^G(R \wedge S^V)$ such that $\pi_0^G(v^\flat)(u) = 1$ in $\pi_0^G(R)$.*
(c) *For every closed subgroup H of G, the class $\Phi^H(v)$ is a homogeneous graded unit in the geometric fixed point ring $\Phi_*^H(R)$.*

Proof. (a)$\Longrightarrow$(b) Since the morphism $v^\flat \colon R \wedge S^V \longrightarrow R$ is an equivalence of G-spectra, the induced map of homotopy groups

$$\pi_0^G(v^\flat) \ : \ \pi_0^G(R \wedge S^V) \ \longrightarrow \ \pi_0^G(R)$$

is bijective. In particular, the class $1 \in \pi_0^G(R)$ has a preimage.

(b)$\Longrightarrow$(c) We let H be a closed subgroup of G. Applying the geometric fixed point homomorphism $\Phi^H \colon \pi_0^G(R) \longrightarrow \Phi_0^H(R)$ to the relation $\pi_0^G(v^\flat)(u) = 1$ yields

$$\Phi_0^H(v^\flat)(\Phi^H(u)) \ = \ \Phi^H(\pi_0^G(v^\flat)(u)) \ = \ \Phi^H(1) \ = \ 1 \tag{1.6}$$

in $\Phi_0^H(R)$. A choice of orientation of V^H induces a well-defined homotopy class of equivalence $S^l \sim S^{V^H}$. This, in turn, induces isomorphisms

$$\Phi_{m-l}^H(R) \ \cong \ \Phi_m^H(R \wedge S^l) \ \cong \ \Phi_m^H(R \wedge S^{V^H}) \ \cong \ \Phi_m^H(R \wedge S^V) \ . \tag{1.7}$$

Moreover, composition of this isomorphism with the map $\Phi_m^H(v^\flat) \colon \Phi_m^H(R \wedge S^V) \longrightarrow \Phi_m^H(R)$ is multiplication by the element $\Phi^H(v)$.

For $m = 0$, we use this isomorphism to turn the class $\Phi^H(u) \in \Phi_0^H(R \wedge S^V)$ into a class $\bar{u} \in \Phi_{-l}^H(R)$. The relation (1.6) then becomes the relation $\Phi^H(v) \cdot \bar{u} = 1$. So the class $\Phi^H(v)$ has a right inverse. Since the homotopy G-ring spectrum R is commutative, the graded ring $\Phi_*^H(R)$ is graded-commutative, so $\Phi^H(v)$ also has a left inverse, and is thus a graded unit.

(c)$\Longrightarrow$(a) For every closed subgroup H of G and choice of orientation of V^H, the composite of the isomorphism (1.7) with $\Phi_m^H(v^\flat) \colon \Phi_m^H(R \wedge S^V) \longrightarrow \Phi_m^H(R)$ is multiplication by the class $\Phi^H(v)$. Because $\Phi^H(v)$ is a graded unit, multiplication by $\Phi^H(v)$ is bijective, hence so is the map $\Phi_m^H(v^\flat)$. So the morphism $v^\flat$ induces isomorphisms of H-geometric fixed point groups for all closed subgroups H of G. Hence $v^\flat$ is an equivalence of G-spectra, which means that v is invertible. $\square$

The next proposition establishes some expected properties of invertible classes. For an orthogonal G-ring spectrum R, we can construct multiplication maps on representation-graded homotopy groups

$$\cdot \ : \ \pi_V^G(R) \times \pi_W^G(R) \ \longrightarrow \ \pi_{V \oplus W}^G(R) \tag{1.8}$$

by multiplying representatives. More precisely, if $f \colon S^{U \oplus V} \longrightarrow R(U)$ and $g \colon S^{U' \oplus W} \longrightarrow R(U')$ are G-maps that represent classes $x = [f] \in \pi_V^G(R)$ and $y = [g] \in \pi_W^G(R)$, respectively, then $x \cdot y$ is the class represented by the following composite:

$$S^{U \oplus U' \oplus V \oplus W} \xrightarrow{(U \oplus \tau_{U',V} \oplus W)_*} S^{U \oplus V \oplus U' \oplus W} \xrightarrow{f \wedge g} R(U) \wedge R(U') \xrightarrow{\mu_{U,U'}} R(U \oplus U')$$

Here $\mu_{U,U'}$ is a component of the multiplication morphism, in 'external' form.

Proposition 1.9. *Let V be a representation of a compact Lie group G, let R be a commutative homotopy G-ring spectrum, and let $v \in \pi_V^G(R)$ be a representation-graded homotopy class.*

(i) *If the class v is invertible, then for every morphism $i \colon R \longrightarrow S$ of homotopy G-ring spectra, the class $\pi_V^G(i)(v) \in \pi_V^G(S)$ is invertible.*
(ii) *If the class v is invertible, and H is a closed subgroup of G, then the class $\mathrm{res}_H^G(v) \in \pi_V^H(R)$ is invertible.*

(iii) *Let $w \in \pi_W^G(R)$ be another representation-graded homotopy class. Then the product $v \cdot w$ is invertible if and only if both v and w are invertible.*

Proof. (i) For every closed subgroup H of G, the morphism i induces a homomorphism of graded rings $\Phi^H_*(i)\colon \Phi^H_*(R) \longrightarrow \Phi^H_*(S)$. Since $\Phi^H(v)$ is a graded unit by Proposition 1.5, the class

$$\Phi^H_*(i)(\Phi^H(v)) \ = \ \Phi^H(\pi_V^G(i)(v))$$

is a graded unit in the graded ring $\Phi^H_*(S)$. Since this holds for all closed subgroups H of G, Proposition 1.5 shows that the class $\pi_V^G(i)(v)$ is invertible.

(ii) The morphism $\mathrm{res}^G_H(v)^\flat\colon R \wedge S^V \longrightarrow R$ is obtained from $v^\flat\colon R \wedge S^V \longrightarrow R$ by applying the restriction functor $\mathrm{res}^G_H\colon \mathcal{S}p_G \longrightarrow \mathcal{S}p_H$. So if $v^\flat$ is an equivalence of G-spectra, then $\mathrm{res}^G_H(v)^\flat$ is an equivalence of H-spectra.

(iii) We let H be a closed subgroup of G. We choose orientations of V^H and W^H, and then orient $(V \oplus W)^H$ by concatenating the chosen orientations. Then

$$\Phi^H(v \cdot w) \ = \ \Phi^H(v) \cdot \Phi^H(w)$$

in $\Phi^H_{k+l}(R)$, where $k = \dim(V^H)$ and $l = \dim(W^H)$. So $\Phi^H(v \cdot w)$ is invertible in the graded ring $\Phi^H_*(R)$ if and only if $\Phi^H(v)$ and $\Phi^H(w)$ are invertible. Since this holds for all closed subgroups H of G, Proposition 1.5 shows that $v \cdot w$ is invertible if and only if both v and w are invertible. □

Construction 1.10 (Restriction homomorphisms)**.** We let X be an orthogonal spectrum, and we let $\gamma\colon K \longrightarrow G$ be a continuous homomorphism of compact Lie groups. We define the *restriction homomorphism*

$$\gamma^* \ : \ \pi_V^G(X) \ \longrightarrow \ \pi^K_{\gamma^*(V)}(X) \tag{1.11}$$

by sending the class of a G-map $f\colon S^{U \oplus V} \longrightarrow X(U)$ to the class of the K-map

$$S^{\gamma^*(U)\oplus\gamma^*(V)} \ = \ \gamma^*(S^{U\oplus V}) \ \longrightarrow \ \gamma^*(X(U)) \ = \ X(\gamma^*(U)) \ .$$

As in the special case $V = 0$ in [38, Construction 3.1.15], we exploit here that elements of $\pi^K_{\gamma^*(V)}(X)$ can be unambiguously represented by K-maps $S^{W\oplus\gamma^*(V)} \longrightarrow X(W)$ for any K-representation W, not necessarily a subrepresentation of the chosen complete universe, see [38, Construction 3.1.13]. The restriction homomorphisms γ^* are clearly contravariantly functorial for composition of group homomorphisms. When γ is the inclusion of a subgroup $H \leq G$, then γ^* specializes to the restriction homomorphism res^G_H considered above. The restriction homomorphism γ^* is a natural transformation between two functors from orthogonal spectra to abelian groups that both invert global equivalences. So γ^* descends uniquely to a natural transformation between the descended functors from the ∞-category of global spectra to abelian groups; we abuse notation and denote the descended transformation by the same symbol.

By a *global homotopy ring spectrum* we mean an algebra in the symmetric monoidal global stable homotopy category $h(\mathcal{S}p^\wedge_{\mathrm{gl}})$. The strong symmetric monoidal functor $\mathrm{res}^\wedge_G\colon \mathcal{S}p^\wedge_{\mathrm{gl}} \longrightarrow \mathcal{S}p^\wedge_G$ from (B.4) that extracts the G-homotopy type of a global spectrum induces a strong symmetric monoidal functor of homotopy categories $h(\mathrm{res}^\wedge_G)\colon h(\mathcal{S}p^\wedge_{\mathrm{gl}}) \longrightarrow h(\mathcal{S}p^\wedge_G)$. The functor $\mathrm{Alg}(h(\mathrm{res}^\wedge_G))$ thus takes global homotopy ring spectra to homotopy G-ring spectra. We define the representation-graded homotopy groups and the notion of 'invertibility' for global homotopy ring spectra on the underlying homotopy G-ring spectra. We shall now show that invertible classes are then stable under restriction along continuous homomorphisms of compact Lie groups.

Proposition 1.12. *Let $\gamma\colon K \longrightarrow G$ be a continuous homomorphism of compact Lie groups. Let R be a commutative global homotopy ring spectrum, and let $v \in \pi_V^G(R)$ be an invertible representation-graded homotopy class. Then the class $\gamma^*(v)$ in $\pi^K_{\gamma^*(V)}(R)$ is invertible.*

Proof. We let L be a closed subgroup of K, and we set $H = \gamma(L)$, a closed subgroup of G. The restriction of γ to L factors as a continuous epimorphism $\beta\colon L \longrightarrow H$, followed by the inclusion of H into G. Since v is invertible, so is $\mathrm{res}^G_H(v)$ by Proposition 1.9 (ii). A choice of orientation $\mathbb{R}^l \cong V^H = (\gamma^*(V))^L$ yields a commutative square of inflation and geometric fixed point homomorphisms:

$$\begin{tikzcd}
\pi^H_V(R) \arrow[r, "\beta^*"] \arrow[d, "\Phi^H"'] & \pi^L_{\gamma^*(V)}(R) \arrow[d, "\Phi^L"] \\
\Phi^H_l(R) \arrow[r, "\beta^*"'] & \Phi^L_l(R)
\end{tikzcd}$$

The lower horizontal morphism is defined in (B.8), and the square commutes at the level of representatives by direct inspection.

Since v is invertible, the class $\Phi^H(v)$ is a graded unit in the ring $\Phi^H_*(R)$. Since the inflation homomorphism $\beta^*\colon \Phi^H_*(R) \longrightarrow \Phi^L_*(R)$ is a morphism of graded rings, the class

$$\beta^*(\Phi^H(v)) \ = \ \Phi^L(\beta^*(\mathrm{res}^G_H(v))) \ = \ \Phi^L(\gamma^*(v))$$

is invertible in the graded ring $\Phi^L_*(R)$. Since this holds for all closed subgroups of K, the class $\gamma^*(v)$ is invertible by Proposition 1.5. □

By a *commutative global ring spectrum* we mean a commutative algebra in the symmetric monoidal ∞-category $\mathcal{S}p^\wedge_{\mathrm{gl}}$ of global spectra, see Definition B.25 for more details. As we recall in Construction B.3, every global spectrum has a underlying genuine G-spectrum, and this process is implemented by a strong symmetric monoidal functor $\mathrm{res}^\wedge_G\colon \mathcal{S}p^\wedge_{\mathrm{gl}} \longrightarrow \mathcal{S}p^\wedge_G$ whose underlying functor $\mathrm{res}_G\colon \mathcal{S}p_{\mathrm{gl}} \longrightarrow \mathcal{S}p_G$ has adjoints on either side, and thus preserves limits and colimits, see Theorem B.6. We obtain an induced functor of commutative algebras

$$\mathrm{CAlg}(\mathrm{res}^\wedge_G) \ : \ \mathrm{CAlg}(\mathcal{S}p^\wedge_{\mathrm{gl}}) \ \longrightarrow \ \mathrm{CAlg}(\mathcal{S}p^\wedge_G)$$

from commutative global ring spectra to commutative G-ring spectra. We define the representation-graded homotopy groups and the pairing (1.8) as the one for the underlying G-spectrum.

In the following a *set of representation-graded equivariant homotopy classes* of a global spectrum will mean a set $\mathcal{X} = \{v\}$ of classes in $\pi^G_V(R)$ for varying pairs (G, V) consisting of a compact Lie group G and a G-representation V. We will say that the class $v \in \pi^G_V(R)$ is *supported* at the group G, or at the pair (G, V). The main example for this paper will be the set $\{b_n\}_{n\geq 1}$ of *pre-Bott classes* of the ultra-commutative ring spectrum $\Sigma^\infty_+\mathbf{P}$ defined in Construction 2.4 below; the class b_n is supported at the unitary group $U(n)$ and its tautological representation ν_n on $\mathbb{C}^n$.

Definition 1.13 (Global localization)**.** Let R be a commutative global ring spectrum. Let $\mathcal{X}$ be a set of representation-graded equivariant homotopy classes of R.

(i) A morphism $i\colon R \longrightarrow S$ of commutative global ring spectra *inverts* $\mathcal{X}$ if for every element $v \in \mathcal{X}$ supported at (G, V), the class $i_*(v) \in \pi^G_V(S)$ is invertible in the sense of Definition 1.3.

(ii) A *global localization by* $\mathcal{X}$ is an initial object of the full subcategory of the slice category $R\backslash \mathrm{CAlg}(\mathcal{S}p^\wedge_{\mathrm{gl}})$ spanned by the $\mathcal{X}$-inverting morphisms.

Remark 1.14. The kind of localizations of ring spectra we study first arose in non-equivariant homotopy theory. A special feature in that context is that the localization of R by a homotopy class $u \in \pi_n(R)$, or more precisely its underlying spectrum, can always be calculated as the colimit, classically modeled by a mapping telescope, of the sequence

$$R \ \xrightarrow{\cdot u} \ R \wedge S^{-n} \ \xrightarrow{\cdot u} \ R \wedge S^{-2n} \xrightarrow{\cdot u} \ \dots$$

of iterations of the graded endomorphism of R that lifts multiplication by u, compare [31, Proposition 4.3.17]. Consequently, the topological localization always realizes the algebraic localization on graded homotopy rings.

We alert the reader that global localizations are *essentially never* given by a sequential colimit over endomorphisms that resemble 'multiplication by the elements' to be inverted. The issue is already present for localization at class in the equivariant homotopy rings of degree 0. Much as in the G-equivariant context, the functor $\pi_0^G\colon \mathcal{S}p_{\mathrm{gl}} \longrightarrow \mathcal{A}b$ is also represented by a global spectrum. However, a key difference is that the representing object $\Sigma^\infty_+ B_{\mathrm{gl}}G$, the unreduced suspension spectrum of the global classifying space of G, is *not* invertible in the global stable homotopy category (unless the group is trivial). In fact, the Picard group of the global stable homotopy category is isomorphic to $\mathbb{Z}$, generated by the suspension of the global sphere spectrum, see [38, Theorem 4.5.5]. In other words: the only invertible global spectra are the 'trivial spheres'. So if the class we wish to invert is supported at a nontrivial group, there is not even any sequence of 'graded endomorphisms' one might wish to take the colimit of.

The situation changes as soon as we fix a compact Lie group G and pass to the underlying G-spectra. As we show in Theorem 5.9, the spectrum $\mathbf{KU}_G$ is a 'Bott-inverted telescopic localization' of $\Sigma^\infty_+\mathbf{P}$, i.e., a sequential colimit of copies of $\Sigma^\infty_+\mathbf{P}$ shifted by the spheres of an exhaustive sequence of unitary representations, along morphisms that represent multiplication by pre-Bott classes.

The following easy characterization of initial objects will be used several times to recognize certain morphisms as localizations.

Lemma 1.15. *For an object I of an ∞-category $\mathcal{D}$, the following two conditions are equivalent.*

(a) *The object I is initial in $\mathcal{D}$.*

(b) *For every $\mathcal{D}$-object F, there is coproduct presentation $(j\colon F \longrightarrow Z, g\colon I \longrightarrow Z)$ of F and I in $\mathcal{D}$ in which $j\colon F \xrightarrow{\sim} Z$ is an equivalence.*

Proof. If I is initial and F any $\mathcal{D}$-object, we let $g\colon I \longrightarrow F$ be any morphism. For every $\mathcal{D}$-object X, the anima $\mathcal{D}(I,X)$ is contractible, because I is initial. So the map

$$(\mathcal{D}(\mathrm{Id}_F, X), \mathcal{D}(g,X)) \ : \ \mathcal{D}(F,X) \ \longrightarrow \ \mathcal{D}(F,X) \times \mathcal{D}(I,X)$$

is an equivalence of animae, showing that (Id_F, g) is the desired coproduct presentation of condition (b).

Now we assume condition (b). Given a $\mathcal{D}$-object F, we choose a coproduct presentation $(j\colon F \xrightarrow{\sim} Z, g\colon I \longrightarrow Z)$ of F and I with j an equivalence. Then the map

$$(\mathcal{D}(j,F), \mathcal{D}(g,F)) \ : \ \mathcal{D}(Z,F) \ \longrightarrow \ \mathcal{D}(F,F) \times \mathcal{D}(I,F)$$

is an equivalence of animae. Since the first component $\mathcal{D}(j,F)$ is an equivalence between non-empty animae, the anima $\mathcal{D}(I,F)$ must be contractible. So I is indeed an initial object of $\mathcal{D}$. $\square$

The following theorem is our main tool for recognizing global localizations. The composite functor

$$\mathcal{S}p_{\mathrm{gl}}^\wedge \ \xrightarrow{\mathrm{res}_K^\wedge} \ \mathcal{S}p_K^\wedge \ \xrightarrow{\Phi^K} \ \mathcal{S}p^\wedge$$

is a strong symmetric monoidal left adjoint. To simplify the exposition we abusively abbreviate the induced functor on categories of commutative algebras to

$$\Phi^K \ = \ \mathrm{CAlg}(\Phi^K \circ \mathrm{res}_K^\wedge) \ : \ \mathrm{CAlg}(\mathcal{S}p_{\mathrm{gl}}^\wedge) \ \longrightarrow \ \mathrm{CAlg}(\mathcal{S}p^\wedge) \ .$$

In the proof of the following theorem, and in many arguments that follow, we shall systematically blur the distinction between two different, but naturally isomorphic, geometric fixed point rings associated with a homotopy K-ring spectrum R. On the one hand, 'geometric fixed point ring' could refer to the homotopy groups of the geometric fixed point ring spectrum $\Phi^K(R)$ in the ∞-category $\mathrm{CAlg}(\mathcal{S}p^\wedge)$. On the other hand, it could refer to the explicit point-set definition from [38, Definition 3.3.1], which we recall in (A.11) below, descended from the 1-category of orthogonal spectra to the ∞-category of spectra. These two interpretations are naturally and multiplicatively isomorphic, by the isomorphism (B.14) that is characterized by its effect on inflated K-spectra. So whenever we pretend that the rings $\pi_*(\Phi^K(R))$ and $\Phi^K_*(R)$ were equal, we are implicitly applying the preferred natural isomorphism of Corollary B.16.

Theorem 1.16 (Geometric fixed point criterion)**.** *Let $i\colon R \longrightarrow S$ be a morphism of commutative global ring spectra, and let $\mathcal{X}$ be a set of representation-graded equivariant homotopy classes of R. Suppose that for every compact Lie group K, the morphism of K-geometric fixed point rings $\Phi^K_*(i)\colon \Phi^K_*(R) \longrightarrow \Phi^K_*(S)$ is a localization by the set of classes $\Phi^K(\beta^*(x))$ for all compact Lie groups G, all continuous homomorphisms $\beta\colon K \longrightarrow G$, and all classes $x \in \mathcal{X}$ supported at G. Then the morphism $i\colon R \longrightarrow S$ is a global localization by $\mathcal{X}$ in* $\mathrm{CAlg}(\mathcal{S}p^\wedge_{\mathrm{gl}})$.

Proof. We let $f\colon R \longrightarrow P$ be any morphism of commutative global ring spectra that inverts $\mathcal{X}$. We choose a pushout in $\mathrm{CAlg}(\mathcal{S}p^\wedge_{\mathrm{gl}})$:

$$\begin{tikzcd} R \arrow[r, "i"] \arrow[d, "f"'] & S \arrow[d, "g"] \\ P \arrow[r, "j"'] & Q \end{tikzcd}$$

We will argue that the morphism $j\colon P \longrightarrow Q$ is a global equivalence. To this end we let K be any compact Lie group. The geometric fixed point functor $\Phi^K\colon \mathrm{CAlg}(\mathcal{S}p^\wedge_{\mathrm{gl}}) \longrightarrow \mathrm{CAlg}(\mathcal{S}p^\wedge)$ preserves colimits, so the square

$$\begin{tikzcd} \Phi^K(R) \arrow[r, "\Phi^K(i)"] \arrow[d, "\Phi^K(f)"'] & \Phi^K(S) \arrow[d, "\Phi^K(g)"] \\ \Phi^K(P) \arrow[r, "\Phi^K(j)"'] & \Phi^K(Q) \end{tikzcd}$$

is a pushout of commutative ring spectra. By hypothesis, the morphism $\Phi^K(i)\colon \Phi^K(R) \longrightarrow \Phi^K(S)$ is a localization by the set of classes $\{\Phi^K(\beta^*(x))\}$ for all continuous homomorphisms $\beta\colon K \longrightarrow G$ and all $x \in \mathcal{X}$ supported at G. So its cobase change $\Phi^K(j)\colon \Phi^K(P) \longrightarrow \Phi^K(Q)$ is a localization by the set of classes

$$\Phi^K_*(f)(\Phi^K(\beta^*(x))) \;=\; \Phi^K(f_*(\beta^*(x))) \;=\; \Phi^K(\beta^*(f_*(x))) \ ,$$

for the same pairs (β, x). Since the morphism $f\colon R \longrightarrow P$ inverts $\mathcal{X}$, also the classes $\beta^*(f_*(x))$ are invertible, by Proposition 1.12, and hence so are the classes $\Phi^K(\beta^*(f_*(x)))$. So $\Phi^K_*(j)\colon \Phi^K_*(P) \longrightarrow \Phi^K_*(Q)$ is a localization by a set of classes that are already invertible, and hence an isomorphism. This shows that the morphism $j\colon P \longrightarrow Q$ induces equivalences of geometric fixed points for all compact Lie groups K, so j is an equivalence in $\mathrm{CAlg}(\mathcal{S}p^\wedge_{\mathrm{gl}})$.

The original pushout square witnesses the diagonal composite $jf \sim gi\colon R \longrightarrow Q$ as a coproduct of f and i in $(R\backslash \mathrm{CAlg}(\mathcal{S}p^\wedge_{\mathrm{gl}}))^{\mathcal{X}}$, the full subcategory of $R\backslash \mathrm{CAlg}(\mathcal{S}p^\wedge_{\mathrm{gl}})$ spanned by the $\mathcal{X}$-inverting morphisms. Since $j\colon P \longrightarrow Q$ is an equivalence, Lemma 1.15 shows that i is an initial object of $(R\backslash \mathrm{CAlg}(\mathcal{S}p^\wedge_{\mathrm{gl}}))^{\mathcal{X}}$, and thus a localization by $\mathcal{X}$. $\square$

The ∞-category of *ultra-commutative ring spectra* $\mathrm{URing}_{\mathrm{gl}}$ is the localization of the 1-category of commutative orthogonal ring spectra at the class of global equivalences, see Definition B.25 for more details. In (B.27) we exhibit a forgetful functor

$$u = u_{\mathrm{gl}} \ : \ \mathrm{URing}_{\mathrm{gl}} \ \longrightarrow \ \mathrm{CAlg}(\mathcal{S}p_{\mathrm{gl}}^{\wedge})$$

from the ∞-category of ultra-commutative ring spectra to the ∞-category of commutative global ring spectra. This functor is both monadic and comonadic, see Theorem B.28 (iii). We define the representation-graded homotopy groups for ultra-commutative ring spectra as those of the underlying commutative global ring spectra, and hence as the ones for the underlying G-spectrum, and similarly with the notion of invertibility.

Definition 1.17 (Ultra-commutative localization)**.** Let R be an ultra-commutative ring spectrum, and let $\mathcal{X}$ be a set of representation-graded equivariant homotopy classes of R. An *ultra-commutative localization by* $\mathcal{X}$ is an initial object of the full subcategory of the slice category $R\backslash \mathrm{URing}_{\mathrm{gl}}$ spanned by the $\mathcal{X}$-inverting morphisms.

The process of 'localization at a set of representation-graded homotopy classes' does *not* in general commute with forgetting from ultra-commutative to globally commutative ring spectra:

- if $i\colon R \longrightarrow S$ is an ultra-commutative localization by $\mathcal{X}$, then the morphism of commutative global ring spectra $ui\colon uR \longrightarrow uS$ need not be a localization by $\mathcal{X}$ in $\mathrm{CAlg}(\mathcal{S}p_{\mathrm{gl}}^{\wedge})$;
- if $j\colon uR \longrightarrow T$ is a global localization by $\mathcal{X}$ in $\mathrm{CAlg}(\mathcal{S}p_{\mathrm{gl}}^{\wedge})$, then the localization T need not admit any ultra-commutative structure.

We will illustrate this phenomenon in Section 6, where we consider the localization of $\Sigma^\infty_+\mathbf{P}$ by the first pre-Bott class b_1 in the two different senses, i.e., in $\mathrm{URing}_{\mathrm{gl}}$ and in $\mathrm{CAlg}(\mathcal{S}p_{\mathrm{gl}}^{\wedge})$. The symmetric group Σ_3 on three letters witnesses the difference between the two localizations: as we explain in Example 6.5, the global localization by b_1 in $\mathrm{CAlg}(\mathcal{S}p_{\mathrm{gl}}^{\wedge})$ has nontrivial Σ_3-geometric fixed points, whereas the ultra-commutative localization by b_1 in $\mathrm{URing}_{\mathrm{gl}}$ has trivial Σ_3-geometric fixed points.

The phenomenon is related to the additional operations, i.e., power operations and norm maps, that are supported by ultra-commutative ring spectra. More specifically, inverting homotopy elements for ultra-commutative ring spectra not only inverts all their restrictions along continuous homomorphisms of compact Lie groups, but also all norms and power operations of such restrictions, see Proposition D.7. We mention without proof that a morphism $i\colon R \longrightarrow S$ is an ultra-commutative localization by $\mathcal{X}$ if and only the underlying morphism of commutative global ring spectra $ui\colon uR \longrightarrow uS$ is a localization in $\mathrm{CAlg}(\mathcal{S}p_{\mathrm{gl}}^{\wedge})$ by the set $\{P^m(x)\colon m \geq 1, x \in \mathcal{X}\}$ of all powers of all elements in $\mathcal{X}$.

A morphism $a\colon A \longrightarrow B$ of anima is a *monomorphism* if it is fully faithful, i.e., an inclusion of some of the path components, compare [32, Tag 04VE]. Equivalently, the commutative square of animae

$$\begin{tikzcd} A \arrow[r, "\mathrm{Id}_A"] \arrow[d, "\mathrm{Id}_A"'] & A \arrow[d, "a"] \\ A \arrow[r, "a"'] & B \end{tikzcd}$$

is a pullback. A morphism $i\colon R \longrightarrow S$ in an ∞-category $\mathcal{C}$ is an *epimorphism* if for every object T of $\mathcal{C}$, the map $\mathcal{C}(i,T)\colon \mathcal{C}(S,T) \longrightarrow \mathcal{C}(R,T)$ is a monomorphism of animae. Equivalently, the commutative square

$$\begin{tikzcd} R \arrow[r, "i"] \arrow[d, "i"'] & S \arrow[d, "\mathrm{Id}_S"] \\ S \arrow[r, "\mathrm{Id}_S"'] & S \end{tikzcd}$$

is a pushout in $\mathcal{C}$. For this paper, an important class of examples of epimorphisms are all localizations of commutative global or of ultra-commutative ring spectra at some set of representation-graded equivariant homotopy classes.

By Theorem B.28 (iii), the forgetful functor $u\colon \mathrm{URing}_{\mathrm{gl}} \longrightarrow \mathrm{CAlg}(\mathcal{S}p^{\wedge}_{\mathrm{gl}})$ has a left adjoint $\lambda\colon \mathrm{CAlg}(\mathcal{S}p^{\wedge}_{\mathrm{gl}}) \longrightarrow \mathrm{URing}_{\mathrm{gl}}$. We write $\epsilon_R\colon \lambda(uR) \longrightarrow R$ for the counit of the adjunction.

Proposition 1.18. *Let $i\colon R \longrightarrow S$ be a morphism of ultra-commutative ring spectra such that the underlying morphism of commutative global ring spectra $ui\colon uR \longrightarrow uS$ is an epimorphism. Then the following commutative square is a pushout in* $\mathrm{URing}_{\mathrm{gl}}$*:*

$$\begin{tikzcd} \lambda(uR) \arrow[r, "\lambda(ui)"] \arrow[d, "\epsilon_R"'] & \lambda(uS) \arrow[d, "\epsilon_S"] \\ R \arrow[r, "i"'] & S \end{tikzcd}$$

Proof. We choose a pushout of ϵ_R and $\lambda(ui)$ in $\mathrm{URing}_{\mathrm{gl}}$:

$$\begin{tikzcd} \lambda(uR) \arrow[r, "\lambda(ui)"] \arrow[d, "\epsilon_R"'] & \lambda(uS) \arrow[d, "g"] \\ R \arrow[r, "j"'] & Q \end{tikzcd} \tag{1.19}$$

Since the functor λ is a left adjoint, it preserves pushouts and thus epimorphisms, so $\lambda(ui)$ is an epimorphism in $\mathrm{URing}_{\mathrm{gl}}$. Hence also its cobase change j is an epimorphism. Since the functor u is a left adjoint, it preserves pushouts and thus epimorphisms, so $u(\lambda(ui))$ and uj are epimorphisms in $\mathrm{CAlg}(\mathcal{S}p^{\wedge}_{\mathrm{gl}})$.

The universal property of the pushout (1.19) provides a unique morphism $h\colon Q \longrightarrow S$ equipped with homotopies

$$h \circ j \ \sim\ i \qquad \text{and} \qquad h \circ g \ \sim\ \epsilon_S \ .$$

Since $(uh) \circ (uj) \sim ui$ and both uj and ui are epimorphisms, also $uh\colon uQ \longrightarrow uS$ is an epimorphism. The sequence of homotopies

$$\begin{aligned} (ug) \circ \eta_{uS} \circ (uh) \circ (uj) \ &\sim\ (ug) \circ \eta_{uS} \circ (ui) \\ &\sim\ (ug) \circ (u(\lambda(ui))) \circ \eta_{uR} \ \sim\ (uj) \circ (u(\epsilon_R)) \circ \eta_{uR} \ \sim\ uj \end{aligned}$$

and the fact that uj is an epimorphism show that the composite

$$uQ \xrightarrow{\ uh\ } uS \xrightarrow{\ \eta_{uS}\ } u(\lambda(uS)) \xrightarrow{\ ug\ } uQ$$

is homotopic to the identity. So uh is an epimorphism with a retraction, and thus an equivalence. Since u is conservative, also $h\colon Q \longrightarrow S$ is an equivalence. Since the square (1.19) is a pushout, so is the square of the proposition. $\square$

While in general, the localizations as ultra-commutative ring spectra and as commutative global ring spectra can differ, there are examples that are simultaneously localizations in both worlds. The next theorem describes a class of such examples.

Theorem 1.20. *Let $\mathcal{X}$ be a set of representation-graded equivariant homotopy classes of an ultra-commutative ring spectrum R.*

(i) *Let $i\colon uR \longrightarrow T$ be a global localization by $\mathcal{X}$ in* $\mathrm{CAlg}(\mathcal{S}p_{\mathrm{gl}}^{\wedge})$. *Then for every pushout square in* $\mathrm{URing}_{\mathrm{gl}}$

$$\begin{tikzcd} \lambda(uR) \arrow[r, "\lambda(i)"] \arrow[d, "\epsilon_R"'] & \lambda(T) \arrow[d, "g"] \\ R \arrow[r, "j"'] & Q \end{tikzcd}$$

the morphism $j\colon R \longrightarrow Q$ is an ultra-commutative localization by $\mathcal{X}$ in $\mathrm{URing}_{\mathrm{gl}}$.

(ii) *Let $i\colon R \longrightarrow S$ be a morphism of ultra-commutative ring spectra such that the underlying morphism of commutative global ring spectra $ui\colon uR \longrightarrow uS$ is a global localization by $\mathcal{X}$ in* $\mathrm{CAlg}(\mathcal{S}p_{\mathrm{gl}}^{\wedge})$. *Then i is a global localization by $\mathcal{X}$ in* $\mathrm{URing}_{\mathrm{gl}}$.

Proof. (i) For every ultra-commutative ring spectrum E, the upper rectangle in the square of animae

$$\begin{tikzcd} \mathrm{URing}_{\mathrm{gl}}(Q,E) \arrow[r, "j^*"] \arrow[d, "g^*"'] & \mathrm{URing}_{\mathrm{gl}}(R,E) \arrow[d, "\epsilon_R^*"] \arrow[dd, bend left, "u"] \\ \mathrm{URing}_{\mathrm{gl}}(\lambda(T),E) \arrow[r, "\lambda(i)^*"'] \arrow[d, "\sim"'] & \mathrm{URing}_{\mathrm{gl}}(\lambda(uR),E) \arrow[d, "\sim"] \\ \mathrm{CAlg}(\mathcal{S}p_{\mathrm{gl}}^{\wedge})(T,uE) \arrow[r, "i^*"'] & \mathrm{CAlg}(\mathcal{S}p_{\mathrm{gl}}^{\wedge})(uR,uE) \end{tikzcd}$$

is a pullback, by the universal property of pushouts. The two lower vertical maps are the adjunction equivalences. So the outer rectangle is a pullback, too. Since i is a localization by $\mathcal{X}$ in $\mathrm{CAlg}(\mathcal{S}p_{\mathrm{gl}}^{\wedge})$, the lower horizontal map is an equivalence onto the path components of $\mathcal{X}$-inverting morphisms. A morphism $R \longrightarrow E$ in $\mathrm{URing}_{\mathrm{gl}}$ is $\mathcal{X}$-inverting if and only if the underlying morphism in $\mathrm{CAlg}(\mathcal{S}p_{\mathrm{gl}}^{\wedge})$ is $\mathcal{X}$-inverting. So the upper horizontal map is an equivalence onto the path components of $\mathcal{X}$-inverting morphisms. So j has the property that characterizes an ultra-commutative localization by $\mathcal{X}$.

(ii) Since localizations are epimorphisms, the square

$$\begin{tikzcd} \lambda(uR) \arrow[r, "\lambda(ui)"] \arrow[d, "\epsilon_R"'] & \lambda(uS) \arrow[d, "\epsilon_S"] \\ R \arrow[r, "i"'] & S \end{tikzcd}$$

is a pushout in $\mathrm{URing}_{\mathrm{gl}}$ by Proposition 1.18. Since $ui\colon uR \longrightarrow uS$ is a global localization by $\mathcal{X}$ in $\mathrm{CAlg}(\mathcal{S}p_{\mathrm{gl}}^{\wedge})$ by hypothesis, i is a global localization by $\mathcal{X}$ in $\mathrm{URing}_{\mathrm{gl}}$ by part (i). $\square$

2. The pre-Bott classes

To make sense of the slogan that global K-theory is obtained by 'inverting the Bott classes', we need to specify the classes we are inverting. A naive attempt would be to invert the non-equivariant Bott class in $\pi_2(\Sigma_+^\infty \mathbb{C}P^\infty)$ that features in Snaith's theorem. However, inverting the classical Bott class does *not* yield the global K-theory spectrum, not even at abelian compact Lie groups. In this section we introduce the *pre-Bott classes* $b_n \in \pi_{\nu_n}^{U(n)}(\Sigma_+^\infty \mathbf{P})$, certain representation-graded equivariant homotopy classes of $\Sigma_+^\infty \mathbf{P}$, see (2.7). The name is justified by the fact that the morphism $\eta\colon \Sigma_+^\infty \mathbf{P} \longrightarrow \mathbf{KU}$ takes the pre-Bott class b_n to the equivariant Bott class of the tautological $U(n)$-representation, see Remark 4.5. One of our main results, Theorem 4.8, will be that the morphism η is a localization, both as commutative global ring spectra and as ultra-commutative ring spectra, by the classes b_n for $n \geq 1$.

The first pre-Bott class b_1 in $\pi_{\nu_1}^T(\Sigma_+^\infty \mathbf{P})$ is somewhat more distinguished than the higher pre-Bott classes. On the one hand, the class b_1 is constructed first, and then the class b_n for $n \geq 2$ is defined from b_1 as a power operation transferred up to $U(n)$, see (2.7). On the other hand, b_1 is the unique class in $\pi_{\nu_1}^T(\Sigma_+^\infty \mathbf{P})$ that maps to the equivariant Bott class of the tautological T-representation, whereas for $n \geq 2$, the map $\eta_*\colon \pi_{\nu_n}^{U(n)}(\Sigma_+^\infty \mathbf{P}) \longrightarrow \pi_{\nu_n}^{U(n)}(\mathbf{KU})$ has a nontrivial kernel, so the $U(n)$-equivariant Bott class of ν_n has other preimages besides b_n.

Inverting only b_1 already goes quite a long way towards global equivariant K-theory: as we show in Theorem 6.2, the globally commutative localization of $\Sigma_+^\infty \mathbf{P}$ by b_1 is $\mathcal{A}b$-globally equivalent to $\mathbf{KU}$, where $\mathcal{A}b$ is the global family of abelian compact Lie groups. And we show in Theorem 6.3 that the ultra-commutative localization of $\Sigma_+^\infty \mathbf{P}$ by b_1 is even $\mathcal{A}b^\circ$-globally equivalent to $\mathbf{KU}$, where $\mathcal{A}b^\circ$ is the global family of compact Lie groups whose identity component is abelian; this family in particular includes all finite groups and all abelian compact Lie groups.

A central player in this paper is a particular global homotopy type with a highly structured multiplication, the ultra-commutative monoid $\mathbf{P}$; we recall its definition in Construction 2.1. We have settled on the notation $\mathbf{P}$ because it is short and the letter 'P' also features in $\mathbb{C}P^\infty$, which is its underlying non-equivariant homotopy type. As $\mathbf{P}$ is a global classifying space for the circle group $T = U(1)$, other popular names for this global homotopy type are $B_{\mathrm{gl}}T$, $\mathbb{B}T$ or $*//T$. It is important for our purposes that the global classifying space of the circle group (and in fact of any abelian compact Lie group) can be endowed with an ultra-commutative global multiplication. In informal terms, an ultra-commutative multiplication consists of compatible E_∞-multiplications on the underlying G-spaces, natural for restriction along continuous homomorphisms of compact Lie groups, together with transfer morphisms for finite index inclusions of compact Lie groups. The full truth is that $\mathbf{P}$ has an even more rigid structure, namely as 'strict abelian group object' in global spaces; we shall not elaborate on this structure.

An *ultra-commutative monoid* is a commutative orthogonal monoid space in the sense of [38, Definition 2.1.1]. In other words, a commutative monoid in the category of orthogonal spaces under the box product (convolution product). By the universal property of convolution products, this data is equivalent to a lax symmetric monoidal functor from the linear isometries category (under orthogonal direct sum) to the category of spaces (under cartesian product). We use the term 'ultra-commutative monoid' to emphasize that we consider objects in this 1-category through the eyes of global equivalences in the sense of [38, Definition 1.1.1]. Consequently, the ∞-category of *ultra-commutative monoids* is the higher categorical localization of the 1-category of commutative orthogonal monoid spaces at the class of global equivalences.

Construction 2.1 (The global ultra-commutative monoid $\mathbf{P}$)**.** We recall the ultra-commutative monoid $\mathbf{P}$. In [38, (2.3.20)], we use the notation $\mathbf{P}^{\mathbb{C}}$ to distinguish this version made from complex projective spaces from the real version. In this paper, the real version plays no role, so we simplify notation and drop the superscript '$\mathbb{C}$'. We let

$$\mathrm{Sym}(V_{\mathbb{C}}) \;=\; \bigoplus\nolimits_{i\geq 0} \mathrm{Sym}^i(V_{\mathbb{C}})$$

denote the symmetric algebra of the complexification of an inner product space V. The value of $\mathbf{P}$ at the inner product space V is then

$$\mathbf{P}(V) \;=\; P(\mathrm{Sym}(V_{\mathbb{C}}))\ ,$$

the complex projective space of the symmetric algebra. The structure map $\mathbf{P}(\varphi)\colon \mathbf{P}(V) \longrightarrow \mathbf{P}(W)$ induced by a linear isometric embedding $\varphi\colon V \longrightarrow W$ is given by

$$\mathbf{P}(\varphi)(L) \;=\; \mathrm{Sym}(\varphi_{\mathbb{C}})(L)\ ,$$

where $\mathrm{Sym}(\varphi_{\mathbb{C}})\colon\ \mathrm{Sym}(V_{\mathbb{C}}) \longrightarrow \mathrm{Sym}(W_{\mathbb{C}})$ is the induced map of symmetric algebras. This defines $\mathbf{P}$ as an orthogonal space.

The symmetric algebra $\mathrm{Sym}(V_{\mathbb{C}})$ is a commutative $\mathbb{C}$-algebra without zero divisors. So the multiplication of $\mathrm{Sym}(V_{\mathbb{C}})$ descends to a well-defined abelian monoid structure

$$\begin{array}{ccc} \mathbf{P}(V)\times\mathbf{P}(V) = P(\mathrm{Sym}(V_{\mathbb{C}}))\times P(\mathrm{Sym}(V_{\mathbb{C}})) & \longrightarrow & P(\mathrm{Sym}(V_{\mathbb{C}})) = \mathbf{P}(V) \\ (\mathbb{C}\cdot x, \mathbb{C}\cdot y) & \longmapsto & \mathbb{C}\cdot xy \end{array}$$

on the projective space. The map $\mathbf{P}(\varphi)$ associated to a linear isometric embedding $\varphi\colon V \longrightarrow W$ is a monoid homomorphism, so $\mathbf{P}$ becomes a continuous functor from the linear isometries category to topological abelian monoids. As explained in [38, Proposition 2.3.5], this structure gives rise to an ultra-commutative multiplication on $\mathbf{P}$, whose (V,W)-component is the composite

$$\mu_{V,W}\ :\ \mathbf{P}(V)\times\mathbf{P}(W) \xrightarrow{\mathbf{P}(i_V)\times\mathbf{P}(i_W)} \mathbf{P}(V\oplus W)\times\mathbf{P}(V\oplus W) \xrightarrow{\text{multiply}} \mathbf{P}(V\oplus W)\ .$$

Here $i_V\colon V \longrightarrow V\oplus W$ and $i_W\colon W \longrightarrow V\oplus W$ are the direct summand embeddings. The multiplicative unit is the unique element of $\mathbf{P}(0) = P(\mathrm{Sym}(0)) = \{\mathbb{C}\cdot 1\}$.

The underlying global space of $\mathbf{P}$ is a global classifying space, in the sense of [38, Definition 1.1.27], for the circle group $T = U(1)$; in particular, the associated Rep-functor $\underline{\pi}_0(\mathbf{P})$ is represented by T, see [38, Proposition 1.5.12]. Since we will need the representability isomorphism, we review it in more detail. We let ν_1 denote tautological unitary T-representation on $\mathbb{C}$ by scalar multiplication. We write $\mathbb{L}$ for the complex line in the linear summand of $\mathrm{Sym}(u(\nu_1)_{\mathbb{C}})$ spanned by the vector $1\otimes 1 - i\otimes i$. This line $\mathbb{L}$ is T-invariant, and thus a T-fixed point of $\mathbf{P}(u(\nu_1))$. We write

$$u_T \;=\; [\mathbb{L}] \;\in\; \pi_0^T(\mathbf{P})$$

for the class of this T-fixed point, the *unstable tautological class.* Then for every compact Lie group G, the map

$$\mathrm{Hom}(G,T) \;\longrightarrow\; \pi_0^G(\mathbf{P})\ ,\quad \alpha \;\longmapsto\; \alpha^*(u_T) \tag{2.2}$$

is bijective, where $\mathrm{Hom}(G,T)$ is the set of continuous homomorphisms.

The *stable tautological class*

$$e_T \;\in\; \pi_0^T(\Sigma^\infty_+\mathbf{P}) \tag{2.3}$$

is the stabilization of the class u_T, i.e., the one represented by the T-equivariant map

$$S^{\nu_1} \xrightarrow{-\wedge\mathbb{L}} S^{\nu_1}\wedge\mathbf{P}(u(\nu_1))_+ \;=\; (\Sigma^\infty_+\mathbf{P})(u(\nu_1))\ .$$

Because $\mathbf{P}$ is a global classifying space for the circle group T, the pair $(\Sigma^\infty_+\mathbf{P}, e_T)$ represents the functor π_0^T on the global stable homotopy category, compare [38, Theorem 4.4.3].

An *ultra-commutative ring spectrum* is a commutative orthogonal ring spectrum; as in the unstable situation, the terminology emphasizes that we view objects in this 1-category through the eyes of global equivalences in the sense of [38, Definition 4.1.3]. The ∞-*category of ultra-commutative ring spectra* $\mathrm{URing}_{\mathrm{gl}}$ is thus the localization of the 1-category of orthogonal commutative ring spectra at the class of global equivalences, see Definition B.25 for more details. The unreduced suspension spectrum functor Σ^∞_+ from orthogonal spaces to orthogonal spectra [38, Construction 4.1.7] is strong symmetric monoidal. So the unreduced suspension spectrum of a commutative orthogonal monoid space becomes a commutative orthogonal ring spectrum. In particular, the orthogonal spectrum $\Sigma^\infty_+\mathbf{P}$ comes with commutative multiplication, thereby modeling an ultra-commutative ring spectrum.

Construction 2.4 (Pre-Bott classes)**.** We continue to write $T = U(1)$ for the circle group, and ν_1 denotes the tautological T-representation on $\mathbb{C}$ by scalar multiplication. The cofiber sequence of based T-spaces

$$T_+ \longrightarrow S^0 \xrightarrow{\ i\ } S^{\nu_1} \tag{2.5}$$

induces a long exact sequence in any T-equivariant cohomology theory; here $i\colon S^0 \longrightarrow S^{\nu_1}$ is the inclusion of T-fixed points. For global spectra, the restriction homomorphism $\mathrm{res}^T_{\{1\}}$ is a split epimorphism, by inflation; so the long exact sequence splits off a short exact sequence:

$$0 \longrightarrow \pi^T_{\nu_1}(\Sigma^\infty_+\mathbf{P}) \xrightarrow{\ i^*\ } \pi^T_0(\Sigma^\infty_+\mathbf{P}) \xrightarrow{\ \mathrm{res}^T_{\{1\}}\ } \pi_0(\Sigma^\infty_+\mathbf{P}) \longrightarrow 0$$

The class $1 = p^*_T(\mathrm{res}^T_{\{1\}}(e_T))$ is the multiplicative unit of the ring $\pi^T_0(\Sigma^\infty_+\mathbf{P})$, where p^*_T is inflation (1.11) along the unique homomorphism $p_T\colon T \longrightarrow \{1\}$. Because

$$\mathrm{res}^T_{\{1\}}(1 - e_T) \ = \ 0 \ ,$$

there is a unique class, the first *pre-Bott class*

$$b_1 \ \in \ \pi^T_{\nu_1}(\Sigma^\infty_+\mathbf{P}), \text{ such that } i^*(b_1) \ = \ 1 - e_T \ \in \pi^T_0(\Sigma^\infty_+\mathbf{P}) \ .$$

For $n \geq 2$, we define the nth pre-Bott class by transferring the nth power operation of b_1 from $\Sigma_n \wr T$ to $U(n)$. To this end we identify $\Sigma_n \wr T$ with a maximal torus normalizer of $U(n)$ via the preferred embedding

$$\Sigma_n \wr T \ \longrightarrow \ U(n) \tag{2.6}$$

that embeds T^n as diagonal matrices, and that sends elements of Σ_n to the associated permutation matrices. The image of this embedding then consists of the *monomial matrices*, i.e., those unitary matrices in $U(n)$ that have precisely one nonzero entry in every row and in every column. The power operation

$$P^n \ : \ \pi^T_{\nu_1}(\Sigma^\infty_+\mathbf{P}) \ \longrightarrow \ \pi^{\Sigma_n \wr T}_{\nu_1^n}(\Sigma^\infty_+\mathbf{P})$$

is the straightforward generalization of the power operation [38, (5.1.2)] of zeroth homotopy groups to representation-graded homotopy groups; we recall its definition in (D.2). The *nth pre-Bott class* is then defined as

$$b_n \ = \ \mathrm{tr}^{U(n)}_{\Sigma_n \wr T}(P^n(b_1)) \ \in \ \pi^{U(n)}_{\nu_n}(\Sigma^\infty_+\mathbf{P}) \ . \tag{2.7}$$

The gradings work out because the restriction of the tautological $U(n)$-representation ν_n to $\Sigma_n \wr T$ is the representation ν_1^n.

The pre-Bott class b_n for $n \geq 2$ is defined by a transfer; so identifying its restriction to a subgroup of $U(n)$ involves a double coset formula. The following double coset formula will take care of all cases we need in this paper; it ought to be well-known to experts, but I do not know a reference. The case $m = n = 1$ is classical because $T \times T$ is a maximal torus of $U(2)$, and $\Sigma_2 \wr T$ is the maximal torus normalizer; so for $m = n = 1$, the following double coset formula is a special case of [25, IV Corollary 6.7 (i)]. In the following we write

$$U(m,n) \ = \ \{ \left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right) : (A,B) \in U(m) \times U(n) \}$$

for the block subgroup of $U(m+n)$. The embedding (2.6) takes $(\Sigma_m \wr T) \times (\Sigma_n \wr T)$ into $U(m,n)$.

Proposition 2.8. *For all $m,n\geq 1$, the following double coset formula holds:*

$$\mathrm{res}^{U(m+n)}_{U(m,n)}\circ\mathrm{tr}^{U(m+n)}_{\Sigma_{m+n}\wr T} \ = \ \mathrm{tr}^{U(m,n)}_{(\Sigma_m\wr T)\times(\Sigma_n\wr T)}\circ\mathrm{res}^{\Sigma_{m+n}\wr T}_{(\Sigma_m\wr T)\times(\Sigma_n\wr T)}$$

Proof. We refer to [25, IV Theorem 6.3] or [38, Theorem 3.4.9] for the general double coset formula for $\mathrm{res}^G_K\circ\mathrm{tr}^G_H$ for two closed subgroups H and K of a compact Lie group G; we need to specialize it to the situation at hand.

We identify the coset space $U(m+n)/(\Sigma_{m+n}\wr T)$ with the space of sets $\{L_1,\dots,L_{m+n}\}$ of $m+n$ pairwise orthogonal lines in $\mathbb{C}^{m+n}$ by the homeomorphism that sends $A\cdot(\Sigma_{m+n}\wr T)$ to the set of lines spanned by the columns of A. Now we let $\{L_1,\dots,L_{m+n}\}$ be any set of $m+n$ pairwise orthogonal lines in $\mathbb{C}^{m+n}$.

Now we consider the special case where at least one line L_i is neither contained in $\mathbb{C}^m\oplus 0^n$ nor in $0^m\oplus\mathbb{C}^n$. We claim that then the intersection of the stabilizer of the set $\{L_1,\dots,L_{m+n}\}$ with the center of $U(m,n)$ is 1-dimensional. The center $Z(U(m,n))$ of $U(m,n)$ consists of the matrices $(\lambda E_m)\oplus(\mu E_n)$ for all $\lambda,\mu\in T$. Since some line L_i is neither contained in $\mathbb{C}^m\oplus 0^n$ nor in $0^m\oplus\mathbb{C}^n$, it is not stabilized by the matrix $(\lambda E_m)\oplus(\mu E_n)$ for any $\lambda,\mu\in T$ with $\lambda\neq\mu$. So the intersection of the stabilizer of L_i with $Z(U(m,n))$ equals the center of $U(m+n)$. Since any line is stabilized by the center of $U(m+n)$, also the intersection of the stabilizer of the tuple $(L_1,\dots,L_{m+n})$ of lines with $Z(U(m,n))$ equals the center of $U(m+n)$, and is thus 1-dimensional. The stabilizer of the tuple $(L_1,\dots,L_{m+n})$ has finite index in the stabilizer of the set $\{L_1,\dots,L_{m+n}\}$. So the intersection of the stabilizer of the set $\{L_1,\dots,L_{m+n}\}$ with $Z(U(m,n))$ is also 1-dimensional, as claimed.

For every compact Lie group G, the center $Z(G)$ of G normalizes any given closed subgroup H, so the group $Z(G)/(H\cap Z(G))$ injects into the Weyl group $W_GH=(N_GH)/H$. The center of $G=U(m,n)$ is isomorphic to $T\times T$, and hence 2-dimensional. We take $H=\mathrm{Stab}\{L_1,\dots,L_{m+n}\}\cap U(m,n)$ as the intersection of the stabilizer of the set $\{L_1,\dots,L_{m+n}\}$ with $U(m,n)$; the claim we just established says that $H\cap Z(U(m,n))$ is 1-dimensional. So the group $Z(U(m,n))/(H\cap Z(U(m,n)))$ is 1-dimensional, and so the Weyl group of $\mathrm{Stab}\{L_1,\dots,L_{m+n}\}\cap U(m,n)$ inside $U(m,n)$ is infinite. This means that the $U(m,n)$-orbit of the set $\{L_1,\dots,L_n\}$ does not contribute to the double coset formula.

We conclude that all contributions to the double coset formula come from sets $\{L_1,\dots,L_{m+n}\}$ of pairwise orthogonal lines for which each line is either contained in $\mathbb{C}^m\oplus 0^n$ or in $0^m\oplus\mathbb{C}^n$. The group $U(m,n)$ acts transitively on these, so they only contribute the principal summand to the double coset formula. □

Construction 2.9 (Pre-Bott classes of unitary representations)**.** We let V be a unitary representation of a compact Lie group G, of complex dimension n. We let $\alpha\colon G\longrightarrow U(n)$ classify V, i.e., V is isomorphic to $\alpha^*(\nu_n)$. We let $\psi\colon V\cong\alpha^*(\nu_n)$ be a G-equivariant $\mathbb{C}$-linear isometry. We write $\psi^\flat\colon\pi_{\alpha^*(\nu_n)}(\Sigma^\infty_+\mathbf{P})\longrightarrow\pi^G_V(\Sigma^\infty_+\mathbf{P})$ for the isomorphism given by precomposition with the equivariant homeomorphism $S^\psi\colon S^V\cong S^{\alpha^*(\nu_n)}$. The *pre-Bott class* of V is

$$b_{G,V} \ = \ \psi^\flat(\alpha^*(b_n)) \ \in \ \pi^G_V(\Sigma^\infty_+\mathbf{P}) \ .$$

The group of equivariant self-isometries of a unitary representation is path connected. So any two choices of ψ are homotopic through equivariant isometries. Hence the map $\psi^\flat$, and thus also the class $b_{G,V}$, is independent of the choice of ψ.

For an orthogonal ring spectrum R, representing a global ring spectrum, we can 'externalize' the homotopy group pairing (1.8) in the group variable by using the global structure. Given two compact Lie groups G and K, a G-representation V, and a K-representation W, we define a pairing

$$\times \ : \ \pi^G_V(R)\times\pi^K_W(R) \ \longrightarrow \ \pi^{G\times K}_{V\times W}(R)$$

as the composite

$$\pi_V^G(R) \times \pi_W^K(R) \xrightarrow{p_G^* \times p_K^*} \pi^{G\times K}_{p_G^*(V)}(R) \times \pi^{G\times K}_{p_K^*(W)}(R) \xrightarrow{\ \cdot\ } \pi^{G\times K}_{V\times W}(R) \ .$$

Here $p_G \colon G \times K \longrightarrow G$ and $p_K \colon G \times K \longrightarrow K$ are the projections, and $V \times W = p_G^*(V) \oplus p_K^*(W)$ is the external product, a representation of $G \times K$. For $G = K$, the 'internal' product (1.8) can be obtained from the external product by restricting to the diagonal in $G \times G$.

By the following theorem, the class b_{n-1} is a factor of a restriction of b_n. So every morphism of commutative global ring spectra from $\Sigma^\infty_+ \mathbf{P}$ that inverts b_n also inverts $b_1, b_2, \dots, b_{n-1}$. We shall show in Theorem 6.1 below that the ultra-commutative localization by b_{n-1} does *not* invert b_n. Hence the localizations by the classes b_n are all mutually distinct for different values of n, and all distinct from the localization by the entire set $\{b_n\}_{n\geq 1}$.

Theorem 2.10 (Multiplicativity of pre-Bott classes)**.**

(i) *For all $m, n \geq 1$, the relation*

$$\mathrm{res}^{U(m+n)}_{U(m)\times U(n)}(b_{m+n}) \ = \ b_m \times b_n$$

holds in the group $\pi^{U(m)\times U(n)}_{\nu_m \times \nu_n}(\Sigma^\infty_+ \mathbf{P})$.

(ii) *For all unitary representations V and W of a compact Lie group G, the relation $b_{G,V} \cdot b_{G,W} = b_{G,V\oplus W}$ holds in $\pi^G_{V\oplus W}(\Sigma^\infty_+ \mathbf{P})$.*

Proof. (i) We calculate from the definitions:

$$\begin{aligned}
\mathrm{res}^{U(m+n)}_{U(m,n)}(b_{m+n}) \ &= \ \mathrm{res}^{U(m+n)}_{U(m,n)}(\mathrm{tr}^{U(m+n)}_{\Sigma_{m+n}\wr T}(P^{m+n}(b_1))) \\
&= \ \mathrm{tr}^{U(m,n)}_{(\Sigma_m\wr T)\times(\Sigma_n \wr T)}(\mathrm{res}^{\Sigma_{m+n}\wr T}_{(\Sigma_m\wr T)\times(\Sigma_n\wr T)}(P^{m+n}(b_1))) \\
&= \ \mathrm{tr}^{U(m,n)}_{(\Sigma_m\wr T)\times(\Sigma_n \wr T)}(P^m(b_1) \times (P^n(b_1))) \\
&= \ \mathrm{tr}^{U(m)}_{\Sigma_m \wr T}(P^m(b_1)) \times \mathrm{tr}^{U(n)}_{\Sigma_n\wr T}(P^n(b_1)) \ = \ b_m \times b_n \ .
\end{aligned}$$

The second relation is the double coset formula of Proposition 2.8. The third relation is a general property of power operations, see (D.3).

(ii) We may assume that $V = \alpha^*(\nu_m)$ and $W = \beta^*(\nu_n)$ for continuous homomorphisms $\alpha \colon G \longrightarrow U(m)$ and $\beta \colon G \longrightarrow U(n)$. Then $V \oplus W$ is the restriction of the tautological $U(m+n)$-representation ν_{m+n} along the composite

$$G \xrightarrow{(\alpha,\beta)} U(m) \times U(n) \xrightarrow{\text{incl}} U(m+n) \ .$$

Hence

$$\begin{aligned}
b_{G,V} \cdot b_{G,W} \ &= \ \alpha^*(b_m) \cdot \beta^*(b_n) \ = \ (\alpha, \beta)^*(b_m \times b_n) \\
&= \ (\alpha,\beta)^*(\mathrm{res}^{U(m+n)}_{U(m)\times U(n)}(b_{m+n})) \ = \ b_{G,V\oplus W} \ .
\end{aligned}$$

□

In equivariant K-theory, the Thom classes of the Atiyah–Bott–Shapiro orientation are compatible with power operations, see [5]. The Bott classes of unitary representations are particular cases of such Thom classes, so a consequence is the fact that in equivariant K-theory, for a finite index subgroup H of a compact Lie group G, the norm operation N_H^G takes the H-equivariant Bott class of a unitary H-representation to the G-equivariant Bott class of the induced G-representation.

We alert the reader that our pre-Bott classes are *not* compatible in the same way with power operations and norms: for a finite index subgroup H of a compact Lie group G, and a unitary H-representation W, the class $N_H^G(b_{H,W})$ is typically different from the class $b_{G,\mathrm{ind}_H^G W}$. We exhibit a specific witness in Example 3.6 below.

3. Geometric fixed points of pre-Bott classes

In this section we shall acquire a firm grip on the images of the pre-Bott classes $b_{G,V}$ in the G-geometric fixed points of $\Sigma^\infty_+\mathbf{P}$. This is the technical heart of this paper that makes our main results work. It is well-known that the G-geometric fixed points of the equivariant K-theory spectrum are trivial for all non-cyclic compact Lie groups G. What we need in the remainder of the paper is that our pre-Bott classes are sufficiently plentiful so that inverting them enforces the same property, i.e., kills the geometric fixed points of $\Sigma^\infty_+\mathbf{P}$ for all non-cyclic compact Lie groups. This feature is ensured by the main result of this section, Theorem 3.21: for every non-cyclic compact Lie group G, it provides a unitary G-representation V such that $\Phi^G(b_{G,V})=0$ in $\Phi^G_0(\Sigma^\infty_+\mathbf{P})$.

A key ingredient for the proof of Theorem 3.21 is a formula for $\Phi^G(b_{G,V})$ in the case when V is irreducible. Our formula in Theorem 3.10 is a sum indexed by all ways to write V as induced from a 1-dimensional representation of some finite index subgroup of G. Theorem 3.10 in particular says that if V is an irreducible G-representation that is not *monomial*, i.e., not induced from a 1-dimensional representation of any finite index subgroup, then $\Phi^G(b_{G,V})=0$. This takes care of all compact Lie groups that have non-monomial irreducible representations. Additional arguments are needed to deal with monomial groups, i.e., ones where every irreducible representation is monomial. For those we provide ad hoc arguments involving some explicit group theoretic manipulations.

The place where our main calculations live is the geometric fixed point ring $\Phi^G_0(\Sigma^\infty_+\mathbf{P})$, for varying compact Lie groups G; the next proposition exhibits a monomial basis of this ring. The G-equivariant homotopy ring $\pi^G_0(\Sigma^\infty_+\mathbf{P})$ is free as an abelian group with a basis given by the classes $\mathrm{tr}^G_H(\alpha^*(e_T))$ for all G-conjugacy classes of pairs (H,α) consisting of a closed subgroup H of G with finite Weyl group, and a continuous homomorphism $\alpha\colon H\longrightarrow T$, see [38, Corollary 4.1.13]. We write

$$\alpha^\sharp \ = \ \Phi^G(\alpha^*(e_T)) \ \in \ \Phi^G_0(\Sigma^\infty_+\mathbf{P}) \ ,$$

where $\Phi^G\colon \pi^G_0(\Sigma^\infty_+\mathbf{P})\longrightarrow \Phi^G_0(\Sigma^\infty_+\mathbf{P})$ is the geometric fixed point homomorphism (1.4) or (A.14).

Example 3.1. Let $\alpha\colon G\longrightarrow T$ be a non-trivial character of a compact Lie group G. Then the geometric fixed point homomorphism $\Phi^G\colon \pi^G_\alpha(\Sigma^\infty_+\mathbf{P})\longrightarrow \Phi^G_0(\Sigma^\infty_+\mathbf{P})$ from (1.4) factors as the composite

$$\pi^G_\alpha(\Sigma^\infty_+\mathbf{P}) \ \xrightarrow{\ i^*\ } \ \pi^G_0(\Sigma^\infty_+\mathbf{P}) \ \xrightarrow{\ \Phi^G\ } \ \Phi^G_0(\Sigma^\infty_+\mathbf{P}) \ ,$$

where $i\colon S^0\longrightarrow S^\alpha$ is the inclusion of G-fixed points. We record the relation

$$i^*(b_{G,\alpha}) \ = \ i^*(\alpha^*(b_1)) \ = \ \alpha^*(i^*(b_1)) \ = \ \alpha^*(1-e_T) \ = \ 1-\alpha^*(e_T) \ .$$

Hence

$$\Phi^G(b_{G,\alpha}) \ = \ \Phi^G(i^*(b_{G,\alpha})) \ = \ 1-\Phi^G(\alpha^*(e_T)) \ = \ 1-\alpha^\sharp \ . \tag{3.2}$$

The inflation homomorphism

$$p^*_G \ : \ \pi_k(\Sigma^\infty\mathbf{P}) \ = \ \Phi^{\{1\}}_k(\Sigma^\infty_+\mathbf{P}) \ \longrightarrow \ \Phi^G_k(\Sigma^\infty\mathbf{P})$$

associated with the unique homomorphism $p_G\colon G\longrightarrow\{1\}$ is defined in [38, Construction 3.3.4], see also (B.8).

Proposition 3.3. *Let G be a compact Lie group.*

(i) *The map*

$$\mathrm{Hom}(G,T) \ \longrightarrow \ \Phi^G_0(\Sigma^\infty_+\mathbf{P}) \ , \quad \alpha \ \longmapsto \ \alpha^\sharp \ = \ \Phi^G(\alpha^*(e_T))$$

is multiplicative, and its extension to a ring homomorphism

$$\mathbb{Z}[\mathrm{Hom}(G,T)] \ \longrightarrow \ \Phi^G_0(\Sigma^\infty_+\mathbf{P})$$

is an isomorphism of rings.

(ii) *The morphism of graded-commutative rings*

$$\mathbb{Z}[\mathrm{Hom}(G,T)]\otimes \pi_*(\Sigma^\infty_+\mathbf{P}) \longrightarrow \Phi^G_*(\Sigma^\infty_+\mathbf{P}) \ , \quad \alpha\otimes y \longmapsto \alpha^\sharp\cdot p_G^*(y)$$

is an isomorphism.

Proof. (i) By [38, Proposition 2.2.23], the bijection (2.2) is multiplicative in the sense that

$$\alpha^*(u_T)\cdot\beta^*(u_T) \ = \ (\alpha\cdot\beta)^*(u_T)$$

in $\pi_0^G(\mathbf{P})$, for every pair of continuous homomorphisms $\alpha,\beta\colon G\longrightarrow T$. The stabilization map [38, (3.3.12)]

$$\sigma^T \ : \ \pi_0^T(\mathbf{P})=\pi_0(\mathbf{P}^T) \longrightarrow \pi_0^T(\Sigma^\infty_+\mathbf{P})$$

takes the unstable tautological class u_T to the stable tautological class e_T, by definition of the latter. The stabilization map is multiplicative and natural for restriction homomorphisms, and thus

$$\begin{aligned}\alpha^*(e_T)\cdot\beta^*(e_T) \ &= \ \alpha^*(\sigma^T(u_T))\cdot\beta^*(\sigma^T(u_T))\\ &= \ \sigma^G(\alpha^*(u_T)\cdot\beta^*(u_T)) \ = \ \sigma^G((\alpha\cdot\beta)^*(u_T))\\ &= \ (\alpha\cdot\beta)^*(\sigma^G(u_T)) \ = \ (\alpha\cdot\beta)^*(e_T)\ .\end{aligned}$$

This relation and the multiplicativity of the geometric fixed point map yield

$$\alpha^\sharp\cdot\beta^\sharp \ = \ \Phi^G(\alpha^*(e_T))\cdot\Phi^G(\beta^*(e_T)) \ = \ \Phi^G(\alpha^*(e_T)\cdot\beta^*(e_T)) \ = \ \Phi^G((\alpha\cdot\beta)^*(e_T)) \ = \ (\alpha\cdot\beta)^\sharp\ .$$

For every compact Lie group G and every G-space A, the abelian group $\Phi_0^G(\Sigma^\infty_+ A)$ is free, with a basis indexed by the set $\pi_0(A^G)$. More precisely, the composite

$$\pi_0(A^G) \xrightarrow{\sigma^G} \pi_0^G(\Sigma^\infty_+ A) \xrightarrow{\Phi^G} \Phi_0^G(\Sigma^\infty_+ A)$$

of the stabilization map [38, (3.3.12)] and the geometric fixed point map identifies $\pi_0(A^G)$ with a $\mathbb{Z}$-basis of $\Phi_0^G(\Sigma^\infty_+ A)$. For the underlying G-space of the global space $\mathbf{P}$, we have $\pi_0(\mathbf{P}^G)=\pi_0^G(\mathbf{P})=\mathrm{Hom}(G,T)$. Moreover,

$$\Phi^G(\sigma^G(\alpha^*(u_T))) \ = \ \Phi^G(\alpha^*(\sigma^T(u_T))) \ = \ \Phi^G(\alpha^*(e_T)) \ = \ \alpha^\sharp$$

for every continuous homomorphism $\alpha\colon G\longrightarrow T$, and hence these classes form a $\mathbb{Z}$-basis of $\pi_0^G(\Sigma^\infty_+\mathbf{P})$.

(ii) Part (i) shows that the graded ring $\Phi_0^G(\Sigma^\infty_+\mathbf{P})\otimes\pi_*(\Sigma^\infty_+\mathbf{P})$ is a free module over $\pi_*(\Sigma^\infty_+\mathbf{P})$ with basis the classes $\alpha^\sharp\otimes 1$ for all $\alpha\in\mathrm{Hom}(G,T)$. Suspension spectra and geometric fixed points commute, in the sense that the composite

$$\pi_*(\Sigma^\infty_+\mathbf{P}^G) \xrightarrow{\sigma^G} \pi_*^G(\Sigma^\infty_+\mathbf{P}) \xrightarrow{\Phi^G} \Phi_*^G(\Sigma^\infty_+\mathbf{P})$$

of the stabilization maps and the geometric fixed point map is an isomorphism. The fixed point space $\mathbf{P}^G=P(\mathcal{U}_G^{\mathbb{C}})^G$ is the space of G-invariant lines in a complete complex G-universe, and hence the disjoint union over all $\alpha\in\mathrm{Hom}(G,T)$, of copies of $\mathbb{C}P^\infty$. More precisely, choices of linear isometries $\alpha_\flat\colon\mathbb{C}^\infty\cong\mathrm{Hom}^G(\alpha,\mathcal{U}_G^{\mathbb{C}})$ to the α-isotypical summands in the complete universe assemble into a homeomorphism

$$\mathrm{Hom}(G,T)\times\mathbb{C}P^\infty \ \cong \ P(\mathcal{U}_G^{\mathbb{C}})^G=\mathbf{P}^G\ .$$

Stable homotopy groups takes disjoint unions to direct sums, so the induced map

$$\mathbb{Z}[\mathrm{Hom}(G,T)]\otimes\pi_*(\Sigma^\infty_+\mathbf{P}) \longrightarrow \pi_*(\Sigma^\infty_+\mathbf{P}^G)$$

is an isomorphism. □

The structure of ultra-commutative monoid on $\mathbf{P}$ gives the Rep-functor $\underline{\pi}_0(\mathbf{P})$ the structure of a *global power monoid* in the sense of [38, Definition 2.2.8]. Roughly speaking, this means that $\underline{\pi}_0(\mathbf{P})$ becomes a functor to abelian monoids that is additionally equipped with additive power operations. By [38, Proposition 2.2.23], the represented functor $\mathrm{Hom}(-,T)$—and hence also the Rep-functor $\underline{\pi}_0(\mathbf{P})$—in fact has a *unique* structure of global power monoid. Namely, the abelian monoid structure $\mathrm{Hom}(G,T)$ is elementwise multiplication of homomorphisms; and for $m \geq 1$, the power operation

$$[m] \ : \ \mathrm{Hom}(G,T) \ \longrightarrow \ \mathrm{Hom}(\Sigma_m \wr G, T)$$

sends $\alpha\colon G \longrightarrow T$ to $[m](\alpha)\colon \Sigma_m \wr G \longrightarrow T$ defined by

$$[m](\alpha)(\sigma; \lambda_1, \dots, \lambda_m) \ = \ \alpha(\lambda_1) \cdot \dots \cdot \alpha(\lambda_m) \ .$$

For a finite index subgroup H of G, the transfer

$$\mathrm{tr}_H^G \ : \ \mathrm{Hom}(H,T) \ \longrightarrow \ \mathrm{Hom}(G,T) \tag{3.4}$$

takes a continuous homomorphism $\chi\colon H \longrightarrow T$ to the composite

$$G \ \xrightarrow{\Psi_{\bar{g}}} \ \Sigma_m \wr H \ \xrightarrow{[m](\chi)} \ T \ ,$$

where $\Psi_{\bar{g}}\colon G \longrightarrow \Sigma_m \wr H$ is the monomorphism (D.6) associated to any choice of transversal of H in G. This transfer is the straightforward generalization to compact Lie groups of the transfer in group cohomology $H^1(-;T) = \mathrm{Hom}(-,T)$.

Like for any ultra-commutative ring spectrum, the 0-th equivariant homotopy groups of $\Sigma^\infty_+\mathbf{P}$ support norm operations $N_H^G\colon \pi_0^H(\Sigma^\infty_+\mathbf{P}) \longrightarrow \pi_0^G(\Sigma^\infty_+\mathbf{P})$ as defined in [38, Remark 5.1.7], for every finite index subgroup H of a compact Lie group G, see also Construction D.4. Part (ii) of the following proposition determines some of these norms.

Proposition 3.5. *Let H be a finite index subgroup of a compact Lie group G, and let $\chi\colon H \longrightarrow T$ be a continuous homomorphism.*

(i) *The relation*
$$N_H^G(\chi^*(e_T)) \ = \ (\mathrm{tr}_H^G(\chi))^*(e_T)$$
holds in the ring $\pi_0^G(\Sigma^\infty_+\mathbf{P})$, where $\mathrm{tr}_H^G(\chi)\colon G \longrightarrow T$ is the transfer (3.4) of χ.

(ii) *If χ is non-trivial, then the relation*
$$\Phi^G(N_H^G(b_{H,\chi})) \ = \ 1 - \mathrm{tr}_H^G(\chi)^\sharp$$
holds in the ring $\Phi_0^G(\Sigma^\infty_+\mathbf{P})$.

Proof. (i) Unraveling the definition (3.4) of the transfer and of $[m](\chi)$ reveals $\mathrm{tr}_H^G(\chi)$ as the composite

$$G \ \xrightarrow{\Psi_{\bar{g}}} \ \Sigma_m \wr H \ \xrightarrow{\Sigma_m \wr \chi} \ \Sigma_m \wr T \ \xrightarrow{p_m} \ T \ .$$

Here $\Psi_{\bar{g}}$ is the monomorphism (D.6) associated to a choice of transversal of H in G, and

$$p_m(\sigma; \lambda_1, \dots, \lambda_m) \ = \ \lambda_1 \cdot \dots \cdot \lambda_m \ .$$

Like for any ultra-commutative monoid, the stabilization map [38, (3.3.12)]

$$\sigma^H \ : \ \pi_0^H(\mathbf{P}) \ \longrightarrow \ \pi_0^H(\Sigma^\infty_+\mathbf{P})$$

sends power operations in the global power monoid $\underline{\pi}_0(\mathbf{P})$ to power operations in the global power functor $\underline{\pi}_0(\Sigma^\infty_+\mathbf{P})$, by direct inspection. Because the unstable and stable tautological classes are related by $e_T = \sigma^T(u_T)$, we deduce the relation

$$P^m(e_T) \ = \ P^m(\sigma^T(u_T)) \ = \ \sigma^{\Sigma_n \wr T}([m](u_T)) \ = \ \sigma^{\Sigma_m \wr T}(p_m^*(u_T)) \ = \ p_m^*(\sigma^T(u_T)) \ = \ p_m^*(e_T)$$

in $\pi_0^{\Sigma_m\wr T}(\Sigma^\infty_+\mathbf{P})$. This yields the desired relation:

$$\begin{aligned} N_H^G(\chi^*(e_T)) \ &= \ \Psi_{\bar g}^*(P^m(\chi^*(e_T))) \ = \ \Psi_{\bar g}^*((\Sigma_m\wr\chi)^*(P^m(e_T)))\\ &= \ \Psi_{\bar g}^*((\Sigma_m\wr\chi)^*(p_m^*(e_T))) \ = \ (p_m\circ(\Sigma_m\wr\chi)\circ\Psi_{\bar g})^*(e_T) \ = \ (\mathrm{tr}_H^G(\chi))^*(e_T) \ . \end{aligned}$$

(ii) The class $N_H^G(b_{H,\chi})$ has representation-grading $\mathrm{ind}_H^G(\chi)$. Since $\chi\colon H\longrightarrow T$ is nontrivial, the G-fixed points of $\mathrm{ind}_H^G(\chi)$ are trivial. So the geometric fixed point homomorphism $\Phi^G\colon \pi^G_{\mathrm{ind}_H^G(\chi)}(\Sigma^\infty_+\mathbf{P})\longrightarrow \Phi_0^G(\Sigma^\infty_+\mathbf{P})$ factors as the composite

$$\pi^G_{\mathrm{ind}_H^G(\chi)}(\Sigma^\infty_+\mathbf{P}) \ \xrightarrow{\ i^*\ } \ \pi_0^G(\Sigma^\infty_+\mathbf{P}) \ \xrightarrow{\ \Phi^G\ } \ \Phi_0^G(\Sigma^\infty_+\mathbf{P}) \ .$$

We record the relation

$$i^*(b_{H,\chi}) \ = \ i^*(\chi^*(b_1)) \ = \ \chi^*(i^*(b_1)) \ = \ \chi^*(1-e_T) \ = \ 1-\chi^*(e_T) \ .$$

The composite $\Phi^G\circ N_H^G\colon \pi_0^H(\Sigma^\infty_+\mathbf{P})\longrightarrow\Phi_0^G(\Sigma^\infty_+\mathbf{P})$ is additive, see Proposition D.8. So we obtain the desired relation:

$$\begin{aligned} \Phi^G(N_H^G(b_{H,\chi})) \ &= \ \Phi^G(i^*(N_H^G(b_{H,\chi}))) \ = \ \Phi^G(N_H^G(i^*(b_{H,\chi})))\\ &= \ \Phi^G(N_H^G(1-\chi^*(e_T))) \ = \ 1-\Phi^G(N_H^G(\chi^*(e_T)))\\ &=_{\text{(i)}} \ 1-\Phi^G((\mathrm{tr}_H^G(\chi))^*(e_T)) \ = \ 1-\mathrm{tr}_H^G(\chi)^\sharp \ . \end{aligned}$$ $\square$

Example 3.6. An example to illustrate the incompatibility of pre-Bott classes with norms is $G=C_4=\{\pm1,\pm i\}$, the cyclic subgroup of order 4 of T, and the sign representation σ of the unique index 2 subgroup $H=C_2=\{\pm1\}$ of $G=C_4$. In this case

$$\mathrm{ind}_{C_2}^{C_4}\sigma \ = \ \zeta\oplus\zeta^3 \ ,$$

where $\zeta\colon C_4=\{\pm1,\pm i\}\longrightarrow T$ is the inclusion. On the one hand, multiplicativity of pre-Bott classes yields the relation

$$\Phi^{C_4}(b_{C_4,\zeta\oplus\zeta^3}) \ = \ \Phi^{C_4}(b_{C_4,\zeta})\cdot\Phi^{C_4}(b_{C_4,\zeta^3}) \ =_{(3.2)} \ (1-\zeta^\sharp)\cdot(1-(\zeta^3)^\sharp) \ = \ 2-\zeta^\sharp-(\zeta^3)^\sharp$$

in the ring $\Phi_0^{C_4}(\Sigma^\infty_+\mathbf{P})$. On the other hand, Proposition 3.5 (ii) yields

$$\Phi^{C_4}(N_{C_2}^{C_4}(b_{C_2,\sigma})) \ = \ 1-\mathrm{tr}_{C_2}^{C_4}(\sigma)^\sharp \ = \ 1-(\zeta^2)^\sharp \ .$$

Since the classes $\Phi^{C_4}(b_{C_4,\mathrm{ind}_{C_2}^{C_4}\sigma})$ and $\Phi^{C_4}(N_{C_2}^{C_4}(b_{C_2,\sigma}))$ are different, $b_{C_4,\mathrm{ind}_{C_2}^{C_4}\sigma}$ is different from $N_{C_2}^{C_4}(b_{C_2,\sigma})$.

Now we embark on the journey towards Theorem 3.10 which provides a formula for $\Phi^G(b_{G,V})$ when V is irreducible. The theorem refers to the ways to write an irreducible G-representation as induced a 1-dimensional representation of a finite index subgroup; we set up some terminology and notion for dealing with this.

Construction 3.7 (Monomial structures)**.** We let H be a finite index subgroup of a compact Lie group G. For a continuous homomorphism $\chi\colon H\longrightarrow T$, the *induced representation* is

$$\mathrm{ind}_H^G(\chi) \ = \ \{f\colon G\longrightarrow\mathbb{C} \mid f(hg)=\chi(h)\cdot f(g) \text{ for all } h\in H,\ g\in G\} \ ,$$

with pointwise $\mathbb{C}$-vector space structure, with G-action by $(\gamma\cdot f)(g)=f(g\gamma)$, and with inner product

$$\langle f,f'\rangle \ = \ \sum_{Hg\in H\backslash G}\langle f(g),f'(g)\rangle \ .$$

Now we let V be a unitary representation of G. A *monomial datum* for V is a pair (H,χ) consisting of a finite index subgroup H of G, and a continuous homomorphism $\chi\colon H\longrightarrow T$, such that V is isomorphic to the induced representation $\mathrm{ind}_H^G(\chi)$. For $g \in G$, let ${}^gH = gHg^{-1}$ be the conjugate subgroup, and $c_g \colon {}^gH \longrightarrow H$ is the conjugation isomorphism defined by $c_g(h) = g^{-1}hg$. Then the G-representation $\mathrm{ind}_{{}^gH}^G(\chi \circ c_g)$ is isomorphic to $\mathrm{ind}_H^G(\chi)$. So the group G acts on the set of monomial data for V by

$$^g(H, \chi) \ = \ ({}^gH, \chi \circ c_g) \ .$$

A *monomial structure* for V is a G-orbit of monomial data. We write $\mathrm{mon}(V)$ for the set of monomial structures on V, and $[H, \chi] \in \mathrm{mon}(V)$ for the monomial structure represented by a monomial datum (H, χ).

Remark 3.8 (Finiteness of $\mathrm{mon}(V)$)**.** If (H, χ) is a monomial datum for V, then in particular, the following two properties must be satisfied:

- Since $\dim(\mathrm{ind}_H^G(\chi)) = [G : H]$, the index of H in G equals the dimension of V.
- Since the character χ is a summand in $\mathrm{res}_H^G(\mathrm{ind}_H^G(\chi))$, it is also a summand in $\mathrm{res}_H^G(V)$, the restriction of V to H.

A compact Lie group G has only finitely many subgroups of finite index, and $\mathrm{res}_H^G(V)$ has only finitely many non-zero isotypical summands. So V has only finitely many monomial data. Hence also the set $\mathrm{mon}(V)$ of monomial structures is finite.

Every 1-dimensional representation has a unique monomial datum, and a unique monomial structure. Tautologically, the pair (H, χ) is a monomial datum for the induced representation $\mathrm{ind}_H^G(\chi)$. We alert the reader that even irreducible representations can have several monomial structures, see Example 3.15. The next proposition exhibits a situation where monomial structures are unique.

Proposition 3.9. *Let V be a unitary representation of a compact Lie group G. Suppose that G has a unique subgroup of index $\dim(V)$. Then V has at most one monomial structure.*

Proof. Every monomial datum is supported on the unique subgroup H of G of index $\dim(V)$. By uniqueness, the group H is in particular normal in G, so if (H, χ) is a monomial datum, then

$$\mathrm{res}_H^G(V) \ \cong \ \mathrm{res}_H^G(\mathrm{ind}_H^G(\chi)) \ \cong \ \bigoplus_{gH \in G/H} \chi \circ c_g \ .$$

So any other monomial datum is of the form $(H, \chi \circ c_g)$ for some $g \in G$, and hence conjugate to (H, χ). □

Now we can establish our formula for the geometric fixed points of the pre-Bott class of an irreducible representation. There are irreducible G-representations V without monomial structure, for example if G does not have any subgroup of index $\dim(V)$. In such cases, the following theorem says that $\Phi^G(b_{G,V}) = 0$.

Theorem 3.10. *For every non-trivial irreducible unitary representation V of a compact Lie group G, the relation*

$$\Phi^G(b_{G,V}) \ = \ \sum_{[H,\chi] \in \mathrm{mon}(V)} (1 - \mathrm{tr}_H^G(\chi)^\sharp)$$

holds in the ring $\Phi_0^G(\Sigma_+^\infty \mathbf{P})$. The sum on the right hand side runs over a set of representatives of the monomial structures on V.

Proof. We start with the special case where G is a closed subgroup of $U(n)$, and $V = \mathrm{res}_G^{U(n)}(\nu_n)$ is the restriction of the tautological representation. We shall exhibit a bijection between the G-fixed point of the coset space $U(n)/(\Sigma_n \wr T)$ and the set of monomial structures on V.

We write $\mathcal{L}_n$ for the space of sets $\ell = \{L_1, \dots, L_n\}$ of n pairwise orthogonal 1-dimensional subspaces of $\mathbb{C}^n$. The tautological $U(n)$-action $\mathbb{C}^n$ induces a transitive action on $\mathcal{L}_n$, and the stabilizer of the set $\{\mathbb{C}\cdot e_1, \dots, \mathbb{C}\cdot e_n\}$ is the subgroup $\Sigma_n \wr T$, embedded into $U(n)$ by (2.6). So the map

$$U(n)/(\Sigma_n \wr T) \ \cong\ \mathcal{L}_n \ , \quad A\cdot(\Sigma_n \wr T) \ \longmapsto\ \{\mathbb{C}\cdot A\cdot e_1, \dots, \mathbb{C}\cdot A\cdot e_n\}$$

is a $U(n)$-equivariant homeomorphism. Hence the induced map on G-fixed points

$$(U(n)/(\Sigma_n \wr T))^G \ \cong\ (\mathcal{L}_n)^G$$

is a homeomorphism, too.

An element ℓ of $\mathcal{L}_n$ is G-fixed precisely if G permutes the n lines in ℓ. Since G acts irreducibly on ν_n, the action of G on such a set $\ell \in (\mathcal{L}_n)^G$ must be transitive. Hence the stabilizer H of every element $L \in \ell$ has index n in G. We let $\chi\colon H \longrightarrow T$ be the character that encodes the H-action on L, i.e.,

$$h\cdot x \ =\ \chi(h)\cdot x$$

for all $(h,x) \in H\times L$. We let $x \in L$ be a unit vector in L. Then the map

$$V \ \xrightarrow{\ \cong\ }\ \operatorname{ind}_H^G(\chi) \ , \quad x \ \longmapsto\ \{g \mapsto \langle g\cdot x, x\rangle\} \tag{3.11}$$

is an isomorphism of G-representations, so (H,χ) is a monomial datum for V. If L' is another element of ℓ, then $L' = gL$ for some $g \in G$, so the stabilizer of L' is ${}^gH$, and ${}^gH$ acts on L' through the character $\chi\circ c_g$. So the G-conjugacy class of the monomial datum is independent of the choice of line in $\ell \in (\mathcal{L}_n)^G$. In other words, the map

$$(\mathcal{L}_n)^G \ \longrightarrow\ \operatorname{mon}(V) \ , \quad \ell \ \longmapsto [H,\chi] \tag{3.12}$$

is well-defined.

Now we let ℓ, ℓ' be two elements of $(\mathcal{L}_n)^G$ such that $[H,\chi] = [H',\chi']$ in $\operatorname{mon}(V)$. Then there are lines $L \in \ell$ and $L' \in \ell'$ such that the associated monomial data (H,χ) and (H',χ') are G-conjugate. So after replacing L' by a suitable G-translate, if necessary, we can assume that L and L' have the same stabilizer group H, and the H-action on L and L' is given by the same character χ. After choosing unit vectors $x \in L$ and $x' \in L'$, each define an isomorphism of G-representations (3.11). By direct inspection, these send x and x', respectively, to the distinguished function $\chi^\dagger\colon G \longrightarrow T$ defined by

$$\chi^\dagger(\gamma) \ =\ \begin{cases} \chi(\gamma) & \text{for } \gamma \in H\text{, and} \\ 0 & \text{for } \gamma \notin H. \end{cases} \tag{3.13}$$

Since V is irreducible, the two isomorphisms differ by multiplication by a scalar in T, so we must have $x' = \lambda x$ for some $\lambda \in T$. In particular $L' = \mathbb{C}\cdot x' = \mathbb{C}\cdot x = L$. Hence also the G-orbits $\ell = G\cdot L$ and $\ell' = G\cdot L'$ are equal in $(\mathcal{L}_n)^G$. In other words, the map (3.12) is injective.

If (H,χ) is any monomial datum on V, then the induced representation $\operatorname{ind}_H^G(\chi)$ comes with a preferred G-invariant set of pairwise orthogonal lines, namely the set

$$\{\mathbb{C}\cdot g\cdot \chi^\dagger\}_{g\in G}$$

of lines spanned by the G-translates of the distinguished function (3.13). Moreover, H is the stabilizer group of the line $\mathbb{C}\cdot\chi^\dagger$, and H acts on this line by the character χ. So the image of this set under any isomorphism of G-representations

$$\operatorname{ind}_H^G(\chi) \ \cong\ V$$

is an element of $(\mathcal{L}_n)^G$, and the image of $\mathbb{C}\cdot\chi^\dagger$ in an element with stabilizer group H and character χ. So the map (3.12) is surjective, and hence bijective. In sum, we have now established a bijection

$$(U(n)/(\Sigma_n \wr T))^G \ \xrightarrow{\ \cong\ }\ \operatorname{mon}(V) \tag{3.14}$$

that sends a G-fixed coset $A\cdot(\Sigma_n\wr T)$ to the monomial structure $[H,\chi]$, where H is the stabilizer group of the line $\mathbb{C}\cdot A\cdot e_1$, and $\chi\colon H\longrightarrow T$ encodes the H-action on this line.

Now we prove the claim in the special case. Proposition 3.4.2 (ii) of [38] shows that

$$\Phi^G\circ \mathrm{tr}_{\Sigma_n\wr T}^{U(n)} \ = \sum_{A(\Sigma_n\wr T)\in(U(n)/(\Sigma_n\wr T))^G} \Phi^G\circ A_\star\circ \mathrm{res}^{\Sigma_n\wr T}_{G^A}\ .$$

We have used that the space $(U(n)/(\Sigma_n\wr T))^G$ is finite by the bijection (3.14) and the finiteness of $\mathrm{mon}(V)$, so that its path components are points and have Euler characteristic equal to 1. Then

$$\begin{aligned}\Phi^G(b_{G,V}) \ &= \ \Phi^G(b_n) \ = \ \Phi^G(\mathrm{tr}_{\Sigma_n\wr T}^{U(n)}(P^n(b_1)))\\ &= \sum_{A(\Sigma_n\wr T)\in(U(n)/(\Sigma_n\wr T))^G} \Phi^G(A_\star(\mathrm{res}^{\Sigma_n\wr T}_{G^A}(P^n(b_1))))\ .\end{aligned}$$

We use the bijection (3.14) to replace the indexing set of the sum by the set of monomial structures on V, and we rewrite the corresponding summand.

Given any G-fixed point $A(\Sigma_n\wr T)\in(U(n)/(\Sigma_n\wr T))^G$, we set $v_i=A\cdot e_i$, where e_i is the ith vector of the standard basis of $\mathbb{C}^n$. Then $(v_1,\dots,v_n)$ is a monomial basis for V, i.e., an orthonormal basis in which the G-action is given entirely by monomial matrices in $\Sigma_n\wr T$. We let H be the stabilizer group of the line $\mathbb{C}\cdot v_1$, and we let $\chi\colon H\longrightarrow T$ encode the H-action on this line, so that $[H,\chi]$ is the monomial structure corresponding to $A(\Sigma_n\wr T)$ under the bijection (3.14).

Because G acts irreducibly on V, it acts transitively on the set of lines $\{\mathbb{C}\cdot v_1,\dots,\mathbb{C}\cdot v_n\}$. So we can choose an H-transversal $\bar g=(g_1,\dots,g_n)$ such that $g_i\cdot v_1=\lambda_i\cdot v_i$ for some $\lambda_i\in T$, where we insist on the normalization $g_1=1$, which forces $\lambda_1=1$. The matrix $B=A\cdot(1;\lambda_1,\dots,\lambda_n)$ is then another representative of the same coset $A(\Sigma_n\wr T)=B(\Sigma_n\wr T)$. Moreover, $B\cdot e_i=A\cdot\lambda_i\cdot e_i=\lambda_i\cdot v_i$. We claim that the following diagram commutes:

$$\begin{tikzcd} G \arrow[r, "\Psi_{\bar g}"] \arrow[d, "c_B"'] & \Sigma_n\wr H \arrow[d, "\Sigma_n\wr\chi"] \\ G^B \arrow[r, "\mathrm{incl}"'] & \Sigma_n\wr T \end{tikzcd}$$

To this end we consider $\gamma\in G$ and set $\Psi_{\bar g}(\gamma)=(\sigma;h_1,\dots,h_n)$, which means that

$$\gamma\cdot g_i \ = \ g_{\sigma(i)}\cdot h_i$$

for all $1\le i\le n$. Then

$$\begin{aligned}c_B(\gamma)\cdot e_i \ &= \ B^{-1}\cdot\gamma\cdot B\cdot e_i \ = \ B^{-1}\cdot\gamma\cdot\lambda_i\cdot v_i \ = \ B^{-1}\cdot\gamma\cdot g_i\cdot v_1\\ &= \ B^{-1}\cdot g_{\sigma(i)}\cdot h_i\cdot v_1 \ = \ B^{-1}\cdot g_{\sigma(i)}\cdot\chi(h_i)\cdot v_1 \ = \ \chi(h_i)\cdot B^{-1}\cdot\lambda_{\sigma(i)}\cdot v_{\sigma(i)}\\ &= \ \chi(h_i)\cdot e_{\sigma(i)} \ = \ (\sigma;\chi(h_1),\dots,\chi(h_n))\cdot e_i \ = \ (\Sigma_n\wr\chi)(\Psi_{\bar g}(\gamma))\cdot e_i\ ,\end{aligned}$$

also for all $1\le i\le n$. Since the matrices $c_B(\gamma)$ and $((\Sigma_n\wr\chi)\circ\Psi_{\bar g})(\gamma)$ have the same effect on a basis of $\mathbb{C}^n$, they agree.

The commutative square then yields

$$\begin{aligned}\Phi^G(B_\star(\mathrm{res}^{\Sigma_n\wr T}_{G^B}(P^n(b_1)))) \ &= \ \Phi^G(\Psi_{\bar g}^*((\Sigma_n\wr\chi)^*(P^n(b_1))))\\ &= \ \Phi^G(\Psi^*_{\bar g}(P^n(\chi^*(b_1))))\\ &= \ \Phi^G(N_H^G(b_{H,\chi})) \ = \ 1-\mathrm{tr}_H^G(\chi)^\sharp\ .\end{aligned}$$

The third equation is the definition of the norm N_H^G, see (D.5). The last equality is Proposition 3.5 (ii). This completes the proof in the special case.

Now we treat the case where G acts faithfully on V. Then there is a continuous monomorphism $j\colon G \longrightarrow U(n)$ such that V is isomorphic to $j^*(\nu_n)$. This case follows by naturality from the previous case, applied to the image $j(G)$, a closed subgroup of $U(n)$.

Now we let $\epsilon\colon K \longrightarrow G$ be a continuous epimorphism, and we let V be a unitary G-representation. We assume that the theorem holds for V, and we will show that it then also holds for the inflated K-representation $\epsilon^*(V)$. If (L,κ) is a monomial datum on the K-representation $\epsilon^*(V)$, then κ is a summand in $\mathrm{res}^K_L(\epsilon^*(V))$, so κ vanishes on $L\cap N$, for $N=\ker(\epsilon)$. So the inflation map

$$\epsilon^* \ : \ \mathrm{mon}(V) \ \longrightarrow \ \mathrm{mon}(\epsilon^*(V)) \ , \quad [H,\chi] \ \longmapsto \ [\epsilon^{-1}(H), \chi\circ\epsilon|_{\epsilon^{-1}(H)}]$$

is bijective. Moreover, transfers commute with inflation, in the sense that

$$\epsilon^*(\mathrm{tr}_H^G(\chi)) \ = \ \mathrm{tr}^K_{\epsilon^{-1}(H)}(\epsilon^*(\chi)) \ = \ \mathrm{tr}^K_{\epsilon^{-1}(H)}(\chi\circ\epsilon|_{\epsilon^{-1}(H)}) \ .$$

Hence

$$\begin{aligned}\Phi^K(b_{K,\epsilon^*(V)}) \ &= \ \Phi^K(\epsilon^*(b_{G,V})) \ = \ \epsilon^*(\Phi^G(b_{G,V})) \ = \ \sum_{[H,\chi]\in\mathrm{mon}(V)} (1-\epsilon^*(\mathrm{tr}^G_H(\chi)^\sharp))\\ &= \ \sum_{[H,\chi]\in\mathrm{mon}(V)} (1-\mathrm{tr}^K_{\epsilon^{-1}(H)}(\chi\circ\epsilon|_{\epsilon^{-1}(H)})^\sharp) \ = \ \sum_{[L,\kappa]\in\mathrm{mon}(\epsilon^*(V))} (1-\mathrm{tr}^K_L(\kappa)^\sharp) \ .\end{aligned}$$

So the theorem holds for the K-representation $\epsilon^*(V)$.

A general unitary G-representation V is inflated from a faithful representation of the quotient group G/N, where N is the kernel of V, the closed normal subgroup of elements that act trivially on V. So the theorem holds in general. □

Example 3.15. An irreducible G-representation can have more than one monomial structure, so the right hand side of the formula in Theorem 3.10 may have more than one summand. As an example we consider the quaternion group $G=Q=\{\pm1,\pm i,\pm j,\pm k\}$, in which $i^2=j^2=k^2=ijk=-1$. The quaternion group Q has four 1-dimensional representations, and one other irreducible representation of dimension 2. The cyclic subgroups generated by the elements i, j and k are all of order 4 and normal in Q, but they are not conjugate. We let $\alpha\colon I=\{\pm1,\pm i\}\longrightarrow T$ be the homomorphism with $\alpha(i)=i$. The induced 2-dimensional representation $\mathrm{ind}_I^Q(\alpha)$ is irreducible. Similarly, we can consider any monomorphism $\beta\colon\{\pm1,\pm j\}\longrightarrow T$ and $\gamma\colon K=\{\pm1,\pm k\}\longrightarrow T$. Inducing β or γ to Q also yields 2-dimensional irreducible representations. Hence $\mathrm{ind}_I^Q(\alpha)$, $\mathrm{ind}_J^Q(\beta)$ and $\mathrm{ind}_K^Q(\gamma)$ are irreducible and isomorphic, while I, J and K are not conjugate. So $[I,\alpha]$, $[J,\beta]$ and $[K,\gamma]$ are three distinct monomial structures on the 2-dimensional irreducible Q-representation, and in fact all such monomial structures.

Now we attack the main goal of this section: showing that for every compact Lie group G that is not cyclic, there is a unitary representation V such that $\Phi^G(b_{G,V})=0$. This implies that for such groups, inverting the pre-Bott classes kills the G-geometric fixed points of $\Sigma^\infty_+\mathbf{P}$. The easy case is when the group G has an irreducible representation that is not monomial, i.e., not induced from a 1-dimensional representation of any subgroup: any such representation serves the purpose by Theorem 3.10. So a large part of the upcoming work goes into dealing with monomial groups, i.e., compact Lie groups all of whose irreducible representations are monomial.

Taketa showed in [49] that all finite monomial groups are solvable, see also [21, Corollary 5.13]; the same arguments also show that monomial compact Lie groups are solvable. The converse is not true, and the smallest witness is the solvable group $SL_2(\mathbb{F}_3)$ of order 24. Every abelian compact Lie group is monomial, because all irreducible representations are 1-dimensional. Examples of non-abelian monomial groups are the symmetric group Σ_3, the alternating group A_4, and the orthogonal group $O(2)$. Further examples of monomial groups are all groups in class (b) of Proposition 3.17.

The following definition will be convenient for our purposes.

Definition 3.16. A finite group G is *near-cyclic* if G is not cyclic, and for every normal subgroup N of G with $N \neq \{1\}$, the quotient group G/N is cyclic.

The point of the previous definition is that every non-cyclic finite group admits an epimorphism onto a near-cyclic group. The next proposition separates the near-cyclic groups into three disjoint classes. Of these, all groups in classes (a) and (b) are monomial, and those in class (c) are not.

Proposition 3.17. *Let G be a finite near-cyclic group. Then G has exactly one of the following three properties.*

(a) *The group G is an elementary abelian p-group of rank 2, for some prime p.*
(b) *The commutator subgroup G' is nontrivial and abelian, and the conjugation action of the abelianization $G_{\mathrm{ab}} = G/G'$ on $G' \setminus \{1\}$ is free.*
(c) *The group G is not solvable.*

Proof. If G is abelian and not cyclic, then it admits an epimorphism onto $C_p \times C_p$, for some prime p. So if G is abelian and near-cyclic, it belongs to class (a).

Now we suppose that G is non-abelian, i.e., its commutator subgroup $G' = [G, G]$ is nontrivial. If the commutator subgroup G' is perfect, then G is not solvable, and it belongs to class (c).

For the rest of the argument we assume that G is near-cyclic, non-abelian and the commutator subgroup G' is not perfect. We will argue in several steps that then G has property (b). Since the commutator subgroup G' is nontrivial and normal, and since G is near-cyclic, the abelianization $G_{\mathrm{ab}} = G/G'$ is cyclic. The iterated commutator subgroup $G'' = [G', G']$ is a normal subgroup of G. Since G' is not perfect, G'' is strictly contained in G'. So G/G'' is a non-abelian quotient of G. Since G is near-cyclic, G'' must be trivial, i.e., G' is abelian.

Now we show that the center of G is trivial. To this end we argue that G' is not central in G. We let $t \in G$ be any element whose residue class generates the cyclic group G_{ab}. If G' were central, then t would commute with G'; since t commutes with itself, it would be central, too. Since G' and t generate G, this would imply that G is abelian, contradicting the hypothesis. Since G' is not central, the center $Z(G)$ does not contain G'. So $G/Z(G)$ is a non-abelian quotient of G. Since G is near-cyclic, the center $Z(G)$ must be trivial.

Now we show that G belongs to class (b). We consider any element $g \in G \setminus G'$, and consider the group

$$N \ = \ C_G(g) \cap G' \ ,$$

the intersection of the centralizer of g with the commutator subgroup G'. We show that N is normal in G. Since the commutator subgroup of G is abelian, the relation

$$[a, g] \ = \ [a, gb] \tag{3.18}$$

holds for all $a, b \in G'$ and $g \in G$. We consider any element $h \in N = C_G(g) \cap G'$. Then for all $\kappa \in G$ we have

$$[{}^\kappa h, g] \ =_{(3.18)} \ [{}^\kappa h, g \cdot [g^{-1}, \kappa]] \ = \ [{}^\kappa h, {}^\kappa g] \ = \ {}^\kappa[h, g] \ = \ 1 \ .$$

So ${}^\kappa h \in N$, and thus N is normal, as claimed.

Now we show that N is strictly smaller than G'. To prove that we argue by contradiction. If $N = G'$, then g would centralize G'. We write $g = t^i \cdot \gamma$ for some $i \geq 0$ and some $\gamma \in G'$. Because G' is abelian, γ centralizes G'. Hence $t^i = g\gamma^{-1}$ centralizes G'. Since t^i also commutes with t, the class t^i is central in G. Since G is centerless, we conclude that $t^i = 1$, and thus $g = \gamma \in G'$, contradicting the hypothesis on g. This proves that N is a proper subgroup of G'. So G/N is non-abelian quotient of G. Since G is near-cyclic, we must have $N = \{1\}$. So conjugation by g does not fix any element of $G' \setminus \{1\}$, i.e., G_{ab} acts freely on $G' \setminus \{1\}$. So G belongs to class (b). □

We can now prove the main result of this section for finite groups.

Theorem 3.19. *Let G be a non-cyclic finite group. Then there is a unitary G-representation V with $V^G = 0$ such that $\Phi^G(b_{G,V}) = 0$ in $\Phi_0^G(\Sigma_+^\infty \mathbf{P})$.*

Proof. We start with the case where G is near-cyclic. Then G has one of the three properties (a), (b) or (c) specified in Proposition 3.17. We give separate constructions of the desired G-representation.

(a) For $G = C_p \times C_p$ for some prime p, we can let V be the reduced regular representation, which is isomorphic to the direct sum of all nontrivial irreducible G-representations; hence

$$b_{G,V} \ = \ \prod_{\chi\neq 1} b_{C_p\times C_p,\chi} \ ,$$

where the product is over all nontrivial homomorphisms $\chi\colon C_p \times C_p \longrightarrow T$. The relation

$$\prod_{\chi\neq 1}(1-\chi) \ = \ 0$$

holds in the group ring $\mathbb{Z}[\mathrm{Hom}(C_p \times C_p, T)]$. Multiplicativity of the pre-Bott classes then yields

$$\Phi^{C_p\times C_p}(b_{C_p\times C_p,V}) \ = \ \prod_{\chi\neq 1}\Phi^{C_p\times C_p}(b_{C_p\times C_p,\chi}) \ =_{(3.2)} \ \prod_{\chi\neq 1}(1-\chi^\sharp) \ = \ 0 \ .$$

(b) We suppose that G is non-abelian, the commutator subgroup G' is abelian, and the conjugation action of $G_{\mathrm{ab}} = G/G'$ on $G' \setminus \{1\}$ is free. In this case the order of the commutator subgroup G' is congruent to 1 modulo the order of G_{ab}, so the orders of G' and G_{ab} are coprime. Because G_{ab} acts freely on $G' \setminus \{1\}$, it also acts freely on the set of nontrivial homomorphism from G' to T. We let $\chi\colon G' \longrightarrow T$ be any nontrivial homomorphism. Then $\chi \circ c_g \neq \chi$ for all $g \in G \setminus G'$. The induced G-representation $\mathrm{ind}_{G'}^G(\chi)$ is thus irreducible.

Since the orders of G' and $G_{\mathrm{ab}} = G/G'$ are coprime, G' is the only subgroup of G of index $[G : G']$. So $[G', \chi]$ is the only monomial structure on $\mathrm{ind}_{G'}^G(\chi)$, by Proposition 3.9. Hence for $V = \mathrm{ind}_{G'}^G(\chi)$, the formula of Theorem 3.10 has only one summand, and it yields the relation

$$\Phi^G(b_{G,V}) \ = \ 1 - \mathrm{tr}_{G'}^G(\chi)^\sharp \ .$$

The transfer $\mathrm{tr}_{G'}^G\colon \mathrm{Hom}(G', T) \longrightarrow \mathrm{Hom}(G, T)$ is a group homomorphism; its source has the same order as G', and its target has the same order as $\mathrm{Hom}(G_{\mathrm{ab}}, T)$, and hence as G_{ab}. Since the orders of G' and G_{ab} are coprime, this transfer homomorphism must be trivial. Hence $\mathrm{tr}_{G'}^G(\chi) = 1$, and thus $\Phi^G(b_{G,V}) = 0$.

(c) If G is not solvable, then G cannot be a monomial group, see [21, Corollary 5.13]. Hence G has an irreducible representation V that is not induced from a 1-dimensional representation of any subgroup of G. Then $\Phi^G(b_{G,V}) = 0$ by Theorem 3.10.

It remains to treat the case of a general non-cyclic finite group. Any non-cyclic group G admits an epimorphism $\epsilon\colon G \longrightarrow G_{\mathrm{nc}}$ onto a near-cyclic group. By the above, there is a G_{nc}-representation W such that $\Phi^{G_{nc}}(b_{G_{\mathrm{nc}},W}) = 0$. Then $V = \epsilon^*(W)$ is the desired G-representation, because the geometric fixed point homomorphisms commute with inflation, see (B.9), and thus

$$\Phi^G(b_{G,V}) \ = \ \Phi^G(\epsilon^*(b_{G_{\mathrm{nc}},W})) \ = \ \epsilon^*(\Phi^{G_{\mathrm{nc}}}(b_{G_{\mathrm{nc}},W})) \ = \ 0 \ . \qquad \square$$

Now that we took care of finite groups, we turn to compact Lie groups of positive dimension. For those, most of the work goes into the case where the identity component is abelian, and the component group is cyclic.

Theorem 3.20. *Let G be a non-abelian compact Lie group whose identity component G° is abelian, and whose component group $\pi_0(G)$ is cyclic.*

(i) *There is a unitary G-representation V with $V^G = 0$ such that $\Phi^G(b_{G,V}) = 0$ in $\Phi_0^G(\Sigma_+^\infty \mathbf{P})$.*

(ii) *There is a finite index subgroup H of G and a nontrivial continuous homomorphism $\chi\colon H \longrightarrow T$ such that* $\mathrm{tr}_H^G(\chi) = 1$.

Proof. We first show a reduction scheme for epimorphisms. Let $\epsilon\colon K \longrightarrow G$ be a continuous epimorphism, and suppose that the conclusion of the theorem holds for G. Then the conclusion of the theorem also holds for K. Indeed, if V is a unitary G-representation with $V^G = 0$ and $\Phi^G(b_{G,V}) = 0$, then restricted K-representation $W = \epsilon^*(V)$ satisfies $W^K = V^G = 0$ and

$$\Phi^K(b_{K,W}) \ = \ \Phi^K(\epsilon^*(b_{G,V})) \ =_{(\mathrm{B.9})} \ \epsilon^*(\Phi^G(b_{G,V})) \ = \ 0 \ .$$

And if $\chi\colon H \longrightarrow T$ is a nontrivial continuous homomorphism from a finite index subgroup of G such that $\mathrm{tr}_H^G(\chi) = 1$, then $L = \epsilon^{-1}(H)$ is a finite index subgroup of K and $\beta = \chi \circ \epsilon|_L\colon L \longrightarrow T$ is a nontrivial continuous homomorphism such that

$$\mathrm{tr}_L^K(\beta) \ = \ \mathrm{tr}_L^K(\epsilon|_L^*(\chi)) \ = \ \epsilon^*(\mathrm{tr}_H^G(\chi)) \ = \ 1 \ .$$

So whenever properties (i) and (ii) holds for G, they also hold for K.

Now we prove the theorem in three steps, for successively more general classes of groups. In the proof of the following first case, we shall need an algebraic fact. We let A be a torsion-free abelian group equipped with a faithful action of a finite cyclic group C of order n. Such an action has a free orbit, i.e., there is an element $a \in A$ whose orbit Ca has n elements. Now let $\tau\colon A \longrightarrow B$ be a homomorphism of abelian groups that is C-equivariant for the trivial C-action on the target. We define another element $b \in A$ by

$$b \ = \ n \cdot a - (a + ta + \dots + t^{n-1}a) \ ,$$

where t is a generator of C. Because A is torsion-free, the C-orbit of b also has n elements, and b lies in the kernel of τ. So the kernel of τ contains a free C-orbit.

Case 1: The conjugation action of $\pi_0(G)$ on G° is faithful. Then also the induced $\pi_0(G)$-action on $\mathrm{Hom}(G^\circ, T)$ is faithful. Since G° is a torus, the abelian group $\mathrm{Hom}(G^\circ, T)$ is torsion-free. Each element of $\pi_0(G)$ acts on G° by the restriction of an inner automorphisms of G. So the transfer homomorphism

$$\mathrm{tr}_{G^\circ}^G \ : \ \mathrm{Hom}(G^\circ, T) \ \longrightarrow \ \mathrm{Hom}(G, T)$$

is equivariant for the trivial $\pi_0(G)$-action on the target. Since the $\pi_0(G)$-action on $\mathrm{Hom}(G^\circ, T)$ is faithful, the kernel of this transfer homomorphism contains a free $\pi_0(G)$-orbit, by the comment above. In other words: there is a continuous homomorphism $\psi\colon G^\circ \longrightarrow T$ such that $\mathrm{tr}_{G^\circ}^G(\chi) = 1$ and such that $\chi \circ c_g \neq \chi$ for all $g \in G \setminus G^\circ$. In particular, condition (ii) holds for $H = G^\circ$.

The fact that $\chi \circ c_g \neq \chi$ for all $g \in G \setminus G^\circ$ means that the induced G-representation $\mathrm{ind}_{G^\circ}^G(\chi)$ is irreducible. Since G° is the only subgroup of G whose index equals $[G : G^\circ]$, $[G^\circ, \chi]$ is the only monomial structure on $\mathrm{ind}_{G^\circ}^G(\chi)$, by Proposition 3.9. Theorem 3.10 then lets us conclude

$$\Phi^G(b_{G,\mathrm{ind}_{G^\circ}^G(\chi)}) \ = \ 1 - \mathrm{tr}_{G^\circ}^G(\chi)^\sharp \ = \ 0 \ .$$

So condition (i) holds for $V = \mathrm{ind}_{G^\circ}^G(\chi)$.

Case 2: The projection $q\colon G \longrightarrow G/G^\circ = \pi_0(G)$ has a multiplicative section. We can assume without loss of generality that $G = C \ltimes_\psi G^\circ$ is the semidirect product of an action of a finite cyclic group C on a torus G°, given by a homomorphism $\psi\colon C \longrightarrow \mathrm{Aut}(G^\circ)$. We let D be the kernel of ψ, i.e., the subgroup of C consisting of the elements that act trivially on G°. Then the C-action descends uniquely to a faithful action $\bar\psi\colon C/D \longrightarrow \mathrm{Aut}(G^\circ)$ of the quotient group, and the group G surjects onto the semidirect product $\bar G = (C/D) \ltimes_{\bar\psi} G^\circ$. Since G is nonabelian, the C-action on G° is nontrivial. Hence also the induced (C/D)-action on G° is nontrivial, and the group $\bar G$ is nonabelian. Moreover the component group $\pi_0(\bar G)$ is isomorphic to C/D, hence cyclic, in a way that identifies the conjugation action on $\bar G^\circ = G^\circ$ with the action $\bar\psi\colon C/D \longrightarrow \mathrm{Aut}(G^\circ)$. Since C/D acts faithfully on G° by design, the group $\bar{G}$ satisfies the hypothesis of the special case 1. Since the theorem holds for $\bar{G}$ by case 1, the epimorphic reduction scheme proves the theorem for the group G.

Case 3, the general case: We let $N = Z(G) \cap G^\circ$ be the intersection of the center of G with the identity component G°. We claim that the quotient group G/N is nonabelian. If G/N were abelian, then the commutator subgroup G' of G would be contained in N, and hence central in G. Since the abelianization $G_{\rm ab} = G/G'$ has a cyclic component group $\pi_0(G_{\rm ab}) = (\pi_0(G))_{\rm ab} = \pi_0(G)$, the abelianization is itself cyclic. But any group that has a central subgroup with cyclic quotient is abelian, contradicting the hypothesis on G. This shows that G/N is nonabelian. The identity component of G/N is G°/N, and thus abelian. Since N is contained in G°, the quotient map $G \longrightarrow G/N$ induces an isomorphism of component groups $\pi_0(G) \cong \pi_0(G/N)$.

We claim that the projection $G/N \longrightarrow \pi_0(G/N)$ has a multiplicative section. To this end we let $t \in G$ be an element whose class $t \cdot G^\circ$ in $G/G^\circ = \pi_0(G)$ generates the cyclic component group. Then the image of t also generates the cyclic group $\pi_0(G/N)$. If $n = [G : G^\circ]$ is the order of $\pi_0(G)$, and hence also of $\pi_0(G/N)$, then $t^n \in G^\circ$. Since G° is abelian, t^n commutes with t and with G°. Since G is generated by t and G°, the element t^n is central, and thus $t^n \in N$. So the quotient map $G/N \longrightarrow \pi_0(G/N)$ has a multiplicative section, as claimed. Since the theorem holds for G/N by case 2, the epimorphic reduction scheme proves the theorem for the group G. $\square$

We recall that a compact Lie group G is called *cyclic* if it has an element whose powers are dense in G. Then G is isomorphic to the product of a torus and a finite cyclic group, see for example [8, I Corollary 4.14]. The following theorem is the main result of this section, implying that inverting pre-Bott classes kills the G-geometric fixed points for all non-cyclic compact Lie groups.

Theorem 3.21. *Let G be a compact Lie group. If G is not cyclic, then there is a unitary G-representation V with $V^G = 0$ such that $\Phi^G(b_{G,V}) = 0$ in $\Phi_0^G(\Sigma^\infty_+\mathbf{P})$.*

Proof. We distinguish three cases.

(a) We suppose that the component group $\pi_0(G)$ is not cyclic. Then Theorem 3.19 provides a $\pi_0(G)$-representation W such that $\Phi^{\pi_0(G)}(b_{\pi_0(G),W}) = 0$. We let $q\colon G \longrightarrow G/G^\circ = \pi_0(G)$ denote the projection. Then the inflation $q^*(W)$ has the desired property, because

$$\Phi^G(b_{G,q^*(W)}) \ = \ \Phi^G(q^*(b_{\pi_0(G),W})) \ = \ q^*(\Phi^{\pi_0(G)}(b_{\pi_0(G),W})) \ = \ 0 \ .$$

(b) We suppose that the identity component G° is non-abelian. Then G° has irreducible representations of arbitrarily large dimensions, see for example [8, Chapter VI, (1.20), Exercise 2]. Since every irreducible G°-representation is a G°-equivariant summand of some irreducible G-representation, G itself has irreducible representations of arbitrarily large dimensions. In particular, there is an irreducible G-representation V whose dimension exceeds the cardinality of the finite group $\pi_0(G)$. Then G has no closed subgroup of finite index equal to the dimension of V, so V is not induced from any 1-dimensional representation of any finite index subgroup of G. Hence $\Phi^G(b_{G,V}) = 0$ by Theorem 3.10.

(c) We suppose that the component group $\pi_0(G)$ is cyclic and the identity component G° is abelian. Then G itself must be non-abelian, for otherwise it would be cyclic. So Theorem 3.20 (i) provides the desired G-representation. $\square$

As already mentioned, one feature of the global K-theory spectrum $\mathbf{KU}$ is that its geometric fixed points vanish for all non-cyclic compact Lie groups. As we shall now show, for commutative global ring spectra with this property, inverting b_1 automatically also inverts the higher pre-Bott classes.

Theorem 3.22. *Let R be a commutative global homotopy ring spectrum, and let $j\colon \Sigma^\infty_+\mathbf{P} \longrightarrow R$ be a morphism of global homotopy ring spectra that inverts the pre-Bott class b_1. Then the following two conditions are equivalent.*

(a) *The morphism j inverts the pre-Bott classes b_n for all $n \geq 1$.*
(b) *The geometric fixed point ring $\Phi_0^H(R)$ is trivial for all non-cyclic compact Lie groups H.*
(c) *The geometric fixed point ring $\Phi_0^H(R)$ is trivial for all non-abelian compact Lie groups H.*

Proof. (a)$\Longrightarrow$(b) We let H be a non-cyclic compact Lie group. Theorem 3.21 provides a unitary H-representation V with $V^H = 0$ such that $\Phi^H(b_{H,V}) = 0$ in $\Phi_0^H(\Sigma_+^\infty \mathbf{P})$. Because j inverts all pre-Bott classes b_n, and because $b_{H,V}$ is a restriction of b_n along a continuous homomorphism, j also inverts the class $b_{H,V}$ by Proposition 1.12. Hence the class $\Phi^H(j_*(b_{H,V})) = j_*(\Phi^H(b_{H,V}))$ is simultaneously invertible and zero. So the ring $\Phi_0^H(R)$ is trivial.

If H is non-abelian, then it is in particular not cyclic, so condition (b) implies condition (c).

(c)$\Longrightarrow$(a) We let H be a closed subgroup of $U(n)$. If H is not abelian, then $\Phi_0^H(R) = 0$ by hypothesis, and hence the entire graded ring $\Phi_*^H(R)$ is trivial. Thus all its elements, in particular the class $\Phi^H(b_n)$, are invertible.

If H is abelian, then it is subconjugate to the diagonal maximal torus T^n. The relation

$$\mathrm{res}^{U(n)}_{T^n}(j_*(b_n)) \;=\; j_*(\mathrm{res}^{U(n)}_{T^n}(b_n)) \;=\; j_*(b_1 \times \cdots \times b_1) \;=\; j_*(b_1) \times \cdots \times j_*(b_1)$$

holds by Theorem 2.10 (i). Because $j_*(b_1)$ is invertible, also $j_*(b_1) \times \cdots \times j_*(b_1)$ is invertible by Proposition 1.9 (iii) and Proposition 1.12. Hence also the restriction of $j_*(b_n)$ to T^n is invertible by Proposition 1.9 (ii). Since H is subconjugate to T^n, also $\Phi^H(j_*(b_n))$ is invertible by Proposition 1.5 (c). Given that all the classes $\Phi^H(j_*(b_n))$ are invertible, the class $j_*(b_n)$ itself is invertible by Proposition 1.5 (a). $\square$

4. Global K-theory as a global localization

In this section we show our main result, Theorem 4.8, namely that the morphism $\eta\colon \Sigma_+^\infty \mathbf{P} \longrightarrow \mathbf{KU}$ is a global localization by the set of pre-Bott classes $\{b_n\}_{n\geq 1}$, both in the ∞-category of commutative global ring spectra and also in the ∞-category of ultra-commutative ring spectra. To set the stage, we first review some facts about the global K-theory spectrum $\mathbf{KU}$ and about the morphism $\eta\colon \Sigma_+^\infty \mathbf{P} \longrightarrow \mathbf{KU}$ of ultra-commutative ring spectra. The key ingredient to our main result is Theorem 4.6, saying that for every compact Lie group G, the morphism $\Phi_*^G(\eta)\colon \Phi_*^G(\Sigma_+^\infty \mathbf{P}) \longrightarrow \Phi_*^G(\mathbf{KU})$ of geometric fixed point rings is a localization of graded-commutative rings by the images of the pre-Bott classes of all unitary G-representations.

The global K-theory spectrum $\mathbf{KU}$ was introduced by Joachim [23, Definition 4.3], see also [38, Construction 6.4.9]; for every compact Lie group G, the underlying genuine G-spectrum of $\mathbf{KU}$ represents G-equivariant complex K-theory, see [23, Theorem 4.4] or [38, Corollary 6.4.23]. In particular, the equivariant homotopy group $\pi_0^G(\mathbf{KU})$ is isomorphic to the complex representation ring of the compact Lie group G; a specific ring isomorphism $R(G) \cong \pi_0^G(\mathbf{KU})$ is described in [38, Theorem 6.4.24]. As the group G varies, these isomorphisms are compatible with all the additional structure in sight, i.e., with restriction homomorphisms, transfers, power operations and norms, see again [38, Theorem 6.4.24].

Construction 4.1 (The T-equivariant Bott class)**.** A specific T-equivariant refinement of the classical Bott class will play an important role for us. In essence, it is the reduced K-theory class of the tautological T-equivariant line bundle γ over $P(\mathbb{C} \oplus \nu_1)$. In more detail, we restrict the class of this tautological line bundle along the T-equivariant homeomorphism

$$\psi \;:\; S^{\nu_1} \xrightarrow{\;\cong\;} P(\mathbb{C} \oplus \nu_1)\;, \quad \lambda \mapsto [1 : \lambda]$$

to obtain a class in $KU_T(S^{\nu_1})$. The T-representation over the fixed point at ∞ of $\psi^*(\gamma)$ is then the tautological representation ν_1. So the formal difference

$$\beta \;=\; [\gamma] - [\underline{\nu_1}] \;\in\; \widetilde{KU}_T(S^{\nu_1}) \tag{4.2}$$

with the trivial line bundle with fiber ν_1 is a reduced T-equivariant K-theory class. Equivariant Bott periodicity says that for every finite based T-CW complex X, reduced product with this class is an isomorphism

$$- \cdot \beta \ : \ \widetilde{KU}_T(X) \xrightarrow{\cong} \widetilde{KU}_T(X \wedge S^{\nu_1}) \ ,$$

see for example [2, Theorem 4.3], [41, Proposition 3.3] or [24, Théorème 3.1]. For $X = S^0$ this says in particular that β freely generates $\widetilde{KU}_T(S^{\nu_1})$ as a module over the representation ring $R(T)$.

We turn the Bott classes into a ν_1-graded T-equivariant homotopy class of the global K-theory spectrum $\mathbf{KU}$ via the isomorphism

$$\widetilde{KU}_T(S^{\nu_1}) \xrightarrow{\cong} \widetilde{\mathbf{KU}}^0_T(S^{\nu_1}) \ = \ \pi^T_{\nu_1}(\mathbf{KU})$$

established in [38, Corollary 6.4.23]. We abuse notation and denote the image of the Bott class (4.2) by the same name

$$\beta \ \in \ \pi^T_{\nu_1}(\mathbf{KU}) \ ,$$

and we refer to it as the *T-equivariant Bott class*. Since the fiber of the T-equivariant line bundle $\psi^*(\gamma)$ over the non-basepoint 0 of S^{ν_1} is the trivial T-representation, the image of $[\gamma] - [\underline{\nu_1}]$ under the restriction homomorphism $i^* \colon \widetilde{KU}_T(S^{\nu_1}) \longrightarrow \widetilde{KU}_T(S^0) = R(T)$ is the class $1 - \nu_1$. Hence the *Euler class*

$$i^*(\beta) \in \ \pi^T_0(\mathbf{KU})$$

corresponds to $1 - \nu_1$ under the preferred ring isomorphism $R(T) \cong \pi^T_0(\mathbf{KU})$ from the complex representation ring.

Construction 4.3. The morphism of non-equivariant spectra from $\Sigma^\infty_+ \mathbb{C}P^\infty$ to KU that classifies the tautological complex line bundle over $\mathbb{C}P^\infty$ has a particularly nice and prominent global-equivariant refinement, a morphism of ultra-commutative ring spectra:

$$\eta \ : \ \Sigma^\infty_+ \mathbf{P} \ \longrightarrow \ \mathbf{KU} \ .$$

The morphism is defined as a composite of two morphisms of ultra-commutative ring spectra

$$\Sigma^\infty_+ \mathbf{P} \ \xrightarrow{\mu} \ \mathbf{ku} \ \xrightarrow{j} \ \mathbf{KU} \ .$$

The intermediate object is the connective global K-theory spectrum [38, Construction 6.3.9], and the first morphism μ is the inclusion of the 'rank 1' part in the rank filtration, compare [38, Construction 6.3.40]. The second morphism j is a 'periodization morphism' $j \colon \mathbf{ku} \longrightarrow \mathbf{KU}$ defined in [38, Construction 6.4.13].

The underlying morphism of global spectra classifies the tautological T-representation on $\mathbb{C}$, in the following sense. As mentioned earlier, $\mathbf{P}$ is a global classifying space for the group T, so the pair $(\Sigma^\infty_+ \mathbf{P}, e_T)$ represents the functor π^T_0, where $e_T \in \pi^T_0(\Sigma^\infty_+ \mathbf{P})$ is the stable tautological class (2.3). Under the preferred identification [38, Theorem 6.4.24] of $\pi^T_0(\mathbf{KU})$ with the complex representation ring $RU(T)$, the element e_T maps to the class of the tautological T-representation on $\mathbb{C}$, i.e.,

$$\eta_*(e_T) \ = \ \nu_1 \quad \text{in } \pi^T_0(\mathbf{KU}) \cong R(T).$$

Because η is a morphism of ultra-commutative ring spectra, its effect on equivariant homotopy groups is not only compatible with restriction, inflations and transfers, but also with multiplicative power operations and norms.

Our main result in particular says that the morphism η inverts the pre-Bott classes, which we show now.

Theorem 4.4. *Let $\eta \colon \Sigma^\infty_+ \mathbf{P} \longrightarrow \mathbf{KU}$ be a morphism of commutative global ring spectra such that $\eta_*(e_T) = \nu_1$ in $\pi^T_0(\mathbf{KU}) = R(T)$.*

(i) *The morphism η sends the pre-Bott class b_1 to the T-equivariant Bott class β.*
(ii) *The morphism η inverts the pre-Bott classes b_n for all $n \geq 1$.*

Proof. (i) Since the restriction map $\mathrm{res}^T_{\{1\}} : \pi_0^T(\mathbf{KU}) \longrightarrow \pi_0(\mathbf{KU})$ is split by the inflation homomorphism, the long exact sequence in T-equivariant $\mathbf{KU}$-cohomology associated to the cofiber sequence of based T-spaces (2.5) becomes a short exact sequence:

$$0 \longrightarrow \pi^T_{\nu_1}(\mathbf{KU}) \xrightarrow{\ i^*\ } \pi_0^T(\mathbf{KU}) \xrightarrow{\mathrm{res}^T_{\{1\}}} \pi_0(\mathbf{KU}) \longrightarrow 0$$

Hence restriction along the inclusion $i: S^0 \longrightarrow S^{\nu_1}$ of T-fixed points is injective. We noted in the construction of the T-equivariant Bott class that $i^*(\beta) = 1 - \nu_1$ in the ring $\pi_0^T(\mathbf{KU}) \cong R(T)$. Because

$$i^*(\eta_*(b_1)) \ = \ \eta_*(i^*(b_1)) \ = \ \eta_*(1 - e_T) \ = \ 1 - \nu_1$$

and i^* is injective, we conclude that $\eta_*(b_1) = \beta$.

(ii) We start by showing that the T-equivariant Bott class β is invertible in the sense of Definition 1.3. While the Thom isomorphism says that exterior multiplication by β is an isomorphism of reduced equivariant K-groups, this is not quite the same statement as invertibility in the sense of Definition 1.3, so we need an extra argument.

We write $a \in \pi_0^T(\mathbf{KU} \wedge S^{\nu_1})$ for the Hurewicz image in T-equivariant K-theory of the pre-Euler class of the tautological T-representation, represented by the fixed point inclusion $i: S^0 \longrightarrow S^{\nu_1}$. Then

$$\beta^\flat_*(a \cdot x) \ = \ \beta^\flat_*(a) \cdot x \ = \ i^*(\beta) \cdot x \ = \ (1 - \nu_1) \cdot x$$

for all $x \in \pi_0^T(\mathbf{KU})$. So the left square in the following diagram of abelian groups commutes:

$$\begin{tikzcd}
0 \arrow[r] & \pi_0^T(\mathbf{KU}) \arrow[r, "a\cdot -"] \arrow[d, equal] & \pi_0^T(\mathbf{KU} \wedge S^{\nu_1}) \arrow[r, "\mathrm{res}^T_{\{1\}}"] \arrow[d, "\beta^\flat_*"'] & \pi_0(\mathbf{KU} \wedge S^2) \arrow[r] \arrow[d, "\cong"', "\beta^\flat_*"] & 0 \\
0 \arrow[r] & \pi_0^T(\mathbf{KU}) \arrow[r, "(1-\nu_1)\cdot -"'] & \pi_0^T(\mathbf{KU}) \arrow[r, "\mathrm{res}^T_{\{1\}}"'] & \pi_0(\mathbf{KU}) \arrow[r] & 0
\end{tikzcd}$$

The right square commutes because $\beta^\flat : \mathbf{KU} \wedge S^{\nu_1} \longrightarrow \mathbf{KU}$ is a morphism of T-spectra. We claim that both rows are exact. The upper row is part of the long exact homology sequence of the cofiber sequence of based T-spaces (2.5), combined with the Wirthmüller isomorphism $\pi_0^T(\mathbf{KU} \wedge T_+ \wedge S^1) \cong \pi_0(\mathbf{KU} \wedge S^2)$. It is exact because the adjacent terms are non-equivariant homotopy groups of $\mathbf{KU}$ of odd degrees, and thus trivial. The lower row is exact because $\pi_0^T(\mathbf{KU}) \cong R(T) = \mathbb{Z}[\nu_1^{\pm}]$ is a Laurent polynomial ring on the class ν_1, and hence a domain whose augmentation ideal is generated by $1 - \nu_1$.

By design the T-equivariant Bott class restricts to the classical non-equivariant Bott class $\beta_{cl} = \mathrm{res}^T_{\{1\}}(\beta)$. This classical Bott class is invertible in the graded ring $\pi_*(\mathbf{KU})$; in the particular model for complex K-theory that we work with, this is proved in [38, Theorem 6.4.29]. So the right vertical map in the above commutative diagram is bijective. The middle vertical map is thus bijective by the five lemma. In particular, there is a class $u \in \pi_0^T(\mathbf{KU} \wedge S^{\nu_1})$ such that $\beta^\flat_*(u) = 1$, and so the Bott class β is invertible by Proposition 1.5.

Because $\eta_*(b_1) = \beta$ by part (i), we have shown that η inverts b_1. Since the geometric fixed point rings $\Phi_0^H(\mathbf{KU})$ vanish for all non-cyclic compact Lie groups H, the morphism η inverts all pre-Bott classes b_n by Theorem 3.22. □

Remark 4.5 (Pre-Bott classes map to Bott classes)**.** We have just shown in Theorem 4.4 that any morphism $\eta: \Sigma^\infty_+ \mathbf{P} \longrightarrow \mathbf{KU}$ of commutative global ring spectra satisfying $\eta_*(e_T) = \nu_1$ inverts all the pre-Bott classes b_n. An invertible class in the same representation-grading as $\eta_*(b_n)$ is the $U(n)$-equivariant Bott class, also called the *Thom class*, of the tautological $U(n)$-representation ν_n, as defined for example by Atiyah [2, Section 4], Segal [41, §3], or Karoubi [24, Section III]. We claim that the class $\eta_*(b_n)$ indeed coincides with this $U(n)$-equivariant Bott class $\beta_{U(n),\nu_n}$ under the isomorphism

$$\widetilde{KU}_{U(n)}(S^{\nu_n}) \xrightarrow{\cong} \tilde{\mathbf{KU}}^0_{U(n)}(S^{\nu_n}) = \pi^{U(n)}_{\nu_n}(\mathbf{KU})$$

established in [38, Corollary 6.4.23]. For $n = 1$ the morphism

$$\pi^T_{\nu_1}(\eta) \ : \ \pi^T_{\nu_1}(\Sigma^\infty_+\mathbf{P}) \ \longrightarrow \ \pi^T_{\nu_1}(\mathbf{KU})$$

is in fact an isomorphism. So b_1 is the unique element that maps to the T-equivariant Bott class of the tautological T-representation. For $n \geq 2$, the map $\pi^{U(n)}_{\nu_n}(\eta)$ is not injective, so the equivariant Bott class $\beta_{U(n),\nu_n}$ has many other preimages besides b_n.

All we need to know about equivariant Bott classes to verify our claim is that they are multiplicative and natural for restriction. This yields

$$\begin{aligned} \mathrm{res}^{U(n)}_{T^n}(\eta_*(b_n)) \ &= \ \eta_*(\mathrm{res}^{U(n)}_{T^n}(b_n)) \ = \ \eta_*(b_1 \times \cdots \times b_1) \\ &= \ \eta_*(b_1) \times \cdots \times \eta_*(b_1) \ = \ \beta \times \cdots \times \beta \\ &= \ \beta_{T^n,\nu_1^n} \ = \ \mathrm{res}^{U(n)}_{T^n}(\beta_{U(n),\nu_n}) \ . \end{aligned}$$

Because the restriction maps of representation rings $\mathrm{res}^{U(n)}_{T^n}\colon R(U(n)) \longrightarrow R(T^n)$ is injective, so is the restriction map $\mathrm{res}^{U(n)}_{T^n}\colon \pi^{U(n)}_{\nu_n}(\mathbf{KU}) \longrightarrow \pi^{T^n}_{\nu_1^n}(\mathbf{KU})$. Hence the previous relation implies that

$$\eta_*(b_n) \ = \ \beta_{U(n),\nu_n} \ .$$

The next theorem identifies the morphism induced by $\eta\colon \Sigma^\infty_+\mathbf{P} \longrightarrow \mathbf{KU}$ on G-geometric fixed points as a specific localization. This property will be the key calculational input for the eventual proof that $\eta\colon \Sigma^\infty_+\mathbf{P} \longrightarrow \mathbf{KU}$ is a localization by the pre-Bott classes.

We will make our statements about the morphism $\eta\colon \Sigma^\infty_+\mathbf{P} \longrightarrow \mathbf{KU}$ independent of the specific construction, thereby making it future-proof in the following sense. At the time of this writing, there does not yet seem to be a model-independent definition of ultra-commutative ring spectra, nor of the morphism η in this category. However, I expect it to just be a matter of time until the technology will catch up and provide an intrinsically higher categorical construction of the ∞-category of ultra-commutative ring spectra. For me, a litmus test for such a construction is whether it accommodates $\mathbf{KU}$ as an ultra-commutative ring spectrum, as well as the morphism $\eta\colon \Sigma^\infty_+\mathbf{P} \longrightarrow \mathbf{KU}$, without reference to models. If so, then our localization results will automatically apply to any such incarnation of η.

Theorem 4.6. *Let $\eta\colon \Sigma^\infty_+\mathbf{P} \longrightarrow \mathbf{KU}$ be a morphism of commutative global ring spectra such that $\eta_*(e_T) = \nu_1$ in $\pi^T_0(\mathbf{KU}) = R(T)$. Then for every compact Lie group G, the morphism of geometric fixed point rings*

$$\Phi^G_*(\eta) \ : \ \Phi^G_*(\Sigma^\infty_+\mathbf{P}) \ \longrightarrow \ \Phi^G_*(\mathbf{KU})$$

is a localization by the set of classes $\Phi^G(b_{G,V})$ for all unitary G-representations V.

Proof. If the group G is not cyclic, then Theorem 3.21 provides a G-representation V with $V^G = 0$ such that $\Phi^G(b_{G,V}) = 0$ in the ring $\Phi^G_0(\Sigma^\infty_+\mathbf{P})$. So the localization of $\Phi^G_*(\Sigma^\infty_+\mathbf{P})$ by $\Phi^G(b_{G,V})$ is trivial. Since the class $\eta_*(b_{G,V})$ is invertible in $\pi^G_V(\mathbf{KU})$ by Theorem 4.4, the class $\Phi^G(\eta_*(b_{G,V})) = \eta_*(\Phi^G(b_{G,V}))$ is both invertible and zero. So the graded ring $\Phi^G_*(\mathbf{KU})$ is trivial.

Now we treat the case where G is cyclic, hence abelian. Then all irreducible G-representations are 1-dimensional, so we must show that $\Phi^G_*(\eta)$ is a localization away from the classes $\Phi^G(b_{G,\lambda})$ for all continuous homomorphisms $\lambda\colon G \longrightarrow T$. Proposition 3.3 (ii) shows that the geometric fixed point ring $\Phi^G_*(\Sigma^\infty_+\mathbf{P})$ is a group ring of the group $\mathrm{Hom}(G,T)$ over the non-equivariant homotopy ring $\pi_*^G(\Sigma_+^\infty \mathbf{P})$. After this identification, the morphism $\Phi_*^G(\eta)$ factors as the composite of three morphisms of graded-commutative rings:

$$\pi_*(\Sigma_+^\infty \mathbf{P})[\mathrm{Hom}(G,T)] \xrightarrow{\pi_*(\eta)[\mathrm{Hom}(G,T)]} KU_*[\mathrm{Hom}(G,T)] \xrightarrow[\cong]{\alpha\mapsto\alpha^*(\nu_1)} \pi_*^G(\mathbf{KU}) \xrightarrow{\Phi^G} \Phi_*^G(\mathbf{KU})$$

In the second morphism, $\pi_*^G(\mathbf{KU})$ is viewed as a KU_*-algebra via inflation. We showed in Theorem 4.4 that η sends the pre-Bott class b_1 to the T-equivariant Bott class β. So the underlying morphism of non-equivariant ring spectra sends the classical non-equivariant (pre-)Bott class $b_{cl} = \mathrm{res}^T_{\{1\}}(b_1)$ in $\pi_2(\Sigma_+^\infty \mathbf{P})$ to the classical Bott class in $\pi_2(\mathbf{KU})$. By Snaith's theorem, the morphism $\eta_*\colon \pi_*(\Sigma_+^\infty \mathbf{P}) \longrightarrow KU_*$ of non-equivariant homotopy rings is a localization by b_{cl}. So the first morphism in the composite is a localization by $b_{cl} = \Phi^G(b_{G,\mathbb{C}})$. The second morphism is an isomorphism because G is cyclic, and hence all irreducible G-representations are 1-dimensional.

If $\lambda\colon G \longrightarrow T$ is nontrivial, then $\Phi^G(b_{G,\lambda}) = 1-\lambda^\sharp$ by (3.2); hence the isomorphism of Proposition 3.3 (i) identifies $\Phi^G(b_{G,\lambda})$ with $1-\lambda$. The first two maps in the above composite send $1-\lambda$ to $1-\lambda^*(\nu_1) \in \pi_0^G(\mathbf{KU})$. It is a well-known fact that the geometric fixed point map $\Phi^G\colon \pi_*^G(\mathbf{KU}) \longrightarrow \Phi_*^G(\mathbf{KU})$ is localization away from the set of classes $1-\lambda$ for all nontrivial continuous homomorphisms $\lambda\colon G \longrightarrow T$; we recall the classical argument. The global K-theory spectrum is G-equivariantly complex oriented, with the Bott classes as Thom classes. So the geometric fixed point morphism $\Phi^G\colon \pi_*^G(\mathbf{KU}) \longrightarrow \Phi_*^G(\mathbf{KU})$ presents the target as the localization of the source by the Euler classes of all nontrivial irreducible complex G-representations, see [50, Proposition 7.4.1] or [18, Proposition 3.20]. Moreover, the Euler class of λ is precisely

$$i^*(\beta_{G,\lambda}) \;=\; i^*(\lambda^*(\beta)) \;=\; \lambda^*(i^*(\beta)) \;=\; \lambda^*(1-\nu_1) \;=\; 1-\lambda^*(\nu_1)\ .$$

We have now factored the morphism $\Phi_*^G(\eta)\colon \Phi_*^G(\Sigma_+^\infty \mathbf{P}) \longrightarrow \Phi_*^G(\mathbf{KU})$ as the composite of two localizations:

- a morphism $\Phi_*^G(\Sigma_+^\infty \mathbf{P}) \longrightarrow \pi_*^G(\mathbf{KU})$ that inverts the class $p_G^*(\beta_{cl}) = \Phi^G(b_{G,\mathbb{C}})$ in $\Phi_2^G(\Sigma_+^\infty \mathbf{P})$ arising from the trivial homomorphism $G \longrightarrow T$;
- followed by the morphism $\Phi^G\colon \pi_*^G(\mathbf{KU}) \longrightarrow \Phi_*^G(\mathbf{KU})$ that inverts the images of the classes $\Phi^G(b_{G,\lambda})$ for all nontrivial homomorphisms $\lambda\colon G \longrightarrow T$.

So the composite $\Phi_*^G(\eta)\colon \Phi_*^G(\Sigma_+^\infty \mathbf{P}) \longrightarrow \Phi_*^G(\mathbf{KU})$ is a localization by the classes $\Phi^G(b_{G,\lambda})$ for all continuous homomorphisms $\lambda\colon G \longrightarrow T$, trivial or not. □

The following theorem is a variation of Theorem 4.6 that we will need in the proof of Theorem 6.3 below. That theorem identifies the ultra-commutative localization of $\Sigma_+^\infty \mathbf{P}$ by the first pre-Bott class b_1 as the left $\mathcal{A}b^\circ$-approximation of $\mathbf{KU}$, where $\mathcal{A}b^\circ$ is the global family of compact Lie groups whose identity component is abelian.

Theorem 4.7. *Let $\eta\colon \Sigma_+^\infty \mathbf{P} \longrightarrow \mathbf{KU}$ be a morphism of ultra-commutative ring spectra such that $\eta_*(e_T) = \nu_1$ in $\pi_0^T(\mathbf{KU}) = R(T)$. Let G be a compact Lie group whose identity component is abelian. Then the morphism of geometric fixed point rings*

$$\Phi_*^G(\eta) \;:\; \Phi_*^G(\Sigma_+^\infty \mathbf{P}) \;\longrightarrow\; \Phi_*^G(\mathbf{KU})$$

is a localization by the set of classes $\Phi^G(\beta^*(P^n(b_1)))$ for all $n \geq 1$ and all continuous homomorphisms $\beta\colon G \longrightarrow \Sigma_n \wr T$.

Proof. We start with the case where G is abelian. Then all irreducible unitary G-representations are 1-dimensional. Hence in this case the morphism $\Phi_*^G(\eta)\colon \Phi_*^G(\Sigma_+^\infty \mathbf{P}) \longrightarrow \Phi_*^G(\mathbf{KU})$ is a localization by the set of classes $\Phi^G(b_{G,\chi}) = \Phi^G(\chi^*(b_1))$ for all continuous homomorphisms $\chi\colon G \longrightarrow T$, by Theorem 4.6. Since η is a morphism of ultra-commutative ring spectra, it is compatible with norms, and so the classes $\Phi^G(\beta^*(P^n(b_1)))$ become invertible in $\Phi^G_*(\mathbf{KU})$. Hence these classes are already invertible after inverting the subset of classes for $n = 1$, $\Phi^G_*(\eta)$ is also a localization by the larger set.

Now we assume that G is not abelian. We claim that then the localization of $\Phi^G_*(\Sigma^\infty_+\mathbf{P})$ by the set of classes $\Phi^G(\beta^*(P^n(b_1)))$ as in the statement is the zero ring, consistent with the fact that $\Phi^G_*(\mathbf{KU}) = 0$ for non-cyclic groups G. We distinguish several subcases. If G is finite and not cyclic, we are in particular inverting the classes

$$\Phi^G(\chi^*(b_1)) \ = \ \Phi^G(b_{G,\chi}) \ =_{(3.2)} \ 1 - \chi^\sharp$$

in the ring $\Phi^G_0(\Sigma^\infty_+\mathbf{P}) \cong \mathbb{Z}[\mathrm{Hom}(G,T)]$ for all nontrivial homomorphisms $\chi\colon G \longrightarrow T$. The relation

$$\prod_{\chi\neq 1}(1-\chi) \ = \ 0$$

in the integral group ring $\mathbb{Z}[\mathrm{Hom}(G,T)]$ shows that the localization is the zero ring.

If G is nonabelian and $\pi_0(G)$ is not cyclic, then the projection $q\colon G \longrightarrow \pi_0(G)$ induces an inflation ring homomorphism

$$q^* \ : \ \Phi_0^{\pi_0(G)}(\Sigma^\infty_+\mathbf{P}) \ \longrightarrow \ \Phi^G_0(\Sigma^\infty_+\mathbf{P}) \ .$$

The classes to be inverted in the target include all images under q^* of the classes to be inverted in the source. Since the localization of the source is the zero ring, so is the localization of the target.

If G is nonabelian and the component group $\pi_0(G)$ is cyclic, then Theorem 3.20 (ii) provides a finite index subgroup H of G and a continuous homomorphism $\chi\colon H \longrightarrow T$ such that $\mathrm{tr}^G_H(\chi) = 1$. Proposition 3.5 (ii) then yields the relation

$$\Phi^G(N^G_H(b_{H,\chi})) \ = \ 1 - \mathrm{tr}^G_H(\chi)^\sharp \ = \ 0 \ .$$

We let $n = [G : H]$ denote the index of H in G, and we let $\Psi_{\bar g}\colon G \longrightarrow \Sigma_n \wr H$ classify a choice of H-transversal $\bar g = (g_1, \dots, g_n)$ in G, so that

$$N^G_H \ = \ \Psi^*_{\bar g} \circ P^n \ : \ \pi^H_\chi(\Sigma^\infty_+\mathbf{P}) \ \longrightarrow \ \pi^G_{\mathrm{tr}^G_H(\chi)}(\Sigma^\infty_+\mathbf{P}) \ ,$$

compare [38, Remark 5.1.7]. Setting $\beta = (\Sigma_n \wr \chi) \circ \Psi_{\bar g}\colon G \longrightarrow \Sigma_n \wr T$ yields

$$N^G_H(b_{H,\chi}) \ = \ N^G_H(\chi^*(b_1)) \ = \ \Psi^*_{\bar g}(P^n(\chi^*(b_1))) \ = \ \Psi^*_{\bar g}((\Sigma_n \wr \chi)^*(P^n(b_1))) \ = \ \beta^*(P^n(b_1)) \ .$$

Hence

$$\Phi^G(\beta^*(P^n(b_1))) \ = \ \Phi^G(N^G_H(b_{H,\chi})) \ = \ 0 \ .$$

So the localization of $\Phi^G_*(\Sigma^\infty_+\mathbf{P})$ by the relevant set of classes is the zero ring. □

Now we have all tools at hand to prove our main result.

Theorem 4.8.

(i) *Let $\eta\colon \Sigma^\infty_+\mathbf{P} \longrightarrow \mathbf{KU}$ be a morphism of commutative global ring spectra such that $\eta_*(e_T) = \nu_1$ in $\pi^T_0(\mathbf{KU}) = R(T)$. Then η is a global localization in $\mathrm{CAlg}(\mathcal{S}p^\wedge_{\mathrm{gl}})$ by the pre-Bott classes $\{b_n\}_{n\geq 1}$.*

(ii) *Let $\eta\colon \Sigma^\infty_+\mathbf{P} \longrightarrow \mathbf{KU}$ be a morphism of ultra-commutative ring spectra such that $\eta_*(e_T) = \nu_1$ in $\pi^T_0(\mathbf{KU})$. Then η is an ultra-commutative localization in $\mathrm{URing}_{\mathrm{gl}}$, by the pre-Bott classes $\{b_n\}_{n\geq 1}$.*

Proof. Theorem 4.6 witnesses the geometric fixed point criterion of Theorem 1.16 for the pre-Bott classes $\{b_n\}_{n\geq 1}$, and for the morphism $\eta\colon \Sigma^\infty_+\mathbf{P} \longrightarrow \mathbf{KU}$, proving (i). In the situation of (ii), the underlying morphism of commutative global ring spectra is a global localization in $\mathrm{CAlg}(\mathcal{S}p^\wedge_{\mathrm{gl}})$ by $\{b_n\}_{n\geq 1}$, by (i). Theorem 1.20 (ii) then shows that η is also an ultra-commutative localization in $\mathrm{URing}_{\mathrm{gl}}$ by $\{b_n\}_{n\geq 1}$. □

5. Equivariant K-theory as a G-equivariant localization

In this section we fix a compact Lie group G. Restricting the global morphism η to underlying genuine G-spectra yields a morphism of ultra-commutative G-ring spectra

$$\eta_G \ : \ \Sigma^\infty_+ \mathbf{P}_G \ \longrightarrow \ \mathbf{KU}_G \ .$$

We will show in this section that also η_G is a localization—in several different ways—by the set of pre-Bott classes of all unitary G-representations; or, equivalently, by the pre-Bott classes of all irreducible unitary G-representations. One of these localization properties is directly analogous to the properties in the global context in Theorem 4.8: η_G is a localization in the ∞-category of commutative G-ring spectra by these pre-Bott classes, see Theorem 5.2. In contrast to the global situation, the morphism η_G is also a telescopic localization (sequential colimit) in the ∞-category of G-spectra, along morphisms that realize multiplication by the pre-Bott classes of all unitary G-representations, see Theorem 5.9. And finally, η_G induces algebraic, $RO(G)$-graded localizations, again by the pre-Bott classes of all unitary G-representations, for $\Sigma^\infty_+\mathbf{P}_G$-modules, at the level of $RO(G)$-graded homotopy rings, and on equivariant cohomology theories, see Theorem 5.10.

In Proposition 5.16 we show in the example of the group of two elements that for an equivariant version of Snaith's localization theorem, one cannot get away with a telescopic localization by only inverting integer-graded equivariant homotopy classes. This shows that the main result of [47] is incorrect.

In the following a *set of representation-graded G-equivariant homotopy classes* of a G-spectrum will mean a set $\mathcal{X}$ of classes in $\pi_V^G(R)$ for varying G-representations V. We will say that the class $v \in \pi_V^G(R)$ is *supported* at V. The main example for this paper is the set $\{b_{G,V}\}$ of pre-Bott classes of the ultra-commutative G-ring spectrum $\Sigma^\infty_+\mathbf{P}_G$, for unitary G-representations V.

Definition 5.1 (Localization in $\mathrm{CAlg}(\mathcal{S}p_G^\wedge)$)**.** Let $\mathcal{X}$ be a set of representation-graded G-equivariant homotopy classes of a commutative G-ring spectrum R. A *localization by $\mathcal{X}$* is an initial object of the full subcategory of the slice category $R\backslash \mathrm{CAlg}(\mathcal{S}p_G^\wedge)$ spanned by the $\mathcal{X}$-inverting morphisms.

The next result is a G-equivariant analog of the global localization result in Theorem 4.8 (i).

Theorem 5.2. *Let G be a compact Lie group. Let $\eta\colon \Sigma^\infty_+\mathbf{P} \longrightarrow \mathbf{KU}$ be a morphism of commutative global ring spectra such that $\eta_*(e_T) = \nu_1$ in $\pi_0^T(\mathbf{KU}) = R(T)$. Then the morphism $\eta_G\colon \Sigma^\infty_+\mathbf{P}_G \longrightarrow \mathbf{KU}_G$ of commutative G-ring spectra underlying η_G is a localization in $\mathrm{CAlg}(\mathcal{S}p_G^\wedge)$ by the pre-Bott classes $b_{G,V}$ for all unitary G-representations V.*

Proof. The proof is similar to the one of the geometric fixed point criterion, Theorem 1.16. We let $f\colon \Sigma^\infty_+\mathbf{P}_G \longrightarrow P$ be any morphism of commutative G-ring spectra that inverts all the classes $b_{G,V}$ for all unitary G-representations V. We choose a pushout in $\mathrm{CAlg}(\mathcal{S}p_G^\wedge)$:

$$\begin{tikzcd} \Sigma^\infty_+\mathbf{P}_G \arrow[r, "\eta_G"] \arrow[d, "f"'] & \mathbf{KU}_G \arrow[d, "g"] \\ P \arrow[r, "j"'] & Q \end{tikzcd}$$

We will argue that the morphism $j\colon P \longrightarrow Q$ is an equivalence. To this end we let H be a closed subgroup of G. The geometric fixed point functor $\Phi^H :\ \mathrm{CAlg}(\mathcal{S}p_G^\wedge) \longrightarrow \mathrm{CAlg}(\mathcal{S}p^\wedge)$ preserves colimits, so the square

$$\begin{tikzcd} \Phi^H(\Sigma^\infty_+ \mathbf{P}) \arrow[r, "\Phi^H(\eta)"] \arrow[d, "\Phi^H(f)"'] & \Phi^H(\mathbf{KU}) \arrow[d, "\Phi^H(g)"] \\ \Phi^H(P) \arrow[r, "\Phi^H(j)"'] & \Phi^H(Q) \end{tikzcd}$$

is a pushout of commutative ring spectra. By Theorem 4.6, the morphism $\Phi^H_*(\eta)$ is a localization by the set of classes $\Phi^H(b_{H,W})$ for all unitary H-representations W. While not every H-representation is underlying a G-representation, every H-representation is a *direct summand* in a G-representation. So a $\Phi^H_*(\eta)$ is also a localization by the set of classes $\Phi^H(b_{G,V})$ for all unitary G-representations V. So its cobase change $\Phi^H(j)\colon \Phi^H(P) \longrightarrow \Phi^H(Q)$ is a localization by the set of classes

$$\Phi^H(f)_*(\Phi^H(b_{G,V})) \ = \ \Phi^H(f_*(b_{G,V})) \ .$$

Since the classes $f_*(b_{G,V})$ are all invertible, so are the classes $\Phi^H(f_*(b_{G,V}))$. So $\Phi^H(j)\colon \Phi^H(P) \longrightarrow \Phi^H(Q)$ is a localization by a set of classes that are already invertible, and hence an equivalence. This shows that the morphism $j\colon P \longrightarrow Q$ induces equivalences of geometric fixed points for all closed subgroups of G, so j is an equivalence in $\mathrm{CAlg}(\mathcal{S}p_G^\wedge)$, as claimed.

The original pushout square witnesses the diagonal composite $jf \sim g\eta_G\colon \Sigma^\infty_+ \mathbf{P}_G \longrightarrow Q$ as a coproduct of f and η_G in $(\Sigma^\infty_+\mathbf{P}_G)\backslash \mathrm{CAlg}(\mathcal{S}p_G^\wedge))^{b_{G,\bullet}}$, the full subcategory of $(\Sigma^\infty_+\mathbf{P}_G)\backslash \mathrm{CAlg}(\mathcal{S}p_G^\wedge)$ spanned by the morphisms that invert all pre-Bott classes $b_{G,V}$. Since $j\colon P \longrightarrow Q$ is an equivalence, Lemma 1.15 shows that η_G is an initial object of $(\Sigma^\infty_+\mathbf{P}_G)\backslash \mathrm{CAlg}(\mathcal{S}p_G^\wedge))^{b_{G,\bullet}}$, and thus a localization by $\{b_{G,V}\}$. □

Remark 5.3 ($\eta_G\colon \Sigma^\infty_+\mathbf{P}_G \longrightarrow \mathbf{KU}_G$ as an ultra-commutative G-localization)**.** Besides commutative G-ring spectra, there are even more highly structured kinds of genuine equivariant ring spectra: the *∞-category of ultra-commutative G-ring spectra* URing_G is the localization of the 1-category of commutative orthogonal G-ring spectra at the class of those morphisms that are $\underline{\pi}_*$-isomorphisms of underlying orthogonal G-spectra. As in the global context, we can define a 'forgetful' functor

$$u_G \ : \ \mathrm{URing}_G \ \longrightarrow \ \mathrm{CAlg}(\mathcal{S}p_G^\wedge) \tag{5.4}$$

from the ∞-category of ultra-commutative G-ring spectra to the ∞-category of commutative G-ring spectra. We write

$$\gamma_G^{uc} \ : \ \mathrm{CAlg}(\mathrm{Sp}_G^{O,\wedge}) \ \longrightarrow \ \mathrm{URing}_G$$

for the localization functor. The monoidal localization $\gamma_G^\wedge\colon \mathrm{Sp}_G^{O,\wedge} \longrightarrow \mathcal{S}p_G^\wedge$ induces a functor

$$\mathrm{CAlg}(\gamma_G^\wedge) \ : \ \mathrm{CAlg}(\mathrm{Sp}_G^{O,\wedge}) \ \longrightarrow \ \mathrm{CAlg}(\mathcal{S}p_G^\wedge)$$

between the respective categories of commutative algebras, i.e., from the 1-category of commutative orthogonal G-ring spectra to the ∞-category of commutative G-ring spectra. Since the underlying functor γ_G inverts $\underline{\pi}_*$-isomorphisms, the functor $\mathrm{CAlg}(\gamma_G^\wedge)$ sends $\underline{\pi}_*$-isomorphisms of commutative orthogonal G-ring spectra to equivalences in $\mathrm{CAlg}(\mathcal{S}p_G^\wedge)$. So the universal property of localizations provide an essentially unique functor (5.4) equipped with a natural equivalence

$$u_G \circ \gamma_G^{uc} \ \sim \ \mathrm{CAlg}(\gamma_G^\wedge) \ .$$

The composite

$$\mathrm{URing}_G \ \xrightarrow{u_G} \ \mathrm{CAlg}(\mathcal{S}p_G^\wedge) \ \xrightarrow{\mathrm{ev}_{\langle 1\rangle}} \ \mathcal{S}p_G$$

is then the functor descended from the 1-categorical forgetful functor

$$\mathrm{ev}_{\langle 1\rangle} \ : \ \mathrm{CAlg}(\mathrm{Sp}_G^{O,\wedge}) \ \longrightarrow \ \mathrm{Sp}_G^O$$

that takes a commutative orthogonal G-ring spectrum to its underlying orthogonal G-spectrum.

The forgetful functor (5.4) is *not* an equivalence; rather, it exhibits the ∞-category of ultra-commutative G-ring spectra as monadic and comonadic over the ∞-category of commutative G-ring spectra. This suggests thinking of an ultra-commutative G-ring spectrum as extra structure on a commutative G-ring spectra. The difference between ultra-commutative and commutative G-ring spectra can already be seen at the level of natural operations: the 0th equivariant homotopy groups of a commutative G-ring spectrum form a G-Green functor. For ultra-commutative G-ring spectra, this Green functor comes with additional norm operations that extend it to a G-Tambara functor, see for example [9, Section 7.2] or [48, Theorem 8.1].

Let $\mathcal{X}$ be a set of representation-graded G-equivariant homotopy classes of an ultra-commutative G-ring spectrum R. A *localization by* $\mathcal{X}$ is an initial object of the full subcategory of the slice category $R\backslash\,\mathrm{URing}_G$ spanned by the $\mathcal{X}$-inverting morphisms. In the G-equivariant context, the analogous warning as in the global context applies: for ultra-commutative G-ring spectra, the two localizations in URing_G and in $\mathrm{CAlg}(\mathcal{S}p_G^\wedge)$ need not coincide in general.

However, one can adapt the proof of Theorem 1.20 (ii) from the global to the G-equivariant context to show the following: Let $i\colon R \longrightarrow S$ be a morphism of ultra-commutative G-ring spectra such that the underlying morphism of commutative G-ring spectra $ui\colon uR \longrightarrow uS$ is a localization by $\mathcal{X}$ in $\mathrm{CAlg}(\mathcal{S}p_G^\wedge)$. Then i is a localization by $\mathcal{X}$ in URing_G. Assuming this, we obtained the G-ultra-commutative analog of Theorem 5.2: Let $\eta\colon \Sigma^\infty_+\mathbf{P} \longrightarrow \mathbf{KU}$ be a morphism of ultra-commutative ring spectra such that $\eta_*(e_T) = \nu_1$ in $\pi_0^T(\mathbf{KU}) = R(T)$. Then the morphism of ultra-commutative G-ring spectra $\eta_G\colon \Sigma^\infty_+\mathbf{P}_G \longrightarrow \mathbf{KU}_G$ is a localization in URing_G by the pre-Bott classes $b_{G,V}$ for all unitary G-representations V.

In fact, the proof of Theorem 1.20 (ii) literally goes through in the G-equivariant context if one uses that the forgetful functor $u_G\colon \mathrm{URing}_G \longrightarrow \mathrm{CAlg}(\mathcal{S}p_G^\wedge)$ bimonadic. This fact, in turn, can be proved in much the same way as its global analog in Theorem B.28 (iii). However, as this paper is already quite long, we will not go into a detailed proof.

Because the automorphism groups of unitary representations are connected, we can descend the representation-graded pre-Bott classes unambiguously to $RO(G)$-graded equivariant homotopy classes, as follows. In Construction E.11 we explain how to define isomorphisms $\kappa_V\colon \mathbb{S}_G^{[V]} \cong \Sigma^\infty S^V$, for *unitary* G-representations, in the homotopy category of G-spectra that are multiplicative in V, and such that for every $\mathbb{C}$-linear isometry $\varphi\colon V \cong W$, the composite

$$\mathbb{S}_G^{[W]} \ = \ \mathbb{S}_G^{[V]} \ \cong \ \Sigma^\infty S^V \ \xrightarrow{\Sigma^\infty S^\varphi} \ \Sigma^\infty S^W$$

coincides with κ_W. This means that the class

$$b_{G,\langle V\rangle} \ = \ \kappa_V^*(b_{G,V}) \ \in \ \pi_{[V]}^G(\Sigma^\infty_+\mathbf{P}) \tag{5.5}$$

only depends on $\langle V\rangle \in RU(G)$, the class of V in the complex representation ring, but not on the chosen representative in its isomorphism class. Here $[V] \in RO(G)$ refers to the class of the underlying orthogonal G-representation of V. We can thus unambiguously speak about the pre-Bott classes $b_{G,\alpha}$ for an effective element of $RU(G)$, i.e., one represented by a unitary G-representation, as an $RO(G)$-graded homotopy class of $\Sigma^\infty_+\mathbf{P}$.

Construction 5.6 (Telescopic localization)**.** We choose an exhaustive sequence

$$U_1 \ , \ U_2 \ , \ U_3 \ , \ \dots \tag{5.7}$$

of finite-dimensional unitary G-representations, i.e., such that every finite-dimensional unitary G-representation embeds, equivariantly and $\mathbb{C}$-linearly, into

$$V_n \ = \ U_1 \oplus \dots \oplus U_n$$

for sufficiently large n. We set

$$\alpha_n \;=\; \langle V_n\rangle \;=\; \langle U_1\oplus\dots\oplus U_n\rangle$$

in $RU(G)$. We represent the pre-Bott class $b_{G,\langle U_n\rangle}\in\pi^G_{[U_n]}(\Sigma^\infty_+\mathbf{P})$ by a morphism

$$h_n \;:\; \Sigma^\infty_+\mathbf{P}_G\wedge\mathbb{S}_G^{-\alpha_{n-1}} \;\longrightarrow\; \Sigma^\infty_+\mathbf{P}_G\wedge\mathbb{S}_G^{-\alpha_n}$$

in $\mathrm{Mod}_{\Sigma^\infty_+\mathbf{P}_G}(\mathcal{S}p^\wedge_G)$, i.e., such that

$$h_n(1\wedge\mathbb{S}_G^{-\alpha_{n-1}}) \;=\; b_{G,\langle U_n\rangle}\wedge\mathbb{S}_G^{-\alpha_n}\ ,$$

where the pairing on the right hand side is the $RO(G)$-graded external multiplication

$$\wedge \;:\; \pi^G_{[U_n]}(\Sigma^\infty_+\mathbf{P})\times\pi^G_{-\alpha_n}(\mathbb{S}_G^{-\alpha_n}) \;\longrightarrow\; \pi^G_{-\alpha_{n-1}}(\Sigma^\infty_+\mathbf{P}_G\wedge\mathbb{S}_G^{-\alpha_{n-1}})$$

defined in (E.7); this defines the morphism h_n uniquely up to homotopy. The sequence of morphisms

$$\Sigma^\infty_+\mathbf{P}_G\xrightarrow{\ h_1\ }\Sigma^\infty_+\mathbf{P}_G\wedge\mathbb{S}_G^{-\alpha_1}\xrightarrow{\ h_2\ }\dots\xrightarrow{\ h_n\ }\Sigma^\infty_+\mathbf{P}_G\wedge\mathbb{S}_G^{-\alpha_n}\dots \tag{5.8}$$

then determines a functor $F\colon\mathbb{N}\longrightarrow\mathrm{Mod}_{\Sigma^\infty_+\mathbf{P}_G}(\mathcal{S}p^\wedge_G)$.

We let $\Sigma^\infty_+\mathbf{P}_G$ act on $\mathbf{KU}_G$ through the morphism η_G of commutative ring G-spectra, making $\mathbf{KU}_G$ an object of $\mathrm{Mod}_{\Sigma^\infty_+\mathbf{P}_G}(\mathcal{S}p^\wedge_G)$. We define

$$k_n \;:\; \Sigma^\infty_+\mathbf{P}_G\wedge\mathbb{S}_G^{-\alpha_n} \;\longrightarrow\; \mathbf{KU}_G$$

as a morphism in $\mathrm{Mod}_{\Sigma^\infty_+\mathbf{P}_G}(\mathcal{S}p^\wedge_G)$, again unique up to homotopy, such that

$$k_n(1\wedge\mathbb{S}^{-\alpha_n}) \;=\; \beta^{-1}_{G,\langle V_n\rangle}\ .$$

The relations

$$\eta_*(b_{G,\langle U_n\rangle})\cdot\beta^{-1}_{G,\alpha_n} \;=\; \beta_{G,\langle U_n\rangle}\cdot\beta^{-1}_{G,\alpha_n} \;=\; \beta^{-1}_{G,\alpha_{n-1}}$$

in $\pi^G_{-\alpha_{n-1}}(\mathbf{KU})$ show that $k_n\circ h_n$ equals k_{n-1} in the homotopy category $h(\mathrm{Mod}_{\Sigma^\infty_+\mathbf{P}_G}(\mathcal{S}p^\wedge_G))$. We choose homotopies witnessing this, and they extend the functor $F\colon\mathbb{N}\longrightarrow\mathrm{Mod}_{\Sigma^\infty_+\mathbf{P}_G}(\mathcal{S}p^\wedge_G)$ to a cocone diagram $\bar F\colon\mathbb{N}^\triangleright\longrightarrow\mathrm{Mod}_{\Sigma^\infty_+\mathbf{P}_G}(\mathcal{S}p^\wedge_G)$.

Theorem 5.9 ($\mathbf{KU}_G$ as a telescopic localization)**.** *The cocone diagram $\bar F\colon\mathbb{N}^\triangleright\longrightarrow\mathrm{Mod}_{\Sigma^\infty_+\mathbf{P}_G}(\mathcal{S}p^\wedge_G)$ presents $\mathbf{KU}_G$ as a colimit in $\mathrm{Mod}_{\Sigma^\infty_+\mathbf{P}_G}(\mathcal{S}p^\wedge_G)$ of the sequence* (5.8).

Proof. The forgetful functor $U\colon\mathrm{Mod}_{\Sigma^\infty_+\mathbf{P}_G}(\mathcal{S}p^\wedge_G)\longrightarrow\mathcal{S}p_G$ preserves colimits and is conservative. The geometric fixed point functors $\Phi^H\colon\mathcal{S}p_G\longrightarrow\mathcal{S}p$ preserve colimits and are jointly conservative as H runs over all closed subgroups of G. So it suffices to show that $\Phi^H\circ U\circ\bar F\colon\mathbb{N}^\triangleright\longrightarrow\mathcal{S}p$ is a colimit, for all $H\le G$. Since the functor $\pi_*\colon\mathcal{S}p\longrightarrow gr\mathcal{A}b$ is conservative and preserves sequential colimits, is suffices to show that $\pi_*\circ\Phi^H\circ U\circ\bar F\colon\mathbb{N}^\triangleright\longrightarrow gr\mathcal{A}b$ is a colimit. The underlying H-representation of $\mathcal{U}_G^{\mathbb{C}}$ is a complete complex H-universe, and it thus contains every irreducible H-representation infinitely often. So the colimit of $\pi_*\circ\Phi^H\circ U\circ F\colon\mathbb{N}\longrightarrow gr\mathcal{A}b$ calculates the localization of the geometric fixed point ring $\Phi^H_*(\Sigma^\infty_+\mathbf{P})$ obtained by inverting the classes $\Phi^H(b_{H,\lambda})$ for all irreducible H-representations λ. By Theorem 4.6, the morphism η exhibits $\Phi^H_*(\mathbf{KU})$ as such a localization. Hence $\pi_*\circ\Phi^H\circ U\circ\bar F\colon\mathbb{N}^\triangleright\longrightarrow gr\mathcal{A}b$ is a colimit for all $H\le G$, and hence $\bar F\colon\mathbb{N}^\triangleright\longrightarrow\mathrm{Mod}_{\Sigma^\infty_+\mathbf{P}_G}(\mathcal{S}p^\wedge_G)$ is a colimit, too. $\square$

We alert the reader that in general, the underlying G-spectrum of a global localization by $\mathcal{X}$ in $\mathrm{CAlg}(\mathcal{S}p_{\mathrm{gl}}^{\wedge})$ need not be a telescopic localization in $\mathcal{S}p_G$ by the 'obvious' G-equivariant classes, i.e., the ones obtained from $\mathcal{X}$ by restriction along all homomorphisms from G.

Theorem 5.10 ($\pi_\star^G(\mathbf{KU})$ as an $RO(G)$-graded localization)**.** *Let G be a compact Lie group. Let $\eta\colon \Sigma_+^\infty \mathbf{P} \longrightarrow \mathbf{KU}$ be a morphism of commutative global ring spectra such that $\eta_*(e_T) = \nu_1$ in $\pi_0^T(\mathbf{KU}) = R(T)$.*

(i) *For every $\Sigma_+^\infty \mathbf{P}_G$-module M in $\mathcal{S}p_G^{\wedge}$, the morphism*

$$\pi_\star^G(\eta \wedge_{\Sigma_+^\infty \mathbf{P}_G} M) \ : \ \pi_\star^G(M) \ \longrightarrow \ \pi_\star^G(\mathbf{KU}_G \wedge_{\Sigma_+^\infty \mathbf{P}_G} M)$$

exhibits the target as a localization, in $RO(G)$-graded $\pi_\star^G(\Sigma_+^\infty \mathbf{P})$-modules, of the source by the set of pre-Bott classes of all unitary G-representations.

(ii) *The morphism of graded-commutative $RO(G)$-graded rings*

$$\pi_\star^G(\eta) \ : \ \pi_\star^G(\Sigma_+^\infty \mathbf{P}) \ \longrightarrow \ \pi_\star^G(\mathbf{KU})$$

exhibits the target as an $RO(G)$-graded localization of the source by the set of pre-Bott classes of all unitary G-representations.

(iii) *For every finite based G-CW complex A, the morphism*

$$\eta_G^\star(A) \ : \ (\Sigma_+^\infty \mathbf{P})_G^\star(A) \ \longrightarrow \ \mathbf{KU}_G^\star(A)$$

exhibits the target as a localization, in $RO(G)$-graded $(\Sigma_+^\infty \mathbf{P})_G^\star$-modules, of the source by the set of pre-Bott classes of all unitary G-representations.

Proof. (i) The functor $- \wedge_{\Sigma_+^\infty \mathbf{P}_G} M\colon \mathrm{Mod}_{\Sigma_+^\infty \mathbf{P}_G}(\mathcal{S}p_G^{\wedge}) \longrightarrow \mathcal{S}p_G$ preserves all colimits. So Theorem 5.9 shows that the cocone diagram $\bar{F} \wedge_{\Sigma_+^\infty \mathbf{P}_G} M\colon \mathbb{N}^{\rhd} \longrightarrow \mathcal{S}p_G$ presents $\mathbf{KU}_G \wedge_{\Sigma_+^\infty \mathbf{P}_G} M$ as a colimit in $\mathcal{S}p_G$ of the sequence

$$M \xrightarrow{h_1 \wedge M} M \wedge \mathbb{S}^{-\alpha_1} \xrightarrow{h_2 \wedge M} \dots \xrightarrow{h_n \wedge M} M \wedge \mathbb{S}^{-\alpha_n} \dots$$

We have exploited

$$(\Sigma_+^\infty \mathbf{P}_G \wedge \mathbb{S}_G^{-\alpha_n}) \wedge_{\Sigma_+^\infty \mathbf{P}_G} M \ \cong \ M \wedge \mathbb{S}_G^{-\alpha_n}$$

By design, the morphism $M \wedge h_n$ induces multiplication by the pre-Bott class $b_{G,[U_n]}$ on $RO(G)$-graded homotopy groups.

For every $\alpha \in RO(G)$, the functor $\pi_\alpha^G\colon \mathcal{S}p_G \longrightarrow \mathcal{A}b$ preserves sequential colimits. So $\pi_\star^G(\mathbf{KU}_G \wedge_{\Sigma_+^\infty \mathbf{P}_G} M)$ is exhibited as a sequential colimit, over multiplications by pre-Bott classes, of appropriate $RO(G)$-graded shifts of instances of $\pi_\star^G(M)$. Since the sequence of G-representations (5.7) is exhaustive, the algebraic effect of the sequential colimit is precisely $RO(G)$-graded localization by the pre-Bott classes of all irreducible G-representations.

(ii) The localization property in the category of commutative $RO(G)$-graded rings is detected by the analogous localization property of the underlying $RO(G)$-graded $\pi_\star^G(\Sigma_+^\infty \mathbf{P})$-modules. So property (ii) follows from (i) for the $M = \mathbf{KU}_G$, the $\Sigma_+^\infty \mathbf{P}_G$-module obtained by restriction along η_G.

(iii) By the finiteness of A, the morphism

$$\mathrm{map}_*(A, \Sigma_+^\infty \mathbf{P}_G) \wedge_{\Sigma_+^\infty \mathbf{P}_G} \mathbf{KU}_G \ \longrightarrow \ \mathrm{map}_*(A, \mathbf{KU}_G)$$

is an equivalence of $\mathbf{KU}_G$-modules. So part (iii) follows from part (ii), applied to the $\Sigma_+^\infty \mathbf{P}_G$-module $\mathrm{map}_*(A, \Sigma_+^\infty \mathbf{P}_G)$. $\square$

Example 5.11. If the group G is finite, we write $\rho = \mathbb{C}[G]$ for its complex regular representation. Every unitary G-representation is a summand in some multiple of the regular representation, so inverting $b_{G,\rho}$ inverts the pre-Bott classes of all unitary G-representations. Theorem 5.10 thus specializes to the statement that the morphisms of $RO(G)$-graded $(\Sigma^\infty_+\mathbf{P})^\star_G$-modules

$$\pi^G_\star(M)[b^{-1}_{G,\rho}] \longrightarrow \pi^G_\star(\mathbf{KU}_G \wedge_{\Sigma^\infty_+\mathbf{P}_G} M)$$

and

$$(\Sigma^\infty_+\mathbf{P})^\star_G(A)[b^{-1}_{G,\rho}] \longrightarrow \mathbf{KU}^\star_G(A)$$

induced by $\eta_G\colon \Sigma^\infty_+\mathbf{P}_G \longrightarrow \mathbf{KU}_G$ are isomorphisms, and that the morphism

$$\pi^G_\star(\eta) \ : \ \pi^G_\star(\Sigma^\infty_+\mathbf{P})[b^{-1}_{G,\rho}] \longrightarrow \pi^G_\star(\mathbf{KU})$$

is an isomorphism of graded-commutative $RO(G)$-graded rings.

Remark 5.12 (Segal–Becker splittings)**.** The effect of the morphism $\eta_G\colon \Sigma^\infty_+\mathbf{P}_G \longrightarrow \mathbf{KU}_G$ on equivariant cohomology theories has previously received some attention under the heading 'Segal–Becker splittings'. Segal [42] and Becker [6] showed in the non-equivariant context that the transformation of cohomology theories induced by η is split surjective; in fact, the induced map of infinite loop spaces $\Omega^\infty\eta\colon \Omega^\infty(\Sigma^\infty_+\mathbb{C}P^\infty) \longrightarrow \Omega^\infty KU \sim \mathbb{Z}\times BU$ admits a section by a loop map (but *not* by a double loop map). The Segal–Becker splitting was generalized to the equivariant context by Iriye and Kono [22] for finite groups, and by Crabb [13] for general compact Lie groups. The result is that for every finite G-CW-complex A, the map

$$\eta_* \ : \ (\Sigma^\infty_+\mathbf{P})^0_G(A) \longrightarrow \mathbf{KU}^0_G(A) \tag{5.13}$$

is a naturally split epimorphism. Crabb [13] gives a particularly slick exposition of the splitting.

By the Segal–Becker splitting, the map (5.13) is surjective already before any localization. So the extension of η_* to the algebraic $RO(G)$-graded localization $(\Sigma^\infty_+\mathbf{P})^\star_G(A)[b^{-1}_{G,V}]$ is also surjective. Hence the new insight provided by the isomorphism of Theorem 5.10 (iii) is the fact that the kernel of (5.13) is 'pre-Bott torsion'. In other words, every element in the kernel of (5.13) is annihilated by a pre-Bott class $b_{G,V}$ for some unitary G-representation V.

As we shall explain now, it is necessary to invert *representation-graded* (as opposed to integer-graded) equivariant homotopy classes to obtain a generalization of Snaith's theorem to genuine equivariant spectra. To see this, we let C be a two-element group. We shall now argue that C-equivariant K-theory *cannot* be obtained from $\Sigma^\infty_+\mathbf{P}_C$, the unreduced suspension spectrum of the classifying space for C-equivariant line bundles, by inverting any integer graded class in $\pi^C_*(\Sigma^\infty_+\mathbf{P}_C)$. In particular, the main result of [47] is incorrect. To prove that claim we need some control over the ring graded $\pi^C_*(\Sigma^\infty_+\mathbf{P}_C)$. The next proposition will be instrumental in calculating this ring modulo its ideal of torsion elements.

For the purpose of that proposition, we call a C-space *homotopically trivial* if it is equivalent in the ∞-category $\mathrm{Fun}(BC, \mathcal{A}n)$ of naive C-spaces, to a naive C-space with trivial action. This hypothesis implies that the action of the non-identity element of C is homotopic to the identity, and that the homotopy orbit space $EC\times_C X$ is weakly equivalent to $BC\times X$. The underlying C-space of every global space is homotopically trivial, so this applies to $X = \mathbf{P}_C$. In the following proposition we also identify the geometric fixed point homotopy groups $\Phi^C_*(\Sigma^\infty_+X)$ of the equivariant suspension spectrum with the non-equivariant stable homotopy groups $\Sigma^\infty_+X^C$, via the natural isomorphism [38, Example 3.3.3]. This allows us to interpret the geometric fixed point homomorphism (A.14) as a homomorphism

$$\Phi^C \ : \ \pi^C_*(\Sigma^\infty_+X) \longrightarrow \pi_*(\Sigma^\infty_+X^C) \ .$$

In the next proposition and below, we write 'torsion' for the torsion subgroup of an abelian group.

Proposition 5.14. *For every homotopically trivial C-CW-complex X, the ring homomorphism*

$$(\mathrm{res}^C_{\{1\}}, \Phi^C) \ : \ \pi^C_*(\Sigma^\infty_+ X)/(\text{torsion}) \ \longrightarrow \ (\pi_*(\Sigma^\infty_+ X) \times \pi_*(\Sigma^\infty_+ X^C))/(\text{torsion})$$

given by restriction and geometric fixed points is injective. Moreover, the residue class of an element $(x,y) \in \pi_*(\Sigma^\infty_+ X) \times \pi_*(\Sigma^\infty_+ X^C)$ lies in the image of this monomorphism if and only if the congruence $x \equiv i_*(y)$ holds modulo 2 and torsion, where $i \colon X^C \longrightarrow X$ is the inclusion of fixed points.

Proof. (i) The ring homomorphism

$$(\mathrm{res}^C_{\{1\}}, \Phi^C) \ : \ \pi^C_*(\Sigma^\infty_+ X) \ \longrightarrow \ \pi_*(\Sigma^\infty_+ X) \times \pi_*(\Sigma^\infty_+ X^C) \tag{5.15}$$

becomes an isomorphism after inverting 2, see for example [38, Corollary 3.4.28]. We have used the hypothesis that C acts trivially on $\pi_*(\Sigma^\infty_+ X)$. Since the map (5.15) is a rational isomorphism, it becomes a monomorphism after dividing out the torsion subgroups on both sides.

(ii) We show first that the congruence is sufficient for an element to be in the image of the map (5.15), modulo torsion. So we let $(x,y) \in \pi_*(\Sigma^\infty_+ X) \times \pi_*(\Sigma^\infty_+ X^C)$ be such that $x \equiv i_*(y) + 2z$ modulo torsion, for some $z \in \pi_*(\Sigma^\infty_+ X)$. We let p^*_C denote inflation along the unique homomorphism $p_C \colon C \longrightarrow \{1\}$. Then the map (5.15) sends the class $i_*(p^*_C(y)) + \mathrm{tr}^C_{\{1\}}(z)$ in $\pi^C_*(\Sigma^\infty_+ X)$ to the class $(i_*(y) + 2z, y)$, proving that indeed (x,y) is in the image, modulo torsion.

Now we show that all elements in the image of (5.15) satisfy the desired congruence. The tom Dieck splitting decomposes $\pi^C_k(\Sigma^\infty_+ X)$ into the direct sum of

$$\pi_k(\Sigma^\infty_+ X^C) \quad \text{and} \quad \pi^C_k(\Sigma^\infty_+ EC \times X) \ .$$

The first summand is embedded via the composite

$$\pi_k(\Sigma^\infty_+ X^C) \xrightarrow{p^*_C} \pi^C_k(\Sigma^\infty_+ X^C) \xrightarrow{i_*} \pi^C_k(\Sigma^\infty_+ X) \ .$$

The second summand is embedded via the effect on equivariant homotopy groups of the projection $q \colon EC \times X \longrightarrow X$. The composite

$$\pi_k(\Sigma^\infty_+ X^C) \times \pi^C_k(\Sigma^\infty_+ EC \times X) \ \xrightarrow[\cong]{\text{tom Dieck}} \ \pi^C_k(\Sigma^\infty_+ X) \ \xrightarrow[(5.15)]{(\mathrm{res}^C_{\{1\}}, \Phi^C)} \ \pi_*(\Sigma^\infty_+ X) \times \pi_*(\Sigma^\infty_+ X^C)$$

takes a pair (a,b) to $(i_*(a) + \mathrm{res}^C_{\{1\}}(q_*(b)), a)$, which proves that all elements $(x,y) \in \pi_*(\Sigma^\infty_+ X) \times \pi_*(\Sigma^\infty_+ X^C)$ that lie in the image of (5.15) satisfy the congruence $x \equiv i_*(y)$ modulo the image of $\mathrm{res}^C_{\{1\}} \circ q_* \colon \pi^C_k(\Sigma^\infty_+ EC \times X) \longrightarrow \pi_k(\Sigma^\infty_+ X)$. To complete the proof we may thus show that the image of $\mathrm{res}^C_{\{1\}} \circ q_*$ equals $2 \cdot \pi_k(\Sigma^\infty_+ X)$, plus a torsion group.

By the hypothesis of a homotopically trivial action, the homotopy orbit space $EC \times_C X$ is homotopy equivalent to $BC \times X$. The Adams isomorphism thus identifies $\pi^C_k(\Sigma^\infty_+ EC \times X)$ with the non-equivariant stable homotopy groups $\pi_k(\Sigma^\infty_+ BC \times X)$. A choice of basepoint of BC splits this group as the direct sum of

$$\pi_k(\Sigma^\infty_+ X) \quad \text{and} \quad \pi_k(\Sigma^\infty BC \wedge X_+) \ .$$

The second of these two summands is 2-power torsion because the reduced suspension spectrum $\Sigma^\infty BC$ becomes trivial in the stable homotopy category after inverting 2. So the second summand contributes a torsion group to the image of $\mathrm{res}^C_{\{1\}} \circ q_*$. Under the Adams isomorphism, the first summand $\pi_k(\Sigma^\infty_+ X)$ corresponds to $\pi^C_k(\Sigma^\infty_+ C \times X)$, with $C \times X$ embedded into $EC \times X$ with some C-map $j \colon C \longrightarrow EC$, which is unique up to homotopy. The composite

$$\pi_k(\Sigma^\infty_+ X) \ \xrightarrow[\cong]{\text{Wirth}} \ \pi^C_k(\Sigma^\infty_+ C \times X) \ \xrightarrow{(j \times X)_*} \ \pi^C_k(\Sigma^\infty_+ EC \times X) \ \xrightarrow{q_*} \ \pi^C_k(\Sigma^\infty_+ X)$$

is the transfer $\mathrm{tr}_{\{1\}}^C$, where the first isomorphism is the Wirthmüller isomorphism. So the image of the first summand under $\mathrm{res}^C_{\{1\}} \circ q_*$ equals the image of $\mathrm{res}^C_{\{1\}} \circ \mathrm{tr}^C_{\{1\}}$, which is precisely $2 \cdot \pi_*(\Sigma^\infty_+ X)$ because C acts trivially on $\pi_*(\Sigma^\infty_+ X)$. This concludes the argument. $\square$

For the two-element group, the abelian group $\pi_0^C(\Sigma^\infty_+ \mathbf{P}_C)$ is free of rank 3, with basis given by the classes 1, $\sigma = \mathrm{res}^T_C(e_T)$ and $t = \mathrm{tr}^C_{\{1\}}(1)$, see for example [38, Corollary 4.1.13]. The ring structure is determined by the relations

$$\sigma^2 \ = \ 1 \ , \quad t^2 \ = \ 2t \quad \text{and} \quad t\sigma \ = \ t \ .$$

We write

$$b_{nv} \ = \ b_{C,\mathbb{C}} \ = \ p_C^*(\mathrm{res}^T_{\{1\}}(b_1)) \ \in \ \pi_2^C(\Sigma^\infty_+ \mathbf{P}_C)$$

for the *naive Bott class*, the pre-Bott class the trivial 1-dimensional C-representation.

Proposition 5.16.

(i) *For every class $w \in \pi_0^C(\Sigma^\infty_+ \mathbf{P}_C)$ such that $\eta_*(w)$ is a unit in $\pi_0^C(\mathbf{KU})$, the class $\Phi^C(w) \cdot (1+\sigma^\sharp)$ is nonzero in the ring $\Phi_0^C(\Sigma^\infty_+ \mathbf{P}_C)$.*

(ii) *The composite map*

$$\pi_0^C(\Sigma^\infty_+ \mathbf{P}_C)[b_{nv}] \ \longrightarrow \ \pi_*^C(\Sigma^\infty_+ \mathbf{P}_C) \ \xrightarrow{\text{proj}} \ \pi_*^C(\Sigma^\infty_+ \mathbf{P}_C)/(\text{torsion})$$

is an isomorphism of graded rings.

(iii) *There is no homogeneous element $y \in \pi_*^C(\Sigma^\infty_+ \mathbf{P}_C)$ such that the morphism $\eta\colon \Sigma^\infty_+ \mathbf{P}_C \longrightarrow \mathbf{KU}_C$ extends to an equivalence of C-spectra from the telescopic localization $(\Sigma^\infty_+ \mathbf{P}_C)[y^{-1}]$ by y.*

Proof. (i) We abuse notation and also write σ for the class $\eta_*(\sigma)$ in $\pi_0^C(\mathbf{KU})$. If $\eta_*(w)$ is a unit in $\pi_0^C(\mathbf{KU}) = \mathbb{Z}[\sigma]/(\sigma^2-1)$, then $\eta_*(w)$ is either equal to ± 1, or to $\pm\sigma$. All these units come from units in $\pi_0^C(\Sigma^\infty_+ \mathbf{P}_C)$ with the same names, so there is a unit u in $\pi_0^C(\Sigma^\infty_+ \mathbf{P}_C)$ such that $\eta_*(u \cdot w) = 1$. The kernel of the ring homomorphism $\eta_*\colon \pi_0^C(\Sigma^\infty_+ \mathbf{P}_C) \longrightarrow \pi_0^C(\mathbf{KU})$ is generated by $1+\sigma-t$. So

$$u \cdot w \ = \ 1 + a(1+\sigma-t)$$

for some $a \in \mathbb{Z}$. Since Φ^C is a ring homomorphism which annihilates the transfer class t, we conclude that

$$\begin{aligned} \Phi^C(u) \cdot \Phi^C(w) \cdot (1+\sigma^\sharp) \ &= \ \Phi^C(1 + a(1+\sigma-t)) \cdot (1+\sigma^\sharp) \\ &= \ (1 + a \cdot (1+\sigma^\sharp)) \cdot (1+\sigma^\sharp) \ = \ (1+2a) \cdot (1+\sigma^\sharp) \ \neq \ 0 \ . \end{aligned}$$

Since u is invertible, so is $\Phi^C(u)$, and thus $\Phi^C(w) \cdot (1+\sigma^\sharp) \neq 0$.

(ii) Since the abelian group $\pi_0^C(\Sigma^\infty_+ \mathbf{P}_C)$ is torsion-free, the result is true tautologically in dimension zero. By design, the graded ring $\pi_0^C(\Sigma^\infty_+ \mathbf{P}_C)[b_{nv}]$ is concentrated in even degrees, and the class b_{nv} acts bijectively. Since the claim holds in dimension 0, it remains to show that the graded ring $\pi_*^C(\Sigma^\infty_+ \mathbf{P}_C)$ is torsion in all odd degrees, and the naive pre-Bott class acts bijectively modulo torsion, i.e.,

$$- \cdot b_{nv} \ : \ \pi_k^C(\Sigma^\infty_+ \mathbf{P}_C)/(\text{torsion}) \ \longrightarrow \ \pi_{k+2}^C(\Sigma^\infty_+ \mathbf{P}_C)/(\text{torsion})$$

is bijective for all $k \geq 0$.

We use the classical fact that the composite

$$\mathbb{Z}[b_{tr}] \ \longrightarrow \ \pi_*(\Sigma^\infty_+ \mathbb{C}P^\infty) \ \xrightarrow{\text{proj}} \ \pi_*(\Sigma^\infty_+ \mathbb{C}P^\infty)/(\text{torsion})$$

is an isomorphism of graded rings. Indeed, the quotient of $\pi_*(\Sigma^\infty_+ \mathbb{C}P^\infty)$ by its torsion ideal embeds into $\mathbb{Q} \otimes \pi_*(\Sigma^\infty_+ \mathbb{C}P^\infty) \cong H_*(\mathbb{C}P^\infty; \mathbb{Q}) = \mathbb{Q}[b_{\mathrm{tr}}]$, and the powers of b_{tr} cannot be divisible because their images in $\pi_*(KU)$ are not divisible. In particular, $\pi_*(\Sigma^\infty_+ \mathbb{C}P^\infty)$ is torsion in odd degrees, and the trivial Bott class acts invertibly modulo torsion.

Since the C-space $\mathbf{P}_C$ underlies a global space, its C-action is homotopically trivial. So Proposition 5.14 shows that the map

$$(\mathrm{res}^C_{\{1\}}, \Phi^C) \ : \ \pi_*^C(\Sigma^\infty_+ \mathbf{P}_C)/(\text{torsion}) \ \longrightarrow \ (\pi_*(\Sigma^\infty_+ \mathbb{C}P^\infty) \times \pi_*(\Sigma^\infty_+ (\mathbf{P}_C)^C))/(\text{torsion})$$

is injective, and it characterizes the image by a specific congruence. The above classical fact and the description of $\pi_*(\Sigma^\infty_+ (\mathbf{P}_C)^C)$ from Proposition 3.3 (ii) show that the target is trivial in all odd degrees, and that the image of the pre-Bott class b_{nv} acts bijectively on the target. So also the source is trivial in odd degrees. Moreover, the congruence that characterizes the image of this injection is purely arithmetic, and independent of the degree, so b_{nv} also acts bijective on the image, and hence also on $\pi_*^C(\Sigma^\infty_+ \mathbf{P}_C)/(\text{torsion})$, as claimed. This completes the proof.

(iii) We argue by contradiction and suppose that $y \in \pi_k^C(\Sigma^\infty_+ \mathbf{P}_C)$ were a homogeneous element such that $\eta\colon \Sigma^\infty_+ \mathbf{P}_C \longrightarrow \mathbf{KU}_C$ extends to an equivalence from the telescopic localization by y to $\mathbf{KU}_C$. Then $\eta_*(y)$ in $\pi_k^C(\mathbf{KU})$ is in particular a graded unit in the ring $\pi_*^C(\mathbf{KU})$. This forces $k = 2l$ to be even. Part (ii) lets us write

$$y \ = \ w \cdot (b_{nv})^l + x \tag{5.17}$$

for some $w \in \pi_0^C(\Sigma^\infty_+ \mathbf{P}_C)$ and some torsion element x of $\pi_{2l}^C(\Sigma^\infty_+ \mathbf{P}_C)$. Since $\pi_{2l}^C(\mathbf{KU})$ is torsion-free and $\eta_*(b_{nv}) = \beta_{nv}$, we obtain

$$\eta_*(y) \ = \ \eta_*(w) \cdot (\beta_{nv})^l \ .$$

Since this element is a graded unit in $\pi_*^C(\mathbf{KU}) = \pi_0^C(\mathbf{KU})[\beta_{nv}^\pm]$, we deduce that $\eta_*(w)$ is a unit in $\pi_0^C(\mathbf{KU})$.

Since $\eta_*(1+\sigma) = 1 + \sigma \in \pi_0^C(\mathbf{KU})$ is a transfer from the trivial subgroup, the class $\Phi^C(1+\sigma) = 1 + \sigma^\sharp \in \Phi_0^C(\Sigma^\infty_+ \mathbf{P}_C)$ lies in the kernel of $\eta_*\colon \Phi_*^C(\Sigma^\infty_+ \mathbf{P}_C) \longrightarrow \Phi_*^C(\mathbf{KU})$. Since we assume that the telescopic localization by y maps by an equivalence to $\mathbf{KU}_C$, this morphism is a localization by $\bar{y} = \Phi^C(y)$, so the class $1 + \sigma^\sharp$ is annihilated by some power of $\bar{y}$, say

$$\bar{y}^n \cdot (1 + \sigma^\sharp) \ = \ 0$$

for some $n \geq 1$. Because

$$\mathbb{Q} \otimes \Phi_*^C(\Sigma^\infty_+ \mathbf{P}_C) \ = \ \mathbb{Q}[\sigma^\sharp, b_{nv}]/((\sigma^\sharp)^2 - 1)$$

by Proposition 3.3 (ii), the annihilator ideal of $1 + \sigma^\sharp$ is generated by $1 - \sigma^\sharp$, and hence a prime ideal of the ring $\mathbb{Q} \otimes \Phi_*^C(\Sigma^\infty_+ \mathbf{P}_C)$. So already $\bar{y} \cdot (1 + \sigma^\sharp) = 0$ in the graded ring $\mathbb{Q} \otimes \Phi_*^C(\Sigma^\infty_+ \mathbf{P}_C)$. We have

$$\Phi^C(w) \cdot \Phi^C(b_{nv})^l \cdot (1 + \sigma^\sharp) \ = \ \bar{y} \cdot (1 + \sigma^\sharp) \ = \ 0$$

in $\mathbb{Q} \otimes \Phi_*^C(\Sigma^\infty_+ \mathbf{P}_C)$ by (5.17). The class $\Phi^C(b_{nv})$ acts bijectively on $\mathbb{Q} \otimes \Phi_*^C(\Sigma^\infty_+ \mathbf{P}_C)$, so $\Phi^C(w) \cdot (1 + \sigma^\sharp) = 0$ in $\mathbb{Q} \otimes \Phi_0^C(\Sigma^\infty_+ \mathbf{P}_C)$. Since the group $\Phi_0^C(\Sigma^\infty_+ \mathbf{P}_C)$ is torsion-free, this implies that $\Phi^C(w) \cdot (1 + \sigma^\sharp) = 0$ already integrally, i.e., in the geometric fixed point ring $\Phi_0^C(\Sigma^\infty_+ \mathbf{P}_C)$. Since $\eta_*(w)$ is a unit, this contradicts (i). □

6. Localization by b_1

In this section we investigate the localizations of $\Sigma^\infty_+ \mathbf{P}$ by an individual class b_n for some fixed $n \geq 1$, with particular emphasis on the case $n = 1$. By Theorem 2.10 (i), the class b_{n-1} is a factor of a restriction of b_n. So every morphism of commutative global ring spectra from $\Sigma^\infty_+ \mathbf{P}$ that inverts b_n also inverts $b_1, b_2, \dots, b_{n-1}$. So each localization by b_n is potentially 'more drastic' than the previous localizations. And Theorem 6.1 below shows that these localizations are indeed all distinct.

The other main results of this section concern localizations of $\Sigma^\infty_+ \mathbf{P}$ by the first pre-Bott class b_1. We show in Theorem 6.2 that this localization in $\mathrm{CAlg}(\mathcal{S}p^\wedge_{\mathrm{gl}})$ is the left $\mathcal{A}b$-approximation of $\mathbf{KU}$, where $\mathcal{A}b$ is the global family of abelian compact Lie groups. And we show in Theorem 6.3 that the ultra-commutative localization by b_1 is the left $\mathcal{A}b^\circ$-approximation of $\mathbf{KU}$, where $\mathcal{A}b^\circ$ is the global family of compact Lie groups whose identity component is abelian. The family $\mathcal{A}b^\circ$ is much larger than $\mathcal{A}b$; for example, $\mathcal{A}b^\circ$ includes all finite groups. In particular, the two localizations of $\Sigma^\infty_+\mathbf{P}$ by b_1 in $\mathrm{CAlg}(\mathcal{S}p^\wedge_{\mathrm{gl}})$ and in $\mathrm{URing}_{\mathrm{gl}}$ are distinct.

The next theorem shows that inverting b_n in $\mathrm{URing}_{\mathrm{gl}}$ does not invert b_{n+1}. Because a localization in the less structured category $\mathrm{CAlg}(\mathcal{S}p^\wedge_{\mathrm{gl}})$ maps to the localization in the more highly structured category $\mathrm{URing}_{\mathrm{gl}}$, inverting b_n in $\mathrm{CAlg}(\mathcal{S}p^\wedge_{\mathrm{gl}})$ does not invert b_{n+1}, either.

Theorem 6.1. *Let $i\colon \Sigma^\infty_+\mathbf{P} \longrightarrow R$ be an ultra-commutative localization in* $\mathrm{URing}_{\mathrm{gl}}$ *by the pre-Bott class b_n for some $n \geq 1$. Then i does not invert b_{n+1}.*

Proof. It suffices to exhibit any morphism of ultra-commutative ring spectra from $\Sigma^\infty_+\mathbf{P}$ that inverts b_n but not b_{n+1}. In our example, the non-invertibility of b_{n+1} will be witnessed by the group $SU(n+1)$. We write $\mathcal{F}_n$ for global class of compact Lie groups G that are isomorphic to a subgroup of $\Sigma_k \wr U(n)$ for some $k \geq 1$. This global family is multiplicative because $(\Sigma_k \wr U(n)) \times (\Sigma_l \wr U(n))$ embeds into $\Sigma_{k+l} \wr U(n)$. And $\mathcal{F}_n$ is closed under wreath products because $\Sigma_k \wr (\Sigma_l \wr U(n))$ embeds into $\Sigma_{kl} \wr U(n)$. By design, $\mathcal{F}_n$ is the minimal multiplicative global class that contains $U(n)$ and is closed under wreath products.

We let $\delta\colon L^{uc}_{\mathcal{F}_n}(\mathbf{KU}) \longrightarrow \mathbf{KU}$ be an $\mathcal{F}_n$-approximation of the global K-theory spectrum in the ∞-category $\mathrm{URing}_{\mathrm{gl}}$ of ultra-commutative ring spectra, in the sense of Remark C.11. So this morphism is terminal in the full subcategory of $\mathrm{URing}_{\mathrm{gl}}/\mathbf{KU}$ spanned by morphisms whose source is left $\mathcal{F}_n$-induced as a global spectrum. Such a morphism exists by Theorem C.10 (iv).

Since $\mathbf{P}$ is a global classifying space of the abelian compact Lie group $U(1)$, the global spectrum $\Sigma^\infty_+\mathbf{P}$ is left induced from the global family $\mathcal{A}b$ of abelian compact Lie groups. Hence $\Sigma^\infty_+\mathbf{P}$ is also left induced from the larger global class $\mathcal{F}_n$. Hence the morphism $\eta\colon \Sigma^\infty_+\mathbf{P} \longrightarrow \mathbf{KU}$ factors uniquely as a morphism of ultra-commutative ring spectra $\psi\colon \Sigma^\infty_+\mathbf{P} \longrightarrow L^{uc}_{\mathcal{F}_n}(\mathbf{KU})$ followed by $\delta\colon L^{uc}_{\mathcal{F}_n}(\mathbf{KU}) \longrightarrow \mathbf{KU}$. Since δ is an $\mathcal{F}_n$-equivalence, since $U(n) \in \mathcal{F}_n$, and since η inverts b_n, the morphism $\psi\colon \Sigma^\infty_+\mathbf{P} \longrightarrow L^{uc}_{\mathcal{F}_n}(\mathbf{KU})$ also inverts b_n. We will show that ψ does *not* invert b_{n+1}. Incidentally, for $n = 1$ the family $\mathcal{F}_1$ is the family $\mathcal{A}b^\circ$ of compact Lie groups with abelian identity component, and we show in Theorem 6.3 below that the morphism $\psi\colon \Sigma^\infty_+\mathbf{P} \longrightarrow L^{uc}_{\mathcal{A}b^\circ}(\mathbf{KU})$ is an ultra-commutative localization by b_1. For $n \geq 2$, I do not know whether $\psi\colon \Sigma^\infty_+\mathbf{P} \longrightarrow L^{uc}_{\mathcal{F}_n}(\mathbf{KU})$ is an ultra-commutative localization by b_n.

To show that $\psi\colon \Sigma^\infty_+\mathbf{P} \longrightarrow L^{uc}_{\mathcal{F}_n}(\mathbf{KU})$ does not invert b_{n+1} we consider the group $SU(n+1)$. Since $SU(n+1)$ is connected, every continuous homomorphism from $SU(n+1)$ to $\Sigma_k \wr U(n)$ takes values in the identity component $U(n)^k$ of the target. Since every nontrivial irreducible $SU(n+1)$-representation has dimension at least $n+1$, any such homomorphism is trivial. Hence the unique homomorphism $p_{SU(n+1)}\colon SU(n+1) \longrightarrow \{1\}$ is initial among morphisms from $SU(n+1)$ to groups in $\mathcal{F}_n$. Since $L^{uc}_{\mathcal{F}_n}(\mathbf{KU})$ is $\mathcal{F}_n$-left induced, and since this global class is multiplicative, the inflation homomorphism

$$p^*_{SU(n+1)} \ : \ \pi_0(L^{uc}_{\mathcal{F}_n}(\mathbf{KU})) = \Phi_0^{\{1\}}(L^{uc}_{\mathcal{F}_n}(\mathbf{KU})) \ \longrightarrow \ \Phi_0^{SU(n+1)}(L^{uc}_{\mathcal{F}_n}(\mathbf{KU}))$$

is a ring isomorphism by Proposition C.5 (ii). Because $L^{uc}_{\mathcal{F}_n}(\mathbf{KU})$ is non-equivariantly equivalent to $\mathbf{KU}$, the source ring $\pi_0(L^{uc}_{\mathcal{F}_n}(\mathbf{KU}))$ is isomorphic to $\pi_0(\mathbf{KU}) = \mathbb{Z}$. We conclude that $\Phi_0^{SU(n+1)}(L^{uc}_{\mathcal{F}_n}(\mathbf{KU})) \cong \mathbb{Z}$. Since the tautological $SU(n+1)$-representation ν_{n+1} is not induced from any 1-dimensional representation of a closed subgroup of $SU(n+1)$, we obtain the relation

$$\Phi^{SU(n+1)}(\mathrm{res}^{U(n+1)}_{SU(n+1)}(b_{n+1})) \ = \ \Phi^{SU(n+1)}(b_{SU(n+1),\nu_{n+1}}) \ = \ 0$$

in $\Phi_0^{SU(n+1)}(\Sigma^\infty_+\mathbf{P})$ by Theorem 3.10. If b_{n+1} became invertible in $L^{uc}_{\mathcal{F}_n}(\mathbf{KU})$, then this class would be invertible, too, contradicting the fact that the ring $\Phi_0^{SU(n+1)}(L^{uc}_{\mathcal{F}_n}(\mathbf{KU}))$ is nontrivial. □

Now we look more closely at the localization of $\Sigma^\infty_+\mathbf{P}$ by the first pre-Bott class b_1 in $\pi^T_{\nu_1}(\Sigma^\infty_+\mathbf{P})$. The class b_1 is more canonical than the other pre-Bott classes in that $\eta_*\colon \pi^T_{\nu_1}(\Sigma^\infty_+\mathbf{P}) \longrightarrow \pi^T_{\nu_1}(\mathbf{KU})$ is an isomorphism, and hence b_1 is the unique preimage of the T-equivariant Bott class β. As we shall now show, inverting b_1 already yields the K-theory spectrum for a large class of compact Lie groups, including all abelian compact Lie groups. When inverting b_1, the level of coherence in the commutative multiplication *does* make a difference; in other words: the localizations as commutative global ring spectra and as ultra-commutative ring spectra are *not* equivalent, by comparing Theorems 6.2 and 6.3. This is in contrast to the localization by $\{b_n\}_{n\geq 1}$, where the ultra-commutative localization is also a localization in the less highly structured category, by Theorem 4.8.

We let $\mathcal{A}b$ denote the global family of abelian compact Lie groups. This global family is multiplicative, i.e., closed under products. This closure property under products implies that the full subcategory of $\mathcal{S}p_{\mathrm{gl}}$ spanned by the $\mathcal{A}b$-left induced global spectra is closed under the global smash product, see Theorem C.6 (i). And it implies that the full subcategory of $\mathrm{CAlg}(\mathcal{S}p^\wedge_{\mathrm{gl}})$ spanned by those commutative global ring spectra whose underlying global spectra are $\mathcal{A}b$-left induced is coreflective in $\mathrm{CAlg}(\mathcal{S}p^\wedge_{\mathrm{gl}})$, i.e., the inclusion has a right adjoint, see Theorem C.6 (v). In particular, for every commutative global ring spectrum R, the full subcategory of the slice category $\mathrm{CAlg}(\mathcal{S}p^\wedge_{\mathrm{gl}})/R$ spanned by those morphisms $A \longrightarrow R$ such that A is $\mathcal{A}b$-left induced has a terminal object. We denote this terminal object by $L_{\mathcal{A}b}(R) \longrightarrow R$.

We alert the reader that if R in $\mathrm{CAlg}(\mathcal{S}p^\wedge_{\mathrm{gl}})$ is underlying an ultra-commutative ring spectrum, then $L_{\mathcal{A}b}(R)$ will typically *not* underlie any ultra-commutative ring spectrum. For example, the commutative global ring spectrum $L_{\mathcal{A}b}(\mathbf{KU})$ does *not* underlie any ultra-commutative ring spectrum. This phenomenon ultimately stems from the fact that the multiplicative global family $\mathcal{A}b$ is not closed under wreath products.

Since $\mathbf{P}$ is a global classifying space of an abelian compact Lie group, the underlying global spectrum of $\Sigma^\infty_+\mathbf{P}$ is left induced from the family $\mathcal{A}b$. The morphism $\eta\colon \Sigma^\infty_+\mathbf{P} \longrightarrow \mathbf{KU}$ thus uniquely factors as a morphism of commutative global ring spectra $\eta^\natural\colon \Sigma^\infty_+\mathbf{P} \longrightarrow L_{\mathcal{A}b}(\mathbf{KU})$, followed by the universal $\mathcal{A}b$-approximation $\epsilon\colon L_{\mathcal{A}b}(\mathbf{KU}) \longrightarrow \mathbf{KU}$ in $\mathrm{CAlg}(\mathrm{Sp}^\wedge_{\mathrm{gl}})$.

Theorem 6.2 (Commutative global localization by b_1)**.** *The morphism of commutative global ring spectra $\eta^\natural\colon u(\Sigma^\infty_+\mathbf{P}) \longrightarrow L_{\mathcal{A}b}(\mathbf{KU})$ is a global localization in* $\mathrm{CAlg}(\mathcal{S}p^\wedge_{\mathrm{gl}})$ *by the pre-Bott class b_1.*

Proof. We let G be a compact Lie group. The projection $q_G\colon G \longrightarrow G_{\mathrm{ab}}$ to the abelianization is initial among continuous homomorphisms to abelian compact Lie groups; so the global family $\mathcal{A}b$ is reflexive in the sense of [38, Definition 4.5.7]. The underlying global spectra of both $\Sigma^\infty_+\mathbf{P}$ and $L_{\mathcal{A}b}(\mathbf{KU})$ are left induced from the global family $\mathcal{A}b$. So [38, Proposition 4.5.8] shows that both vertical inflation morphisms in the commutative square of graded-commutative rings

$$\begin{tikzcd}
\Phi^{G_{\mathrm{ab}}}_*(\Sigma^\infty_+\mathbf{P}) \arrow[r, "\Phi^{G_{\mathrm{ab}}}_*(\eta^\natural)"'] \arrow[d, "q_G^*"', "\cong"] \arrow[rr, bend left, "\Phi^{G_{\mathrm{ab}}}_*(\eta)"] & \Phi^{G_{\mathrm{ab}}}_*(L_{\mathcal{A}b}(\mathbf{KU})) \arrow[r, "\cong", "\Phi^{G_{\mathrm{ab}}}_*(\epsilon)"'] \arrow[d, "\cong"', "q_G^*"] & \Phi^{G_{\mathrm{ab}}}_*(\mathbf{KU}) \\
\Phi^{G}_*(\Sigma^\infty_+\mathbf{P}) \arrow[r, "\Phi^{G}_*(\eta^\natural)"'] & \Phi^{G}_*(L_{\mathcal{A}b}(\mathbf{KU})) &
\end{tikzcd}$$

are isomorphisms. By Theorem 4.6, the morphism $\Phi^{G_{\mathrm{ab}}}_*(\eta)$ is a localization by the classes $\Phi^{G_{\mathrm{ab}}}(b_{G_{\mathrm{ab}},V})$ for all unitary G_{ab}-representations V. Since $\epsilon\colon L_{\mathcal{A}b}(\mathbf{KU}) \longrightarrow \mathbf{KU}$ is an $\mathcal{A}b$-equivalence, it induces an isomorphism of G_{ab}-geometric fixed points, and so also the morphism $\Phi^{G_{\mathrm{ab}}}_*(\eta^\natural)$ is a localization by the same classes. Since G_{ab} is abelian, every G_{ab}-representation is a direct sum of 1-dimensional representations. By the multiplicativity of the pre-Bott classes, the morphism $\Phi^{G_{\mathrm{ab}}}(\eta^\natural)$ is also a localization by the classes $\Phi^{G_{\mathrm{ab}}}(b_{G_{\mathrm{ab}},\chi})$ for all continuous homomorphisms $\chi\colon G_{\mathrm{ab}} \longrightarrow T$. Since $q_G^*(\Phi^{G_{\mathrm{ab}}}(b_{G_{\mathrm{ab}},\chi})) = \Phi^G(b_{G,\chi\circ q_G})$, and since every continuous homomorphism $G \longrightarrow T$ factors uniquely through $q_G\colon G \longrightarrow G_{\mathrm{ab}}$, this shows that the lower horizontal morphism $\Phi^G_*(\eta^\natural)$ is a localization by the classes $\Phi^G(b_{G,\alpha}) = \Phi^G(\alpha^*(b_1))$ for all continuous homomorphisms $\alpha\colon G \longrightarrow T$. The geometric fixed point criterion of Theorem 1.16 thus shows that $\eta^\natural\colon u(\Sigma^\infty_+\mathbf{P}) \longrightarrow L_{\mathcal{A}b}(\mathbf{KU})$ is a global localization by the pre-Bott class b_1 in $\mathrm{CAlg}(\mathcal{S}p^\wedge_{\mathrm{gl}})$. □

Our next aim is to prove a result analogous to Theorem 6.2 for the ultra-commutative localization of $\Sigma^\infty_+\mathbf{P}$ by b_1, see Theorem 6.3 below. We write $\mathcal{A}b^\circ$ for the multiplicative global family of those compact Lie groups G whose identity component G° is abelian. This family in particular contains all finite groups and all abelian compact Lie groups. In contrast to the global family $\mathcal{A}b$, the family $\mathcal{A}b^\circ$ is not only multiplicative, but also closed under wreath products. Indeed, if G is any compact Lie group, then

$$(\Sigma_n \wr G)^\circ \ = \ (G^\circ)^n \ .$$

So if G is in $\mathcal{A}b^\circ$, then G° and hence also $(\Sigma_n \wr G)^\circ$ are abelian, and so the wreath product $\Sigma_n \wr G$ also belongs to $\mathcal{A}b^\circ$.

Because $\mathcal{A}b^\circ$ is multiplicative and closed under wreath products, the full subcategory of $\mathrm{URing}_{\mathrm{gl}}$ spanned by those ultra-commutative ring spectra whose underlying global spectra are $\mathcal{A}b^\circ$-left induced is coreflective in $\mathrm{URing}_{\mathrm{gl}}$, see Theorem C.10 (iv). Hence for every ultra-commutative ring spectrum R, the full subcategory of the slice category $\mathrm{URing}_{\mathrm{gl}}/R$ spanned by those morphisms $S \longrightarrow R$ such that S is $\mathcal{A}b^\circ$-left induced has a terminal object. We denote this terminal object by $\delta\colon L^{uc}_{\mathcal{A}b^\circ}(R) \longrightarrow R$; informally speaking $L^{uc}_{\mathcal{A}b^\circ}(R)$ is closest from the left to R among all $\mathcal{A}b^\circ$-left induced ultra-commutative ring spectra. Another reassuring property is that the forgetful functor $u_{\mathrm{gl}}\colon \mathrm{URing}_{\mathrm{gl}} \longrightarrow \mathrm{CAlg}(\mathcal{S}p^\wedge_{\mathrm{gl}})$ commutes with left $\mathcal{A}b^\circ$-approximations, by the horizontal right adjointability of Theorem C.10 (v). More precisely, the morphism of commutative global ring spectrum underlying $\delta\colon L^{uc}_{\mathcal{A}b^\circ}(R) \longrightarrow R$ is a left $\mathcal{A}b^\circ$-approximation of $u_{\mathrm{gl}}(R)$ in $\mathrm{CAlg}(\mathcal{S}p^\wedge_{\mathrm{gl}})$. Informally, $u_{\mathrm{gl}}(L^{uc}_{\mathcal{A}b^\circ}(R)) = L_{\mathcal{A}b^\circ}(u_{\mathrm{gl}}(R))$.

Since the underlying global spectrum of $\Sigma^\infty_+\mathbf{P}$ is left induced from $\mathcal{A}b$, it is also left induced from the larger global family $\mathcal{A}b^\circ$. So the morphism $\eta\colon \Sigma^\infty_+\mathbf{P} \longrightarrow \mathbf{KU}$ factors uniquely as a morphism of ultra-commutative global ring spectra $\eta^\flat\colon \Sigma^\infty_+\mathbf{P} \longrightarrow L^{uc}_{\mathcal{A}b^\circ}(\mathbf{KU})$, followed by the universal $\mathcal{A}b^\circ$-approximation $\delta\colon L^{uc}_{\mathcal{A}b^\circ}(\mathbf{KU}) \longrightarrow \mathbf{KU}$. The reader might want to contrast the following theorem to Theorem 6.2, which identifies the localization in $\mathrm{CAlg}(\mathcal{S}p^\wedge_{\mathrm{gl}})$, as opposed to $\mathrm{URing}_{\mathrm{gl}}$.

Theorem 6.3 (Ultra-commutative localization by b_1)**.** *The morphism $\eta^\flat\colon \Sigma^\infty_+\mathbf{P} \longrightarrow L^{uc}_{\mathcal{A}b^\circ}(\mathbf{KU})$ is an ultra-commutative localization by the pre-Bott class b_1.*

Proof. In a first step we show that the underlying morphism in $\mathrm{CAlg}(\mathcal{S}p^\wedge_{\mathrm{gl}})$ of $\eta^\flat\colon \Sigma^\infty_+\mathbf{P} \longrightarrow L^{uc}_{\mathcal{A}b^\circ}(\mathbf{KU})$ is a global localization by the set of classes $\{P^n(b_1)\}_{n\geq 1}$. The global family $\mathcal{A}b^\circ$ is multiplicative, and hence reflexive by Proposition C.4, i.e., the inclusion of $\mathcal{A}b^\circ$ into all compact Lie groups has a left adjoint. For a compact Lie group G, the initial homomorphism $p\colon G \longrightarrow G_\diamond$ to a group in $\mathcal{A}b^\circ$ is the projection onto the quotient group $G_\diamond = G/(G^\circ)'$, where $(G^\circ)' = [G^\circ, G^\circ]$ is the commutator subgroup of the identity component G° of G.

Both $\Sigma^\infty_+\mathbf{P}$ and $L^{uc}_{\mathcal{A}b^\circ}(\mathbf{KU})$ are left induced from the global family $\mathcal{A}b^\circ$. So [38, Proposition 4.5.8] or Proposition C.5 (ii) show that both vertical inflation morphisms in the commutative square of graded-commutative rings

$$\begin{tikzcd}
\Phi^{G_\diamond}_*(\Sigma^\infty_+\mathbf{P}) \arrow[r, "\Phi^{G_\diamond}_*(\eta^\flat)"'] \arrow[d, "p^*"', "\cong"] \arrow[rr, bend left, "\Phi^{G_\diamond}_*(\eta)"] & \Phi^{G_\diamond}_*(L^{uc}_{\mathcal{A}b^\circ}\mathbf{KU}) \arrow[r, "\cong", "\Phi^{G_\diamond}_*(\delta)"'] \arrow[d, "\cong"', "p^*"] & \Phi^{G_\diamond}_*(\mathbf{KU}) \\
\Phi^G_*(\Sigma^\infty_+\mathbf{P}) \arrow[r, "\Phi^G_*(\eta^\flat)"'] & \Phi^G_*(L^{uc}_{\mathcal{A}b^\circ}(\mathbf{KU}))
\end{tikzcd} \tag{6.4}$$

are isomorphisms. The right upper horizontal map is an isomorphism because δ is an $\mathcal{A}b^\circ$-equivalence.

By Theorem 4.7 applied to the group $G_\diamond$, the morphism $\Phi^{G_\diamond}(\eta)$ is a localization by the set of classes

$$\Phi^{G_\diamond}(\beta^*(P^n(b_1)))$$

for all $n \geq 1$ and all continuous homomorphisms $\beta\colon G_\diamond \longrightarrow \Sigma_n \wr T$. The commutative diagram (6.4) thus shows that the lower horizontal homomorphism $\Phi^G_*(\eta^\flat)$ is a localization by the set of classes

$$p^*(\Phi^{G_\diamond}(\beta^*(P^n(b_1)))) \;=\; \Phi^G(p^*(\beta^*(P^n(b_1)))) \;=\; \Phi^G((\beta \circ p)^*(P^n(b_1)))$$

for the same set of pairs (n, β). Every continuous homomorphism $\alpha\colon G \longrightarrow \Sigma_n \wr T$ annihilates $(G^\circ)'$, and hence factors uniquely as $\alpha = \beta \circ p$ for a continuous homomorphism $\beta\colon G_\diamond \longrightarrow \Sigma_n \wr T$. So $\Phi^G_*(\eta^\flat)$ is also a localization by the set of classes $\Phi^G(\alpha^*(P^n(b_1)))$ for all $n \geq 1$ and all continuous homomorphisms $\alpha\colon G \longrightarrow \Sigma_n \wr T$. The geometric fixed point criterion of Theorem 1.16 thus proves the claim that $\eta^\flat$ is a global localization in $\mathrm{CAlg}(\mathcal{S}p^\wedge_{\mathrm{gl}})$ by the set $\{P^n(b_1)\}_{n\geq 1}$.

Since $\eta^\flat$ arises as a morphism of ultra-commutative ring spectra, Theorem 1.20 shows that $\eta^\flat$ is also a global localization by $\{P^n(b_1)\}_{n\geq 1}$ in $\mathrm{URing}_{\mathrm{gl}}$. A morphism of ultra-commutative ring spectra $j\colon \Sigma^\infty_+\mathbf{P} \longrightarrow S$ is compatible with power operations, and thus inverts the class b_1 if and only if it inverts $P^n(b_1)$ for all $n \geq 1$, compare Proposition D.7. So every global localization in $\mathrm{URing}_{\mathrm{gl}}$ by b_1 is also a localization in $\mathrm{URing}_{\mathrm{gl}}$ by $\{P^n(b_1)\}_{n\geq 1}$, and vice versa. □

Example 6.5. As we show in Theorem 6.2, the commutative global localization of $\Sigma^\infty_+\mathbf{P}$ by b_1 is $L_{\mathcal{A}b}(\mathbf{KU})$, the left $\mathcal{A}b$-approximation of $\mathbf{KU}$. In this global spectrum we have

$$\Phi^{\Sigma_3}(L_{\mathcal{A}b}(\mathbf{KU})) \sim \Phi^{C_2}(L_{\mathcal{A}b}(\mathbf{KU})) \sim \Phi^{C_2}(\mathbf{KU}) \ .$$

The first equivalence is by inflation along the quotient map $\Sigma_3 \longrightarrow C_2$ to the abelianization, using Proposition C.5 (ii). The second equivalence is induced by the approximation morphism $\delta\colon L_{\mathcal{A}b}(\mathbf{KU}) \longrightarrow \mathbf{KU}$, which is an $\mathcal{A}b$-equivalence. Since the geometric fixed point spectrum $\Phi^{C_2}(\mathbf{KU})$ is nontrivial, we conclude that also $\Phi^{\Sigma_3}(L_{\mathcal{A}b}(\mathbf{KU}))$ is nontrivial.

However, the ultra-commutative localization of $\Sigma^\infty_+\mathbf{P}$ by b_1 in particular inverts the class

$$\Phi^{\Sigma_3}(N^{\Sigma_3}_{A_3}(\zeta^*(b_1))) \;=\; \Phi^{\Sigma_3}(N^{\Sigma_3}_{A_3}(b_{A_3,\zeta})) \;=\; 1 - \mathrm{tr}^{\Sigma_3}_{A_3}(\zeta)^\sharp \;=\; 0 \ .$$

Here $\zeta\colon A_3 \longrightarrow T$ is any of the two nontrivial characters of the alternating group A_3. The second equality is Proposition 3.5 (ii). So the Σ_3-geometric fixed points of the ultra-commutative localization by b_1 are trivial. Hence the group Σ_3 witnesses that the commutative global localization and the ultra-commutative localization of $\Sigma^\infty_+\mathbf{P}$ by b_1 are not equivalent.

7. Real-global K-theory as a Real-global localization

The global K-theory spectrum $\mathbf{KU}$ is underlying a Real-global refinement $\mathbf{KR}$ that also 'knows' about all conjugate-linear phenomena related to complex K-theory. We show in this section that, and how, our localization results can be extended to this Real-global context. The complex-conjugation involution $\psi\colon \mathbf{KU} \longrightarrow \mathbf{KU}$ was first investigated by Halladay and Kamel [19, Section 6], who show that the resulting genuine C-spectrum that they denote $KU_{\mathbb{R}}$ represents Atiyah's Real K-theory. I show in [40, Theorem B.58] that $\mathbf{KR} = (\mathbf{KU}, \psi)$ deserves to be called the *Real-global K-theory spectrum*: for every augmented Lie group $\alpha\colon G \longrightarrow C$, the genuine G-spectrum $\alpha^*(\mathbf{KR})$ represents α-equivariant Real K-theory KR_α. Complex conjugation induces a multiplicative involution on the orthogonal space $\mathbf{P}$, making it an ultra-commutative C-monoid, compare [40, Construction 4.1]. Moreover, the morphism η is C-equivariant, and hence a morphism of ultra-commutative C-ring spectra $\eta\colon \Sigma^\infty_+\mathbf{P} \longrightarrow \mathbf{KR}$. In this appendix we sketch how our main result, the localization properties of the morphism $\eta\colon \Sigma^\infty_+\mathbf{P} \longrightarrow \mathbf{KU}$ from Theorem 4.8, generalize to the Real-global context, see Theorem 7.5. There is only one minor modification necessary: we must lift the first pre-Bott class b_1 to a specific 'Real-global pre-Bott class' in $\pi^{\tilde{T}}_{\nu_1}(\Sigma^\infty_+\mathbf{P})$, see Construction 7.3. As we explain in the proof of Theorem 7.5 (i), the key feature of the refined pre-Bott class $\tilde{b}_1$ is that inverting it kills the geometric fixed point spectra of all surjectively augmented Lie groups $G \longrightarrow C$.

Before we can dive into the specifics of the C-global morphism $\eta\colon \Sigma^\infty_+\mathbf{P} \longrightarrow \mathbf{KR}$, we must give meaning to 'global localizations' in the more context of Real-global homotopy theory. The general theory would work perfectly well for augmentations to some fixed compact Lie groups K. Since we are only interested in the special case where $K = C = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ is the Galois group, we only spell out the definitions in the case. Since the non-identity element of the group C always acts by some kind of complex conjugation, and the C-actions encode conjugate-linear phenomena, we shall also use the term 'Real-global' as synonymous for 'C-global'. In the following, an *augmented Lie group* is a continuous homomorphism $\alpha\colon G \longrightarrow C$ from a compact Lie group to the Galois group.

Construction 7.1 (C-global spectra)**.** A morphism $f\colon X \longrightarrow Y$ of orthogonal C-spectra is a *C-global equivalence* if for every augmented Lie group $\alpha\colon G \longrightarrow C$ and all integers k, the map $\pi^G_k(\alpha^*(f))\colon \pi^G_k(\alpha^*(X)) \longrightarrow \pi^G_k(\alpha^*(Y))$ is an isomorphism. The C-global equivalences are the weak equivalences in a stable model structure on the category Sp^O_C of orthogonal C-spectra, which is moreover symmetric monoidal with cofibrant unit for the smash product, see [17], or Theorem 7.8 and Corollary 7.19 of [7]. The monoidal localization thus yields a symmetric monoidal ∞-category

$$\mathcal{S}p^\wedge_{C\text{-gl}} \ = \ (\mathrm{Sp}^{O,\wedge}_C)_\infty$$

and a lax symmetric monoidal functor

$$\gamma^\wedge_{C\text{-gl}} \ : \ \mathrm{Sp}^{O,\wedge}_C \ \longrightarrow \ \mathcal{S}p^\wedge_{C\text{-gl}}$$

whose underlying functor is a localization of the 1-category of orthogonal C-spectra at the class of C-global equivalences. We refer to the underlying ∞-category $\mathcal{S}p_{C\text{-gl}}$ as the *∞-category of C-global spectra*. The ∞-category $\mathcal{S}p_{C\text{-gl}}$ is stable and compactly generated, and hence presentable, by [39, Corollary A.19].

Every C-global equivalence is in particular a $\underline{\pi}_*$-isomorphism of orthogonal C-spectra, yielding a strong symmetric monoidal functor $\mathcal{S}p^\wedge_{C\text{-gl}} \longrightarrow \mathcal{S}p^\wedge_C$. But like for the trivial group, C-global equivalences are a much finer invariant than $\underline{\pi}_*$-isomorphisms; the functor is in fact a recollement of stable ∞-categories, but far from an equivalence.

The *∞-category of commutative C-global ring spectra* $\mathrm{CAlg}(\mathcal{S}p^\wedge_{C\text{-gl}})$ is the ∞-category of commutative algebras in the symmetric monoidal ∞-category $\mathcal{S}p^\wedge_{C\text{-gl}}$ of C-global spectra. Since $\mathcal{S}p^\wedge_{C\text{-gl}}$ is presentably symmetric monoidal, the ∞-category $\mathrm{CAlg}(\mathcal{S}p^\wedge_{C\text{-gl}})$ is presentable by [30, Corollary 3.2.3.5].

The ∞-category of *ultra-commutative C-global ring spectra* $\mathrm{URing}_{C\text{-gl}}$ is the localization of the 1-category of commutative orthogonal C-ring spectra at the class of C-global equivalences. The C-global equivalences on the category commutative orthogonal C-ring spectra are the weak equivalences in a model structure that generalizes the global model structure from for ultra-commutative ring spectra from [38, Theorem 5.4.3]; however, to my knowledge, this has not been worked out in detail at the time of this writing. Granted this, we deduce that the ∞-category $\mathrm{URing}_{C\text{-gl}}$ is presentable.

Much like in (B.27), we obtain a 'forgetful' functor

$$u_{C\text{-gl}} \;:\; \mathrm{URing}_{C\text{-gl}} \;\longrightarrow\; \mathrm{CAlg}(\mathcal{S}p^{\wedge}_{C\text{-gl}})$$

from the ∞-category of ultra-commutative C-global ring spectra to the ∞-category of commutative C-global ring spectra. Theorem B.28 (iii) can be generalized to show that the functor $u_{C\text{-gl}}$ is both monadic and comonadic.

Definition 7.2.

(i) Let R be a C-global spectrum. A *set of representation-graded equivariant homotopy classes* of R is set $\mathcal{X} = \{v\}$ of classes in $\pi_V^G(\alpha^*(R))$ for varying pairs (α, V) consisting of an augmented Lie group $\alpha\colon G \longrightarrow C$ and a G-representation V. We will say that the class $v \in \pi_V^G(\alpha^*(R))$ is *supported* at the homomorphism α, or at the pair (α, V).

(ii) Let R be a commutative C-global ring spectrum. Let $\mathcal{X}$ be a set of representation-graded equivariant homotopy classes of R. A morphism $i\colon R \longrightarrow S$ of commutative C-global ring spectra *inverts* $\mathcal{X}$ if for every element $v \in \mathcal{X}$ supported at (α, V), the class $i_*(v) \in \pi_V^G(\alpha^*(S))$ is invertible in the sense of Definition 1.3. A *C-global localization by $\mathcal{X}$* is an initial object of the full subcategory of the slice category $R\backslash\,\mathrm{CAlg}(\mathcal{S}p^{\wedge}_{C\text{-gl}})$ spanned by the $\mathcal{X}$-inverting morphisms.

(iii) Let R be an ultra-commutative C-global ring spectrum, and let $\mathcal{X}$ be a set of representation-graded equivariant homotopy classes of R. An *ultra-commutative localization by $\mathcal{X}$* is an initial object of the full subcategory of the slice category $R\backslash\,\mathrm{URing}_{C\text{-gl}}$ spanned by the $\mathcal{X}$-inverting morphisms.

Construction 7.3 (Real-global pre-Bott class)**.** We write $\tilde{T} = T \rtimes C$ for the *extended circle group*, the semi-direct product of the group $C = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ acting by complex conjugation on $T = U(1)$. The group $\tilde{T}$ is augmented to C by the projection. We abuse notation and write ν_1 for the tautological Real $\tilde{T}$-representation on $\mathbb{C}$ by scalar multiplication and complex conjugation. The underlying unitary T-representation of ν_1 is the tautological representation with the same name, justifying the notational abuse. We introduce a *Real-global pre-Bott class* in the $\tilde{T}$-equivariant homotopy groups of $\Sigma^\infty_+\mathbf{P}$ that restrict to the first pre-Bott classes from Construction 2.4 on the underlying 'non-Real' global spectrum.

We embed the group C into $\tilde{T}$ as complex conjugation. The cofiber sequence of based $\tilde{T}$-spaces

$$\tilde{T}/C_+ \;\longrightarrow\; S^0 \;\xrightarrow{\;i\;}\; S^{\nu_1}$$

induces a long exact sequence in any $\tilde{T}$-equivariant cohomology theory. For C-global spectra, the restriction homomorphism $\mathrm{res}^{\tilde{T}}_C$ is a split epimorphism, by inflation along the augmentation $\tilde{T} \longrightarrow C$; so the long exact sequence splits off a short exact sequence:

$$0 \longrightarrow \pi^{\tilde{T}}_{\nu_1}(\Sigma^\infty_+\mathbf{P}) \xrightarrow{\;i^*\;} \pi^{\tilde{T}}_0(\Sigma^\infty_+\mathbf{P}) \xrightarrow{\mathrm{res}^{\tilde{T}}_C} \pi^C_0(\Sigma^\infty_+\mathbf{P}) \longrightarrow 0$$

We write $e_{\tilde{T}} \in \pi^{\tilde{T}}_0(\Sigma^\infty_+\mathbf{P})$ for the stable tautological class, see [40, (4.2)]; its restriction to T is the tautological class e_T from (2.3). The ring $\pi^C_0(\Sigma^\infty_+\mathbf{P})$ is isomorphic to the Burnside ring of the group C. It is additively generated by $1 = \mathrm{res}^{\tilde{T}}_C(e_{\tilde{T}})$, the multiplicative unit, and the class $\mathrm{tr}^C_{\{1\}}(1) = \mathrm{tr}^C_{\{1\}}(\mathrm{res}^{\tilde{T}}_{\{1\}}(e_{\tilde{T}}))$. Because $\mathrm{res}^{\tilde{T}}_C(1 - e_{\tilde{T}}) = 0$, there is a unique class, the *Real-global pre-Bott class*

$$\tilde{b}_1 \;\in\; \pi^{\tilde{T}}_{\nu_1}(\Sigma^\infty_+\mathbf{P}) \quad \text{such that} \quad i^*(\tilde{b}_1) \;=\; 1 - e_{\tilde{T}} \;\in \pi^{\tilde{T}}_0(\Sigma^\infty_+\mathbf{P})\ .$$

The following proposition shows that the Real-global pre-Bott class is indeed a lift of the first pre-Bott class to the $\tilde{T}$-equivariant homotopy group, and that it maps to the Bott class in Real-equivariant K-theory.

Proposition 7.4.

(i) *The relation* $res_T^{\tilde{T}}(\tilde{b}_1) = b_1$ *holds in* $\pi_{\nu_1}^T(\Sigma_+^\infty \mathbf{P})$.

(ii) *The morphism* $\eta\colon \Sigma_+^\infty \mathbf{P} \longrightarrow \mathbf{KR}$ *sends the class* $\tilde{b}_1$ *to the* $\tilde{T}$*-equivariant Bott class.*

Proof. (i) By design we have

$$i^*(\mathrm{res}_T^{\tilde{T}}(\tilde{b}_1)) \ = \ \mathrm{res}_T^{\tilde{T}}(i^*(\tilde{b}_1)) \ = \ \mathrm{res}_T^{\tilde{T}}(1 - e_{\tilde{T}}) \ = \ 1 - e_T \ = \ i^*(b_1)$$

in the group $\pi_0^T(\Sigma_+^\infty \mathbf{P})$. Since the map $i^*\colon \pi_{\nu_1}^T(\Sigma_+^\infty \mathbf{P}) \longrightarrow \pi_0^T(\Sigma_+^\infty \mathbf{P})$ is injective, this proves the desired relation.

(ii) In equivariant K-theory, the restriction homomorphism $\mathrm{res}_T^{\tilde{T}}\colon \pi_{\nu_1}^{\tilde{T}}(\mathbf{KR}) \longrightarrow \pi_{\nu_1}^T(\mathbf{KR})$ is bijective, and it takes the $\tilde{T}$-equivariant Bott class $\tilde{\beta}$ to the T-equivariant Bott class β. Since η is a morphism of C-global spectra, we have

$$\mathrm{res}_T^{\tilde{T}}(\eta_*(\tilde{b}_1)) \ = \ \eta_*(\mathrm{res}_T^{\tilde{T}}(\tilde{b}_1)) \ = \ \eta_*(b_1) \ = \ \beta \ = \ \mathrm{res}_T^{\tilde{T}}(\tilde{\beta}) \ ,$$

where the third equation is Theorem 4.4 (i). Because restriction is injective in this situation, the desired relation $\eta_*(\tilde{b}_1) = \tilde{\beta}$ follows. □

In contrast to the non-Real situation in Construction 2.4, the map

$$\eta_* \ : \ \pi_{\nu_1}^{\tilde{T}}(\Sigma_+^\infty \mathbf{P}) \ \longrightarrow \ \pi_{\nu_1}^{\tilde{T}}(\mathbf{KR}) \cong \widetilde{K}R_{\tilde{T}}(S^{\nu_1})$$

is not injective. So the Real-global Bott class of the tautological Real $\tilde{T}$-representation has other preimages, besides the class $\tilde{b}_1$.

The following theorem extends our main result Theorem 4.8 to the Real-global context. A caveat is that the proof of part (ii) relies on the facts that $\mathrm{URing}_{C\text{-gl}}$ is presentable and the forgetful functor $u_{C\text{-gl}}\colon \mathrm{URing}_{C\text{-gl}} \longrightarrow \mathrm{CAlg}(\mathcal{S}p^\wedge_{C\text{-gl}})$ is monadic and comonadic, generalizing Theorem B.28 from the global to the C-global context. We have not proved these facts, nor am I aware of a reference in the literature. So strictly speaking, our argument below for part (ii) is only a sketch, rather than a full proof. But it will become a full proof if and when the presentability of $\mathrm{URing}_{C\text{-gl}}$ and the bimonadicity of $u_{C\text{-gl}}$ will have been written up and made public.

Theorem 7.5.

(i) *Let* $\eta\colon \Sigma_+^\infty \mathbf{P} \longrightarrow \mathbf{KR}$ *be a morphism of commutative* C*-global ring spectra such that* $\eta_*(e_{\tilde{T}}) = \nu_1$ *in* $\pi_0^{\tilde{T}}(\mathbf{KR}) = RR(\tilde{T})$*. Then* η *is a* C*-global localization in* $\mathrm{CAlg}(\mathcal{S}p^\wedge_{C\text{-gl}})$ *by the class* $\tilde{b}_1$ *and the classes* $\{b_n\}_{n\geq 2}$*.*

(ii) *Let* $\eta\colon \Sigma_+^\infty \mathbf{P} \longrightarrow \mathbf{KR}$ *be a morphism of ultra-commutative* C*-global ring spectra such that* $\eta_*(e_{\tilde{T}}) = \nu_1$ *in* $\pi_0^{\tilde{T}}(\mathbf{KR})$*. Then* η *is an ultra-commutative* C*-global localization in* $\mathrm{URing}_{C\text{-gl}}$*, by the class* $\tilde{b}_1$ *and the classes* $\{b_n\}_{n\geq 2}$*.*

Proof. (i) We let $f\colon \Sigma_+^\infty \mathbf{P} \longrightarrow P$ be any morphism of commutative C-global ring spectra that inverts $\tilde{b}_1$ and $\{b_n\}_{n\geq 2}$. We choose a pushout in $\mathrm{CAlg}(\mathcal{S}p^\wedge_{C\text{-gl}})$:

$$\begin{tikzcd}
\Sigma_+^\infty \mathbf{P} \arrow[r, "\eta"] \arrow[d, "f"'] & \mathbf{KR} \arrow[d, "g"] \\
P \arrow[r, "j"'] & Q
\end{tikzcd}$$

We will argue that the morphism $j\colon P \longrightarrow Q$ is a C-global equivalence. To this end we let $\alpha\colon G \longrightarrow C$ be any augmented Lie group. The composite functor $\Phi^G \circ \mathrm{res}^\wedge_\alpha\colon \mathcal{S}p^\wedge_{C\text{-gl}} \longrightarrow \mathcal{S}p^\wedge$ is a strong symmetric monoidal left adjoint. So the induced functor $\mathrm{CAlg}(\Phi^G \circ \mathrm{res}^\wedge_\alpha)\colon \mathrm{CAlg}(\mathcal{S}p^\wedge_{C\text{-gl}}) \longrightarrow \mathrm{CAlg}(\mathcal{S}p^\wedge)$ preserves colimits. Thus the square

$$\begin{tikzcd}
\Phi^G(\Sigma^\infty_+\mathbf{P}) \arrow[r, "\Phi^G(\eta)"] \arrow[d, "\Phi^G(f)"'] & \Phi^G(\mathbf{KR}) \arrow[d, "\Phi^G(g)"] \\
\Phi^G(P) \arrow[r, "\Phi^G(j)"'] & \Phi^G(Q)
\end{tikzcd}$$

is a pushout of commutative ring spectra.

We claim that the morphism $\Phi^G(j)\colon \Phi^G(P) \longrightarrow \Phi^G(Q)$ is an equivalence. To prove this, we distinguish two cases. If the augmentation $\alpha\colon G \longrightarrow C$ is trivial, then the morphism of graded rings $\Phi^G_*(\eta)\colon \Phi^G_*(\Sigma^\infty_+\mathbf{P}) \longrightarrow \Phi^G_*(\mathbf{KR}) = \Phi^G_*(\mathbf{KU})$ is a localization by the set of classes $\Phi^G(b_{G,V})$ for all unitary G-representations V, by Theorem 4.6. So its cobase change $\Phi^G(j)\colon \Phi^G(P) \longrightarrow \Phi^G(Q)$ is a localization by the set of classes $\Phi^G(f)(\Phi^G(b_{G,V}))$ for all unitary G-representations V. Since the morphism f inverts $\tilde{b}_1$, it also inverts $\mathrm{res}^{\tilde{T}}_T(\tilde{b}_1) = b_1$. So f inverts the pre-Bott classes b_n for all $n \geq 1$, and hence the pre-Bott classes $b_{G,V}$ of all unitary G-representations. So $\Phi^G(j)\colon \Phi^G(P) \longrightarrow \Phi^G(Q)$ is a localization by the set of classes that are already invertible in the source, and thus an equivalence.

To deal with surjective augmentations, we show first that the class $\Phi^C(\tilde{b}_1)$ in $\Phi^C_1(\Sigma^\infty_+\mathbf{P})$ is nilpotent, where C sits inside $\tilde{T}$ as complex conjugation. The underlying C-space of the C-global space $\mathbf{P}$ is $\mathbb{C}P^\infty$, infinite complex projective space with C-action by complex conjugation. So the fixed point space $\mathbf{P}^C$ is homeomorphic to $\mathbb{R}P^\infty$. The geometric fixed point groups of an equivariant suspension spectrum are isomorphic to the stable homotopy groups of the fixed points space, compare [38, Example 3.3.3]. So the ring $\Phi^C_*(\Sigma^\infty_+\mathbf{P})$ is isomorphic to $\pi_*(\Sigma^\infty_+\mathbb{R}P^\infty)$. The non-equivariant stable homotopy groups of $\mathbb{R}P^\infty$ are explicitly known in small degrees, see for example [33]. In particular, $\pi_5(\Sigma^\infty_+\mathbb{R}P^\infty) = 0$, so the fifth power of every class in $\pi_1(\Sigma^\infty_+\mathbb{R}P^\infty)$ is zero. Hence also the fifth power of the class $\Phi^C(\tilde{b}_1)$ is zero.

If the augmentation $\alpha\colon G \longrightarrow C$ is surjective, then the inflation homomorphism $\alpha^*\colon \Phi^C_*(\Sigma^\infty_+\mathbf{P}) \longrightarrow \Phi^G_*(\Sigma^\infty_+\mathbf{P})$ is a ring homomorphism. Since $\Phi^C(\tilde{b}_1)$ is nilpotent, also the class

$$\Phi^G(f)(\alpha^*(\Phi^C(\tilde{b}_1))) \ = \ \Phi^G(\alpha^*(f_*(\tilde{b}_1)))$$

is nilpotent. Because $f\colon \Sigma^\infty_+\mathbf{P} \longrightarrow P$ inverts the class $\tilde{b}_1$, the class $\Phi^G(\alpha^*(f_*(\tilde{b}_1)))$ is both nilpotent and invertible, so the commutative ring spectrum $\Phi^G(P)$ is trivial. Since $\Phi^G(j)\colon \Phi^G(P) \longrightarrow \Phi^G(Q)$ is a morphism of ring spectra, also its target is trivial, and so $\Phi^G(j)$ is an equivalence. Altogether, this shows that the morphism $j\colon P \longrightarrow Q$ induces equivalences of G-geometric fixed points for all augmented Lie groups $\alpha\colon G \longrightarrow C$, so j is an equivalence in $\mathrm{CAlg}(\mathcal{S}p^\wedge_{C\text{-gl}})$.

The original pushout square witnesses the diagonal composite $jf \sim g\eta\colon \Sigma^\infty_+\mathbf{P} \longrightarrow Q$ as a coproduct of f and η in $(\Sigma^\infty_+\mathbf{P}\backslash \mathrm{CAlg}(\mathcal{S}p^\wedge_{C\text{-gl}}))^{\tilde{b}_1,b_n}$, the full subcategory of $\Sigma^\infty_+\mathbf{P}\backslash \mathrm{CAlg}(\mathcal{S}p^\wedge_{C\text{-gl}})$ spanned by the morphisms that invert $\tilde{b}_1$ and b_n for $n \geq 2$. Since $j\colon P \longrightarrow Q$ is a C-global equivalence, Lemma 1.15 shows that η is an initial object of $(\Sigma^\infty_+\mathbf{P}\backslash \mathrm{CAlg}(\mathcal{S}p^\wedge_{C\text{--gl}}))^{\tilde{b}_1,b_n}$, i.e., a C-global localization by $\{\tilde{b}_1, b_n\colon n \geq 2\}$.

(ii) The second claim is then proved by appealing to the K-global generalization of Theorem 1.20 (ii), for any compact Lie group K, which reads as follows: let $i\colon R \longrightarrow S$ be a morphism of ultra-commutative K-global ring spectra, and let $\mathcal{X}$ be a set of representation-graded equivariant homotopy classes of R. Suppose that the underlying morphism of commutative K-global ring spectra

$u_{K\text{-gl}}(i)\colon u_{K\text{-gl}}(R) \longrightarrow u_{K\text{-gl}}(S)$ is a K-global localization by $\mathcal{X}$ in $\mathrm{CAlg}(\mathcal{S}p^{\wedge}_{K\text{-gl}})$. Then i is a K-global localization by $\mathcal{X}$ in $\mathrm{URing}_{K\text{-gl}}$. This version is proved in exactly the same way as Theorem 1.20, replacing 'global' by K-global throughout, and replacing compact Lie groups by augmented compact Lie groups. However, it is in this argument that the unproven presentability of $\mathrm{URing}_{C\text{-gl}}$ and the bimonadicity of $u_{C\text{-gl}}$ are being used. □

Appendix A. Equivariant spectra

In this appendix we collect all the background material on the ∞-categories of genuine G-spectra that we need in the body of the paper. Throughout the following discussion, we let G be a compact Lie group. For our purposes, it is convenient to introduce the ∞-category of genuine G-spectra via the orthogonal spectrum model of Mandell and May [34]. A detailed discussion of genuine G-spectra based on this model is provided by Mathew, Naumann and Noel in [35, Section 5]; we recall some key aspects below, and refer to [35] for more details. An equivalent model-independent definition of genuine G-spectra is via inverting linear representations spheres on genuine G-spaces, see [15, Appendix C]. For finite groups, yet another intrinsically higher categorical definition is as spectral Mackey functors [4], see [11, Appendix A] for a streamlined proof of the equivalence to the other models.

Geometric fixed points of equivariant ring spectra are our main tool in this paper to identify localizations, see the geometric fixed point criterion from Theorem 1.16. A substantial part of this appendix is thus devoted to geometric fixed points spectra, which we recall in Construction A.3. A main result of this appendix is Proposition B.13 where we reconcile two different meanings of 'geometric fixed point groups': the non-equivariant homotopy groups of the geometric fixed point spectrum, and the explicit 1-categorical definition for equivariant orthogonal spectra that can be found for example in [38, Definition 3.3.1], and that we recall in Construction A.10 below. In Proposition B.13 we show that these two interpretations are naturally and multiplicatively isomorphic.

We will frequently use *monoidal localization* to construct lax or strong symmetric monoidal functors, and to construct monoidal transformations between such functors. Given symmetric monoidal ∞-categories $\mathcal{C}$ and $\mathcal{D}$, we write $\mathrm{Fun}_{\mathrm{lax}}(\mathcal{C},\mathcal{D})$ for the ∞-category with objects the lax symmetric monoidal functors from $\mathcal{C}$ to $\mathcal{D}$, and with morphisms the monoidal transformations. We write $\mathrm{Fun}_{\otimes}(\mathcal{C},\mathcal{D})$ for the full subcategory of $\mathrm{Fun}_{\mathrm{lax}}(\mathcal{C},\mathcal{D})$ spanned by the strong symmetric monoidal functors.

Definition A.1 (Monoidal localization)**.** Let $\mathcal{M}^{\otimes}$ be a symmetric monoidal model category with cofibrant unit object. A *monoidal localization* is a lax symmetric monoidal functor

$$\gamma^{\otimes}_{\mathcal{M}} \ :\ N(\mathcal{M}^{\otimes}) \ \longrightarrow\ \mathcal{M}^{\otimes}_{\infty}$$

to a symmetric monoidal ∞-category $\mathcal{M}^{\otimes}_{\infty}$ with the following universal property: for every symmetric monoidal ∞-category $\mathcal{D}$, restriction along $\gamma^{\otimes}_{\mathcal{M}}$ is an equivalence of categories

$$\mathrm{Fun}_{\mathrm{lax}}(\gamma^{\otimes}_{\mathcal{M}},\mathcal{D}) \ :\ \mathrm{Fun}_{\mathrm{lax}}(\mathcal{M}^{\otimes}_{\infty},\mathcal{D}) \ \xrightarrow{\ \sim\ }\ \mathrm{Fun}^{W}_{\mathrm{lax}}(N(\mathcal{M}^{\otimes}),\mathcal{D}) \ .$$

It would be more precise to speak of *symmetric monoidal localizations.* However, since we have no need to consider localizations of (non-symmetric) monoidal categories, we simplify the terminology by dropping the adjective 'symmetric'. Clearly, monoidal localizations are unique (in the sense of higher category theory) whenever they exist. And Nikolaus and Scholze show in [37, Theorem A.7] that every symmetric monoidal model category admits a monoidal localization. In the same theorem, they show moreover that the monoidal localization has the following additional properties:

- The functor of underlying ∞-categories
$$(\gamma^{\otimes}_{\mathcal{M}})_{\langle 1\rangle} \ : \ N\mathcal{M} = N(\mathcal{M}^{\otimes})_{\langle 1\rangle} \ \longrightarrow \ (\mathcal{M}^{\otimes}_{\infty})_{\langle 1\rangle}$$
is a localization of the 1-category $\mathcal{M}$ at the class of weak equivalences.
- Let $\mathcal{M}^{\otimes}_{\mathrm{cof}}$ denote the full symmetric monoidal subcategory of cofibrant objects in $\mathcal{M}^{\otimes}$. Then the composite
$$\gamma^{\otimes}_{\mathcal{M},\mathrm{cof}} \ : \ N(\mathcal{M}^{\otimes}_{\mathrm{cof}}) \ \xrightarrow{\mathrm{incl}} \ N(\mathcal{M}^{\otimes}) \ \xrightarrow{\gamma^{\otimes}_{\mathcal{M}}} \ \mathcal{M}^{\otimes}_{\infty}$$
is strong (as opposed to only lax) symmetric monoidal, and for every symmetric monoidal ∞-category $\mathcal{D}$, restriction along $\gamma^{\otimes}_{\mathcal{M},\mathrm{cof}}$ is an equivalence of categories
$$\mathrm{Fun}_{\mathrm{lax}}(\gamma^{\otimes}_{\mathcal{M},\mathrm{cof}},\mathcal{D}) \ : \ \mathrm{Fun}_{\mathrm{lax}}(\mathcal{M}^{\otimes}_{\infty},\mathcal{D}) \ \xrightarrow{\sim} \ \mathrm{Fun}^{W}_{\mathrm{lax}}(N(\mathcal{M}^{\otimes}_{\mathrm{cof}}),\mathcal{D}) \quad \text{and}$$
$$\mathrm{Fun}_{\otimes}(\gamma^{\otimes}_{\mathcal{M},\mathrm{cof}},\mathcal{D}) \ : \ \mathrm{Fun}_{\otimes}(\mathcal{M}^{\otimes}_{\infty},\mathcal{D}) \ \xrightarrow{\sim} \ \mathrm{Fun}^{W}_{\otimes}(N(\mathcal{M}^{\otimes}_{\mathrm{cof}}),\mathcal{D}) \ .$$
- Since $\gamma^{\otimes}_{\mathcal{M},\mathrm{cof}}$ is both strong symmetric monoidal and essentially surjective on objects, precomposition with it 'detects strongness', in the following sense: let $F^{\otimes}\colon \mathcal{M}^{\otimes}_{\infty} \longrightarrow \mathcal{D}$ be a lax symmetric monoidal functor such that the composite $F^{\otimes}\circ\gamma^{\otimes}_{\mathcal{M},\mathrm{cof}}\colon N(\mathcal{M}^{\otimes}_{\mathrm{cof}}) \longrightarrow \mathcal{D}$ is strong (as opposed to only lax) symmetric monoidal. Then $F^{\otimes}$ is strong (as opposed to only lax) symmetric monoidal.

Construction A.2 (The symmetric monoidal ∞-category of genuine G-spectra)**.** We write Sp^O_G for the 1-category of orthogonal G-spectra. A morphism $f\colon X \longrightarrow Y$ of orthogonal G-spectra is a $\underline{\pi}_*$*-isomorphism* if the induced map of equivariant homotopy groups $\pi^H_k(f)\colon \pi^H_k(X) \longrightarrow \pi^H_k(Y)$ is an isomorphism for all $k \in \mathbb{Z}$ and all closed subgroups H of G. Various stable model structures on the category of orthogonal G-spectra have been established with $\underline{\pi}_*$-isomorphisms as weak equivalences, the first one being due to Mandell–May [34, III Theorem 4.2]. By [34, III Proposition 7.5] the smash product of orthogonal G-spectra is a Quillen bifunctor for the Mandell–May model structure. Since the monoidal unit, the G-sphere spectrum $\mathbb{S}$, is cofibrant in this model structure, we are dealing with a symmetric monoidal model category. The monoidal localization thus yields a symmetric monoidal ∞-category
$$\mathcal{S}p^{\wedge}_G \ = \ (\mathrm{Sp}^{O,\wedge}_G)_{\infty}$$
and a lax symmetric monoidal functor
$$\gamma^{\wedge}_G \ : \mathrm{Sp}^{O,\wedge}_G \ \longrightarrow \ \mathcal{S}p^{\wedge}_G$$
whose underlying functor
$$\gamma_G = (\gamma^{\wedge}_G)_{\langle 1\rangle} \ : \ \mathrm{Sp}^O_G \ \longrightarrow \ (\mathcal{S}p^{\wedge}_G)_{\langle 1\rangle} = \mathcal{S}p_G$$
is a localization of the 1-category of orthogonal G-spectra at the class of $\underline{\pi}_*$-isomorphisms. We refer to the underlying ∞-category $\mathcal{S}p_G$ as the *∞-category of genuine G-spectra*, and to the monoidal structure as the *equivariant smash product.*

The *G-stable homotopy category* $h(\mathcal{S}p_G)$ is the triangulated homotopy category of the stable ∞-category $\mathcal{S}p_G$. For every closed subgroup H of G, the unreduced suspension spectrum $\Sigma^{\infty}_{+}G/H$ of the coset G-space G/H represents the functor $\pi^H_0\colon h(\mathcal{S}p_G) \longrightarrow \mathcal{A}b$, compare [38, Proposition 3.1.46]. Since the functors π^H_* preserve coproducts and jointly detect isomorphisms in $h(\mathcal{S}p_G)$, the set $\{\Sigma^{\infty}_{+}G/H\}_{H\leq G}$ compactly generates $h(\mathcal{S}p_G)$ as a triangulated category. Hence the stable ∞-category $\mathcal{S}p_G$ of genuine G-spectra is also compactly generated by these objects, and hence presentable.

Construction A.3 (Geometric fixed points)**.** We let $\mathcal{S}^{\wedge}_{G,*}$ denote the presentably symmetric monoidal ∞-category of based G-spaces, defined as the monoidal localization of the symmetric monoidal model category $G\mathbf{T}_*$ of based compactly generated topological G-spaces [38, Proposition B.7], with weak equivalences those morphisms that induce weak homotopy equivalence of fixed points for all closed subgroups of G. The monoidal structure is by smash product.

The 1-categorical suspension spectrum functor

$$\Sigma^\infty \ : \ G\mathbf{T}_* \ \longrightarrow \ \mathrm{Sp}_G^O$$

from based G-spaces to orthogonal G-spectra is strong symmetric monoidal and a left Quillen functor for the model structure presently under consideration. So it descends to a symmetric monoidal left adjoint

$$\Sigma^\infty \ : \ \mathcal{S}_{G,*}^\wedge \ \longrightarrow \ \mathcal{S}p_G^\wedge$$

of presentably symmetric monoidal ∞-categories. Gepner–Meier [15, Corollary C.7] and Clausen–Mathew–Naumann–Noel [11, Theorem A.2] show that the suspension spectrum functor $\Sigma^\infty\colon \mathcal{S}_{G,*}^\wedge \longrightarrow \mathcal{S}p_G^\wedge$ is the universal example of a strong symmetric monoidal left adjoint that inverts all representation spheres. More precisely, $\Sigma^\infty\colon \mathcal{S}_{G,*}^\wedge \longrightarrow \mathcal{S}p_G^\wedge$ is initial in the full subcategory of $\mathcal{S}_{G,*}^\wedge\backslash\,\mathrm{CAlg}(\mathrm{Pr}^L)$ spanned those symmetric monoidal left adjoints $F\colon \mathcal{S}_{G,*}^\wedge \longrightarrow \mathcal{D}$ with the property that $F(V)$ is invertible in $\mathcal{D}$ for every orthogonal G-representation V.

The 1-categorical fixed point functor

$$(-)^G \ : \ G\mathbf{T}_* \ \longrightarrow \ \mathbf{T}_*$$

from based G-spaces to based spaces is fully homotopical, preserves pushouts and filtered colimits along closed embeddings, and commutes with geometric realization and smash products, compare [38, Proposition B.1]. So it descends to a symmetric monoidal left adjoint

$$(-)^G \ : \ \mathcal{S}_{G,*}^\wedge \ \longrightarrow \ \mathcal{S}_*^\wedge$$

on the level of presentably symmetric monoidal ∞-categories. Then the composite

$$\mathcal{S}_{G,*}^\wedge \ \xrightarrow{(-)^G} \ \mathcal{S}_*^\wedge \ \xrightarrow{\Sigma^\infty} \ \mathcal{S}p^\wedge$$

is thus a strong symmetric monoidal left adjoint. For every G-representation V, the space $(S^V)^G$ is a sphere, and thus becomes invertible in $\mathcal{S}p$. So by the the above universal property of $\mathcal{S}p_G$ as a monoidal localization of $\mathcal{S}_{G,*}$, there is a unique strong symmetric monoidal left adjoint

$$\Phi^G \ : \ \mathcal{S}p_G^\wedge \ \longrightarrow \ \mathcal{S}p^\wedge \ , \tag{A.4}$$

the *geometric fixed point functor*, equipped with a homotopy that witnesses the commutativity of the following square in $\mathrm{CAlg}(\mathrm{Pr}^L)$:

$$\begin{tikzcd} \mathcal{S}_{G,*}^\wedge \arrow[r, "(-)^G"] \arrow[d, "\Sigma^\infty"'] & \mathcal{S}_*^\wedge \arrow[d, "\Sigma^\infty"] \\ \mathcal{S}p_G^\wedge \arrow[r, "\Phi^G"'] & \mathcal{S}p^\wedge \end{tikzcd}$$

Remark A.5. Mathew, Naumann and Noel introduce the geometric fixed point functor Φ^G somewhat differently in [35, Definition 6.12], more in tune with the classical perspective advocated by the May school. We let $E\mathcal{P}$ denote a universal G-space for the family of proper closed subgroups, and $\tilde{E}\mathcal{P}$ denotes its unreduced suspension. Mathew, Naumann and Noel essentially define Φ^G as the composite of smashing with the space $\tilde{E}\mathcal{P}$, followed by the 'categorical fixed point' functor $(-)^G\colon \mathcal{S}p_G \longrightarrow \mathcal{S}p$, right adjoint to the inflation functor $i_*\colon \mathcal{S}p \longrightarrow \mathcal{S}p_G$. Their geometric fixed point functor is also a strong monoidal left adjoint, and it also makes the diagram (A.4) commute. So the two constructions are equivalent by the universal property of $\Sigma^\infty\colon \mathcal{S}_{G,*} \longrightarrow \mathcal{S}p_G^\wedge$.

Construction A.6 (Inflation)**.** The symmetric monoidal ∞-category of spectra is initial the ∞-category $\mathrm{CAlg}(\mathrm{Pr}^L_{\mathrm{st}})$ of presentably symmetric monoidal stable ∞-categories, see [30, Corollary 4.8.2.19]. So there is a unique strong symmetric monoidal left adjoint

$$i_G \ : \ \mathcal{S}p^{\wedge} \ \longrightarrow \ \mathcal{S}p^{\wedge}_G \ , \tag{A.7}$$

called the *inflation functor*. The composite of inflation and geometric fixed points (A.4) is a strong symmetric monoidal left adjoint endofunctor of $\mathcal{S}p^{\wedge}$; by virtue of $\mathcal{S}p^{\wedge}$ being initial in $\mathrm{CAlg}(\mathrm{Pr}^L_{\mathrm{st}})$, the composite is uniquely monoidally equivalent to the identity:

$$\mathrm{Id}_{\mathcal{S}p^{\wedge}} \ \sim \ \Phi^G \circ i_G \ . \tag{A.8}$$

We call a morphism $f\colon X \longrightarrow Y$ of G-spectra a *geometric fixed point equivalence* if the morphism $f \wedge \tilde{E}\mathcal{P}\colon X \wedge \tilde{E}\mathcal{P} \longrightarrow Y \wedge \tilde{E}\mathcal{P}$ is an equivalence in $\mathcal{S}p_G$. This happens if and only if the morphism $\Phi^G(f)\colon \Phi^G(X) \longrightarrow \Phi^G(Y)$ is an equivalence in in $\mathcal{S}p$. A result of Lewis and May [25, II Theorem 6.9], expressed in more classical language, says that the geometric fixed point functor $\Phi^G\colon \mathcal{S}p_G \longrightarrow \mathcal{S}p$ witnesses the ∞-category of non-equivariant spectra as a localization of the ∞-category of G-spectra by the class of geometric fixed point equivalences. This result was revisited from the higher categorical perspective by Mathew, Naumann and Noel in [35, Theorem 6.11]. We shall now reinterpret this result of Lewis and May as a monoidal localization property, as follows, without any particular claim to originality.

For a symmetric monoidal ∞-category $\mathcal{D}$, we let $\mathrm{Fun}^{\tilde{E}\mathcal{P}}_{\mathrm{lax}}(\mathcal{S}p^{\wedge}_G, \mathcal{D})$ denote full subcategory of the ∞-category $\mathrm{Fun}_{\mathrm{lax}}(\mathcal{S}p^{\wedge}_G, \mathcal{D})$ spanned by those lax symmetric monoidal functors $F^{\otimes}\colon \mathcal{S}p^{\wedge}_G \longrightarrow \mathcal{D}$ whose underlying functor $F = F^{\otimes}_{\langle 1\rangle}$ inverts the morphism $X \wedge i\colon X \longrightarrow X \wedge \tilde{E}\mathcal{P}$ for every G-spectrum X, where $i\colon S^0 \longrightarrow \tilde{E}\mathcal{P}$ is the inclusion of the G-fixed cone points. Equivalently, F sends all geometric fixed point equivalences to equivalences. And we set

$$\mathrm{Fun}^{\tilde{E}\mathcal{P}}_{\otimes}(\mathcal{S}p^{\wedge}_G, \mathcal{D}) \ = \ \mathrm{Fun}_{\otimes}(\mathcal{S}p^{\wedge}_G, \mathcal{D}) \cap \mathrm{Fun}^{\tilde{E}\mathcal{P}}_{\mathrm{lax}}(\mathcal{S}p^{\wedge}_G, \mathcal{D}) \ ,$$

the full subcategory of $\mathrm{Fun}^{\tilde{E}\mathcal{P}}_{\mathrm{lax}}(\mathcal{S}p^{\wedge}_G, \mathcal{D})$ spanned by the strong symmetric monoidal functors.

The following theorem is a higher categorical version of a special case of [25, II Theorem 6.9], and very closely related to [35, Theorem 6.11].

Theorem A.9. *For every symmetric monoidal ∞-category $\mathcal{D}$, the functors*

$$\mathrm{Fun}_{\mathrm{lax}}(\Phi^G, \mathcal{D})\colon \ \mathrm{Fun}_{\mathrm{lax}}(\mathcal{S}p^{\wedge}, \mathcal{D}) \ \rightleftarrows \ \mathrm{Fun}^{\tilde{E}\mathcal{P}}_{\mathrm{lax}}(\mathcal{S}p^{\wedge}_G, \mathcal{D})\colon \ \mathrm{Fun}_{\mathrm{lax}}(i_G, \mathcal{D})$$

are mutually inverse equivalences of ∞-categories, and the functors

$$\mathrm{Fun}_{\otimes}(\Phi^G, \mathcal{D})\colon \ \mathrm{Fun}_{\otimes}(\mathcal{S}p^{\wedge}, \mathcal{D}) \ \rightleftarrows \ \mathrm{Fun}^{\tilde{E}\mathcal{P}}_{\otimes}(\mathcal{S}p^{\wedge}_G, \mathcal{D})\colon \ \mathrm{Fun}_{\otimes}(i_G, \mathcal{D})$$

are mutually inverse equivalences of ∞-categories.

Proof. The morphism of based G-spaces $\tilde{E}\mathcal{P} \wedge i\colon \tilde{E}\mathcal{P} \longrightarrow \tilde{E}\mathcal{P} \wedge \tilde{E}\mathcal{P}$ is an equivalence in $\mathcal{S}_{G,*}$, so the induced morphism of suspension spectra

$$(\Sigma^\infty \tilde{E}\mathcal{P}) \wedge i \ : \ \Sigma^\infty \tilde{E}\mathcal{P} \ \longrightarrow \ (\Sigma^\infty \tilde{E}\mathcal{P}) \wedge (\Sigma^\infty \tilde{E}\mathcal{P})$$

is an equivalence in $\mathcal{S}p_G$; we have implicitly used the strong symmetric monoidal structure of the suspension spectrum functor. This means that the morphism $\Sigma^\infty i\colon \mathbb{S}_G \longrightarrow \Sigma^\infty \tilde{E}\mathcal{P}$ is an idempotent object in the sense of [30, Definition 4.8.2.1].

As in [35, Theorem 6.11], we write $\mathcal{S}p_G[\mathcal{P}^{-1}]$ for the full subcategory of $\mathcal{S}p_G$ spanned by the those G-spectra X such that the morphism $X \wedge i\colon X \longrightarrow X \wedge \tilde{E}\mathcal{P}$ is an equivalence. Because $\Sigma^\infty \tilde{E}\mathcal{P}$ is an idempotent algebra, this subcategory inherits a unique symmetric monoidal structure, denoted $\mathcal{S}p_G^\wedge[\mathcal{P}^{-1}]$, such that the Bousfield localization functor

$$(\Sigma^\infty \tilde{E}\mathcal{P}) \otimes - \ : \ \mathcal{S}p_G^\wedge \ \longrightarrow \ \mathcal{S}p_G^\wedge[\mathcal{P}^{-1}] \ .$$

becomes strong symmetric monoidal, see [30, Proposition 4.8.2.7]. In particular, this functor is a monoidal localization by the class of geometric fixed point equivalences, i.e., the functor

$$\mathrm{Fun}_{\mathrm{lax}}((\Sigma^\infty \tilde{E}\mathcal{P}) \otimes -, \mathcal{D})\colon \ \mathrm{Fun}_{\mathrm{lax}}(\mathcal{S}p_G^\wedge[\mathcal{P}^{-1}], \mathcal{D}) \ \longrightarrow \ \mathrm{Fun}_{\mathrm{lax}}^{\tilde{E}\mathcal{P}}(\mathcal{S}p_G^\wedge, \mathcal{D})$$

is an equivalence.

Mathew, Naumann and Noel show in [35, Theorem 6.11] that the composite strong symmetric monoidal functor

$$\mathcal{S}p^\wedge \ \xrightarrow{\ i_G\ } \ \mathcal{S}p_G^\wedge \ \xrightarrow{\Sigma^\infty \tilde{E}\mathcal{P}\otimes -} \ \mathcal{S}p_G^\wedge[\mathcal{P}^{-1}]$$

is an equivalence of symmetric monoidal ∞-categories. So in the diagram

$$\mathrm{Fun}_{\mathrm{lax}}(\mathcal{S}p_G^\wedge[\mathcal{P}^{-1}], \mathcal{D}) \xrightarrow{\mathrm{Fun}_{\mathrm{lax}}((\Sigma^\infty \tilde{E}\mathcal{P})\otimes -, \mathcal{D})} \mathrm{Fun}_{\mathrm{lax}}^{\tilde{E}\mathcal{P}}(\mathcal{S}p_G^\wedge, \mathcal{D}) \xrightarrow{\qquad \mathrm{Fun}_{\mathrm{lax}}(i_G, \mathcal{D}) \qquad} \mathrm{Fun}_{\mathrm{lax}}(\mathcal{S}p^\wedge, \mathcal{D})$$

the first functor and the composite are equivalences, hence so is the second. Since Φ^G is monoidally left inverse to i_G by (A.8), the functor

$$\mathrm{Fun}_{\mathrm{lax}}(\Phi^G, \mathcal{D})\colon \ \mathrm{Fun}_{\mathrm{lax}}(\mathcal{S}p^\wedge, \mathcal{D}) \ \longrightarrow \ \mathrm{Fun}_{\mathrm{lax}}^{\tilde{E}\mathcal{P}}(\mathcal{S}p_G^\wedge, \mathcal{D})$$

is an equivalence of ∞-categories, too. Since both i_G and Φ^G are strong (as opposed to just lax) symmetric monoidal, composition with them preserves the full subcategories $\mathrm{Fun}_\otimes^{\tilde{E}\mathcal{P}}(\mathcal{S}p_G^\wedge, \mathcal{D})$ and $\mathrm{Fun}_\otimes(\mathcal{S}p^\wedge, \mathcal{D})$, so the restrictions to these full subcategories are equivalences, too. $\square$

Now we will reconcile two possible meanings of 'geometric fixed point groups' for a G-spectrum X. On the one hand, 'geometric fixed point groups' could refer to the homotopy groups of the geometric fixed point spectrum $\Phi^G(X)$ in the ∞-category $\mathcal{S}p$. On the other hand, it could refer to the explicit 1-categorical definition that can be found for example in [38, Definition 3.3.1], and that we recall in Construction A.10 below. The former interpretation has convenient formal properties, while we use the latter interpretation for explicit calculations. So we need to know that the two interpretations are naturally and multiplicatively isomorphic, and the isomorphism (B.14) constructed below witnesses precisely this.

Construction A.10 (Geometric fixed point homotopy groups)**.** We let G be a compact Lie group, X an orthogonal G-spectrum, and $k \in \mathbb{Z}$ an integer. The *k-th G-geometric fixed point group* is defined as

$$\Phi_k^G(X) \ = \ \mathrm{colim}_{m+k\geq 0, V\in s(\mathcal{U}_G)} [S^{V^G\oplus\mathbb{R}^{m+k}}, X(V\oplus\mathbb{R}^m)^G] \ . \tag{A.11}$$

Here $s(\mathcal{U}_G)$ is the poset, under inclusion, of finite-dimensional G-subrepresentations of the complete G-universe $\mathcal{U}_G$. And '$m+k\geq 0$' means that m runs over the poset of those natural numbers such that $m+k$ is non-negative. For $m \leq n$ and $V \leq W$ in $s(\mathcal{U}_G)$, the transition maps $[S^{V^G\oplus\mathbb{R}^{m+k}}, X(V\oplus \mathbb{R}^m)^G] \longrightarrow [S^{W^G\oplus\mathbb{R}^{n+k}}, X(W\oplus\mathbb{R}^n)^G]$ in the colimit system are generalizing those in [38, Definition 3.3.1]. The monoidal natural isomorphism (B.14) gives a different perspective on these geometric fixed point groups, namely as the homotopy groups of a geometric fixed point spectrum $\Phi^G(X)$.

The geometric fixed point groups are also multiplicative, in a sense made precise now. For orthogonal G-spectra X and Y, we define external multiplication maps

$$\times \ : \ \Phi_k^G(X)\times\Phi_l^G(Y) \ \longrightarrow \ \Phi_{k+l}^G(X\wedge Y) \ . \tag{A.12}$$

We let $f\colon S^{V^G\oplus\mathbb{R}^{m+k}} \longrightarrow X(V\oplus\mathbb{R}^m)^G$ and $g\colon S^{W^G\oplus\mathbb{R}^{n+l}} \longrightarrow Y(W\oplus\mathbb{R}^n)^G$ be continuous based maps that represent classes $x=[f]\in\Phi^G_k(X)$ and $y=[g]\in\Phi^G_l(Y)$, respectively. We define $x\times y$ as the class represented by the following composite:

$$\begin{aligned} S^{(V\oplus W)^G\oplus\mathbb{R}^{m+n+k+l}} &\xrightarrow{\text{shuffle}} S^{V^G\oplus\mathbb{R}^{m+k}\oplus W^G\oplus\mathbb{R}^{n+l}} \\ &\xrightarrow{f\wedge g} X(V\oplus\mathbb{R}^m)^G\wedge Y(W\oplus\mathbb{R}^n)^G \;=\; (X(V\oplus\mathbb{R}^m)\wedge Y(W\oplus\mathbb{R}^n))^G \\ &\xrightarrow{(\iota_{V\oplus\mathbb{R}^m,W\oplus\mathbb{R}^n})^G} ((X\wedge Y)(V\oplus\mathbb{R}^m\oplus W\oplus\mathbb{R}^n))^G \\ &\xrightarrow{\text{shuffle}} ((X\wedge Y)(V\oplus W\oplus\mathbb{R}^{m+n}))^G \end{aligned}$$

Here, as before, $\iota_{V\oplus\mathbb{R}^m,W\oplus\mathbb{R}^n}$ is a component of the universal bimorphism.

Essentially the same arguments as for the pairing on equivariant homotopy groups in [38, Theorem 3.5.14] show that the pairing (A.12) is biadditive, unital, associative and commutative, the latter with respect to the Koszul sign $(-1)^{kl}$ on pairs of elements of degree (k,l). We omit the formal proof.

Construction A.13 (Descending equivariant and geometric fixed point homotopy groups)**.** The universal property of the monoidal localization functor $\gamma^\wedge_G\colon \mathrm{Sp}^{O,\wedge}_G \longrightarrow \mathcal{S}p^\wedge_G$ allows us to descend the multiplicative structure of the homotopy group functors π^G_* and the geometric fixed point functors Φ^G_* from the orthogonal spectrum model to the ∞-category of G-spectra, as follows.

We write $\mathcal{G}r\mathcal{A}b^\otimes$ for the symmetric monoidal 1-category of $\mathbb{Z}$-graded abelian groups, under graded tensor product, and with the Koszul sign built into the symmetry isomorphism. We write $\pi^G_*(X)=\{\pi^G_k(X)\}_{k\in\mathbb{Z}}$ for the $\mathbb{Z}$-graded G-equivariant homotopy groups of an orthogonal G-spectrum X. By [38, Theorem 3.5.14], the maps

$$\times \;:\; \pi^G_k(X)\times\pi^G_l(Y) \;\longrightarrow\; \pi^G_{k+l}(X\wedge Y)$$

defined in [38, (3.5.13)] endow the functor $\pi^G_*\colon \mathrm{Sp}^O_G \longrightarrow \mathcal{G}r\mathcal{A}b$ with a lax symmetric monoidal structure. By definition, this functor also inverts $\underline{\pi}_*$-isomorphisms, so it descends uniquely through the monoidal localization $\gamma^\wedge_G\colon \mathrm{Sp}^{O,\wedge}_G \longrightarrow \mathcal{S}p^\wedge_G$. In other words, there is a unique lax symmetric monoidal functor

$$\bar\pi^G_* \;:\; \mathcal{S}p^\wedge_G \;\longrightarrow\; \mathcal{G}r\mathcal{A}b^\otimes$$

equipped with a monoidal natural isomorphism

$$\bar\pi^G_*\circ\gamma^\wedge_G \;\cong\; \pi^G_*$$

between lax symmetric monoidal functors $\mathrm{Sp}^{O,\wedge}_G \longrightarrow \mathcal{G}r\mathcal{A}b^\otimes$.

Similarly, the maps (A.12) endow the functor $\Phi^G_*\colon \mathrm{Sp}^O_G \longrightarrow \mathcal{G}r\mathcal{A}b$ with a lax symmetric monoidal structure. This functor inverts $\underline{\pi}_*$-isomorphisms by [38, Proposition 3.3.10], so it, too, descends uniquely through the monoidal localization $\gamma^\wedge_G\colon \mathrm{Sp}^{O,\wedge}_G \longrightarrow \mathcal{S}p^\wedge_G$: there is a unique lax symmetric monoidal functor

$$\bar\Phi^G_* \;:\; \mathcal{S}p^\wedge_G \;\longrightarrow\; \mathcal{G}r\mathcal{A}b^\otimes$$

equipped with a monoidal natural isomorphism

$$\bar\Phi^G_*\circ\gamma^\wedge_G \;\cong\; \Phi^G_* \;.$$

The *geometric fixed point homomorphism*

$$\Phi^G \;:\; \pi^G_k(X) \;\longrightarrow\; \Phi^G_k(X) \tag{A.14}$$

does what the name suggests: it sends the class of a G-map $f\colon S^{U\oplus\mathbb{R}^{m+k}} \longrightarrow X(U\oplus\mathbb{R}^m)$ to the class of the map

$$f^G \;:\; S^{U^G\oplus\mathbb{R}^{m+k}} = (S^{U\oplus\mathbb{R}^{m+k}})^G \;\longrightarrow\; X(U\oplus\mathbb{R}^m)^G \;,$$

the restriction of f to G-fixed points. As the integer k varies, the geometric fixed point maps (A.14) form a monoidal natural transformation $\Phi^G\colon \pi_*^G \longrightarrow \Phi_*^G$ between lax symmetric monoidal functors that descend through $\gamma_G^\wedge$. So there is a unique monoidal natural transformation

$$\bar{\Phi}^G \;:\; \bar{\pi}_*^G \;\longrightarrow\; \bar{\Phi}_*^G$$

such that $\bar{\Phi}^G \circ \gamma_G^\wedge = \Phi^G$.

Appendix B. Global spectra

In this section, we recall the necessary background material on global spectra and global ring spectra. We work in the orthogonal spectrum model of [38, Chapter 4]. An equivalent and intrinsically higher categorical construction of global spectra was recently provided by Linskens, Nardin and Pol in [27]. The main qualitative features of global spectra are summarized from a higher categorical perspective in Theorem B.2. And the main properties of the functor res_G that extracts the underlying G-homotopy type of a global spectrum are summarized in Theorem B.6.

We also include two rectification results that exhibit 'global' module categories, in the higher categorical sense, as localizations of 1-categorical modules in the orthogonal spectrum model. Theorem B.17 is a version for left modules over associative ring spectra, and Theorem B.20 is a symmetric monoidal refinement for modules over commutative orthogonal ring spectra. The symmetric monoidal rectification in Theorem B.20 will also be needed in the proof of Theorem B.28 below, where we establish the bimonadity of the forgetful functor $u_{\mathrm{gl}}\colon \mathrm{URing}_{\mathrm{gl}} \longrightarrow \mathrm{CAlg}(\mathcal{S}p_{\mathrm{gl}}^\wedge)$ from ultra-commutative to commutative global ring spectra.

Construction B.1 (The symmetric monoidal ∞-category of global spectra)**.** We recall from [38, Definition 4.1.3] that a morphism $f\colon X \longrightarrow Y$ of orthogonal spectra is a *global equivalence* if the induced map of equivariant homotopy groups $\pi_k^G(f)\colon \pi_k^G(X) \longrightarrow \pi_k^G(Y)$ is an isomorphism for all $k \in \mathbb{Z}$ and all compact Lie groups G. We show in [38, Theorem 4.3.18] that the global equivalences are part of a stable, cofibrantly generated, proper and topological model structure on the category of orthogonal spectra. By [38, Proposition 4.3.24], the smash product of orthogonal spectra is a Quillen bifunctor for the global model structure. Since the monoidal unit, the sphere spectrum $\mathbb{S}$, is cofibrant in the global model structure, we are dealing with a symmetric monoidal model category. The monoidal localization thus yields a symmetric monoidal ∞-category

$$\mathcal{S}p_{\mathrm{gl}}^\wedge \;=\; (\mathrm{Sp}^{O,\wedge})_\infty$$

and a lax symmetric monoidal functor

$$\gamma_{\mathrm{gl}}^\wedge \;:\; \mathrm{Sp}^{O,\wedge} \;\longrightarrow\; \mathcal{S}p_{\mathrm{gl}}^\wedge$$

whose underlying functor

$$\gamma_{\mathrm{gl}} = (\gamma_{\mathrm{gl}}^\wedge)_{\langle 1\rangle} \;:\; \mathrm{Sp}^O \;\longrightarrow\; (\mathcal{S}p_{\mathrm{gl}}^\wedge)_{\langle 1\rangle} = \mathcal{S}p_{\mathrm{gl}}$$

is a localization of the 1-category of orthogonal spectra at the class of global equivalences. We refer to the underlying ∞-category $\mathcal{S}p_{\mathrm{gl}}$ as the *∞-category of global spectra*, and to the monoidal structure as the *global smash product.*

The *∞-category of commutative global ring spectra* $\mathrm{CAlg}(\mathcal{S}p_{\mathrm{gl}}^\wedge)$ is the ∞-category of commutative algebras in the symmetric monoidal ∞-category $\mathcal{S}p_{\mathrm{gl}}^\wedge$ of global spectra under the global smash product.

Theorem B.2.

(i) *The ∞-category $\mathcal{S}p_{\mathrm{gl}}$ of global spectra is stable and compactly generated, and hence presentable.*
(ii) *The symmetric monoidal ∞-category $\mathcal{S}p_{\mathrm{gl}}^\wedge$ of global spectra is presentably symmetric monoidal.*
(iii) *The ∞-category $\mathrm{CAlg}(\mathcal{S}p_{\mathrm{gl}}^\wedge)$ of commutative global ring spectra is presentable.*

Proof. (i) Since the global model structure of [38, Theorem 4.3.18] is stable, the suspension functor on its homotopy category is invertible, so the ∞-category $\mathcal{S}p_{\mathrm{gl}}$ is stable, compare [30, Corollary 1.4.2.27]. Moreover, by [38, Theorem 4.4.3] the triangulated homotopy category of the stable ∞-category of global spectra is compactly generated by the unreduced suspension spectra of the global classifying spaces of all compact Lie groups; in particular, the ∞-category $\mathcal{S}p_{\mathrm{gl}}$ of global spectra is presentable by [30, Corollary 1.4.4.2], and hence complete and cocomplete.

(ii) Because the smash product with a cofibrant orthogonal spectrum is a left Quillen functor for the global model structure, the symmetric monoidal smash product on $\mathcal{S}p_{\mathrm{gl}}$ preserves colimits in both variables. Since the underlying ∞-category is presentable by (i), the symmetric monoidal ∞-category $\mathcal{S}p_{\mathrm{gl}}^{\wedge}$ is presentably symmetric monoidal.

Since $\mathcal{S}p_{\mathrm{gl}}^{\wedge}$ is presentably symmetric monoidal by (ii), the category $\mathrm{CAlg}(\mathcal{S}p_{\mathrm{gl}}^{\wedge})$ of its commutative algebras is presentable by [30, Corollary 3.2.3.5]. $\square$

As we now recall, every global spectrum has underlying genuine G-spectra, for all compact Lie groups G. This process is implemented by a strong symmetric monoidal left adjoint from global spectra to genuine G-spectra, and the construction extends to forgetful functors between the respective categories of commutative and ultra-commutative ring objects.

Construction B.3 (Underlying G-spectrum)**.** We consider the functor of 1-categories

$$(-)_G \ : \ \mathrm{Sp}^O \longrightarrow \mathrm{Sp}^O_G \ , \quad X \longmapsto X_G$$

from orthogonal spectra to orthogonal G-spectra that endows an orthogonal spectrum with the trivial G-action. This trivial action functor takes global equivalences to $\underline{\pi}_*$-isomorphisms. The ‘equivariant’ smash product of orthogonal G-spectra is simply the smash product of the underlying non-equivariant orthogonal spectra with diagonal G-action. So the trivial action functor $(-)_G$ is strong symmetric monoidal for the respective smash products. The universal property of the monoidal localization $\gamma_{\mathrm{gl}}^{\wedge} \colon \mathrm{Sp}^{O,\wedge} \longrightarrow \mathcal{S}p_{\mathrm{gl}}^{\wedge}$ thus yields a unique lax symmetric monoidal functor of symmetric monoidal ∞-categories

$$\mathrm{res}_G^{\wedge} \ : \ \mathcal{S}p_{\mathrm{gl}}^{\wedge} \longrightarrow \mathcal{S}p_G^{\wedge} \tag{B.4}$$

equipped with a homotopy witnessing the commutativity of the following square on the left:

$$\begin{tikzcd} \mathrm{Sp}^{O,\wedge} \arrow[r, "\gamma_{\mathrm{gl}}^{\wedge}"] \arrow[d, "(-)_G^{\wedge}"'] & \mathcal{S}p_{\mathrm{gl}}^{\wedge} \arrow[d, "\mathrm{res}_G^{\wedge}"] \\ \mathrm{Sp}_G^{O,\wedge} \arrow[r, "\gamma_G^{\wedge}"'] & \mathcal{S}p_G^{\wedge} \end{tikzcd} \qquad \begin{tikzcd} \mathrm{CAlg}(\mathrm{Sp}^{O,\wedge}) \arrow[r, "\mathrm{CAlg}(\gamma_{\mathrm{gl}}^{\wedge})"] \arrow[d, "\mathrm{CAlg}((-)_G^{\wedge})"'] & \mathrm{CAlg}(\mathcal{S}p_{\mathrm{gl}}^{\wedge}) \arrow[d, "\mathrm{CAlg}(\mathrm{res}_G^{\wedge})"] \\ \mathrm{CAlg}(Sp_G^{O,\wedge}) \arrow[r, "\mathrm{CAlg}(\gamma_G^{\wedge})"'] & \mathrm{CAlg}(\mathcal{S}p_G^{\wedge}) \end{tikzcd}$$

The underlying functor

$$\mathrm{res}_G \ : \ \mathcal{S}p_{\mathrm{gl}} \longrightarrow \mathcal{S}p_G$$

of $\mathrm{res}_G^{\wedge}$ is then descended from the trivial action functor by the universal property of the localizations $\gamma_{\mathrm{gl}} \colon \mathrm{Sp}^O \longrightarrow \mathcal{S}p_{\mathrm{gl}}$. While a priori only lax, we show in the next theorem that $\mathrm{res}_G^{\wedge}$ is in fact strong symmetric monoidal. Lax symmetric monoidal functors induce functors of commutative algebras, yielding the commutative square of ∞-categories on the right.

A commutative square of ∞-categories

$$\begin{tikzcd} \mathcal{B} \arrow[r, "F"] \arrow[d, "v"'] & \mathcal{D} \arrow[d, "u"] \\ \mathcal{A} \arrow[r, "E"'] & \mathcal{C} \end{tikzcd}$$

is *vertically left adjointable* if the vertical functors v and u admit left adjoints $\kappa\colon \mathcal{A} \longrightarrow \mathcal{B}$ and $\lambda\colon \mathcal{C} \longrightarrow \mathcal{D}$, and the *Beck–Chevalley transform*, i.e., the composite natural transformation

$$\lambda \circ E \xrightarrow{(\lambda\circ E)\diamond\eta} \lambda \circ E \circ v \circ \kappa \sim \lambda \circ u \circ F \circ \kappa \xrightarrow{\epsilon\diamond(F\circ\kappa)} F \circ \kappa$$

of functors $\mathcal{A} \longrightarrow \mathcal{D}$ is a natural equivalence. Here $\eta\colon \mathrm{Id}_{\mathcal{A}} \longrightarrow v \circ \kappa$ is the unit of the adjunction (κ, v), and $\epsilon\colon \lambda \circ u \longrightarrow \mathrm{Id}_{\mathcal{A}}$ is the counit of the adjunction (λ, u). There are analogous definitions of *horizontally right adjointable*, and of *vertically left/right adjointable*.

Every lax symmetric monoidal functor $F^{\otimes}\colon \mathcal{C} \longrightarrow \mathcal{D}$ between symmetric monoidal ∞-categories induces a functor between the categories of commutative algebra $\mathrm{CAlg}(F^{\otimes})\colon \mathrm{CAlg}(\mathcal{C}) \longrightarrow \mathrm{CAlg}(\mathcal{D})$, by postcomposition.

Lemma B.5. *Let $F^{\otimes}\colon \mathcal{C} \longrightarrow \mathcal{D}$ be a strong symmetric monoidal functor between symmetric monoidal ∞-categories whose underlying functor $F = F^{\otimes}_{\langle 1\rangle}\colon \mathcal{C}_{\langle 1\rangle} \longrightarrow \mathcal{D}_{\langle 1\rangle}$ is a left adjoint. Then $F^{\otimes}\colon \mathcal{C} \longrightarrow \mathcal{D}$ is a left adjoint relative to* Fin*, the functor $\mathrm{CAlg}(F^{\otimes})\colon \mathrm{CAlg}(\mathcal{C}) \longrightarrow \mathrm{CAlg}(\mathcal{D})$ has a right adjoint, and the commutative square*

$$\begin{tikzcd} \mathrm{CAlg}(\mathcal{C}) \arrow[r, "\mathrm{CAlg}(F^{\otimes})"] \arrow[d, "\mathrm{ev}_{\langle 1\rangle}"'] & \mathrm{CAlg}(\mathcal{D}) \arrow[d, "\mathrm{ev}_{\langle 1\rangle}"] \\ \mathcal{C}_{\langle 1\rangle} \arrow[r, "F"'] & \mathcal{D}_{\langle 1\rangle} \end{tikzcd}$$

is horizontally right adjointable.

Proof. By [30, Corollary 7.3.2.7], the functor $F^{\otimes}$ has a relative right $G^{\otimes}\colon \mathcal{D} \longrightarrow \mathcal{C}$ over Fin that is moreover a morphism of operads. So $G^{\otimes}$ is a lax symmetric monoidal structure on a right adjoint $G = G^{\otimes}_{\langle 1\rangle}\colon \mathcal{D} \longrightarrow \mathcal{C}$ to F. The unit and counit of the relative adjunction then induce the unit and counit of an adjunction between $\mathrm{CAlg}(F^{\otimes})\colon \mathrm{CAlg}(\mathcal{C}) \longrightarrow \mathrm{CAlg}(\mathcal{D})$ and $\mathrm{CAlg}(G^{\otimes})$. The horizontal right adjointability of the square is simply the statement that the underlying functor of the right adjoint $\mathrm{CAlg}(G^{\otimes})$ to $\mathrm{CAlg}(F^{\otimes})$ is the right adjoint G to the functor F. □

Theorem B.6. *Let G be a compact Lie group.*

(i) *The functor $\mathrm{res}_G\colon \mathcal{S}p_{\mathrm{gl}} \longrightarrow \mathcal{S}p_G$ has a left adjoint and a right adjoint.*
(ii) *The functors $\mathrm{res}_G\colon \mathcal{S}p_{\mathrm{gl}} \longrightarrow \mathcal{S}p_G$ are jointly conservative as G runs over representatives for the isomorphism classes of all compact Lie groups.*
(iii) *The functors $\Phi^G \circ \mathrm{res}_G\colon \mathcal{S}p_{\mathrm{gl}} \longrightarrow \mathcal{S}p$ are jointly conservative as G runs over representatives for the isomorphism classes of all compact Lie groups.*
(iv) *The lax symmetric monoidal functor $\mathrm{res}^{\wedge}_G\colon \mathcal{S}p^{\wedge}_{\mathrm{gl}} \longrightarrow \mathcal{S}p^{\wedge}_G$ is strong symmetric monoidal.*
(v) *The functor $\mathrm{CAlg}(\mathrm{res}^{\wedge}_G)$ has a left adjoint and a right adjoint.*

Proof. (i) The functor res_G is an exact functor between stable ∞-categories, and the induced functor of triangulated homotopy categories $h(\mathrm{res}_G)\colon h(\mathcal{S}p_{\mathrm{gl}}) \longrightarrow h(\mathcal{S}p_G)$ has adjoints on either side, and thus preserves limits and colimits, by [38, Theorem 4.5.24]. Hence also the functor res_G of ∞-categories has adjoints on either side, and preserves limits and colimits.

Property (ii) holds by design, since a morphism $f\colon X \longrightarrow Y$ of orthogonal spectra is defined to be a global equivalence if and only if the morphisms of orthogonal G-spectra $f_G\colon X_G \longrightarrow Y_G$ are $\underline{\pi}_*$-isomorphisms of orthogonal G-spectra for all compact Lie groups G. Property (iii) then follows from property (ii) and the fact that the geometric fixed functors $\Phi^H \circ \mathrm{res}^G_H\colon \mathcal{S}p_G \longrightarrow \mathcal{S}p$ are jointly conservative for all closed subgroups H of G, see [35, Proposition 6.13] for an ∞-categorical treatment, and [38, Proposition 3.3.10] for a proof in the orthogonal spectrum model. We also use implicitly that the functor $\Phi^H \circ \mathrm{res}^G_H \circ \mathrm{res}_G\colon \mathcal{S}p_{\mathrm{gl}} \longrightarrow \mathcal{S}p$ is naturally equivalent to $\Phi^H \circ \mathrm{res}_H$.

(iv) Strong (versus lax) monoidality can be tested on homotopy categories; at the level of homotopy categories, the argument is explained immediately before [38, Theorem 4.5.24]. The ultimate reason is the fact that for every flat orthogonal spectrum X (i.e., cofibrant in the global model structure) and every compact Lie group G, the functor $X_G \wedge - \colon \mathrm{Sp}^O_G \longrightarrow \mathrm{Sp}^O_G$ preserves $\underline{\pi}_*$-isomorphisms, see [38, Theorem 3.5.10].

(v) Since $\mathrm{res}^\wedge_G$ is a strong symmetric monoidal left adjoint, $\mathrm{CAlg}(\mathrm{res}^\wedge_G)$ has a right adjoint by Lemma B.5. For every complete and cocomplete symmetric monoidal ∞-category $\mathcal{C}^\otimes$, the forgetful functor $\mathrm{ev}_{\langle 1\rangle} \colon \mathrm{CAlg}(\mathcal{C}^\otimes) \longrightarrow \mathcal{C}$ preserves and creates all limits and sifted colimits by Corollary 3.2.2.5 and Proposition 3.2.3.1 of [30]. Since res_G has adjoints on either side, $\mathrm{ev}_{\langle 1\rangle} \circ \mathrm{CAlg}(\mathrm{res}^\wedge_G) \sim \mathrm{res}_G \circ \mathrm{ev}_{\langle 1\rangle}$ preserves limits and sifted colimits. Since $\mathrm{ev}_{\langle 1\rangle}$ is conservative and creates limits and sifted colimits, we conclude that the functor $\mathrm{CAlg}(\mathrm{res}^\wedge_G)$ preserves limits and sifted colimits. Since the source and target of $\mathrm{CAlg}(\mathrm{res}^\wedge_G)$ are presentable, this functor has a right adjoint by the adjoint functor theorem [28, Corollary 5.5.2.9]. □

Construction B.7 (Inflation homomorphisms)**.** We let X be an orthogonal spectrum, and we let $\gamma \colon K \longrightarrow G$ be a continuous epimorphism of compact Lie groups. For every G-representation V, we have $\gamma^*(V)^K = V^G$ and

$$X(\gamma^*(V) \oplus \mathbb{R}^m)^K \;=\; \gamma^*(X(V \oplus \mathbb{R}^m))^K \;=\; X(V \oplus \mathbb{R}^m)^G \ ,$$

because γ is surjective. We define the *inflation homomorphism*

$$\gamma^* \;:\; \Phi^G_k(X) \;\longrightarrow\; \Phi^K_k(X) \tag{B.8}$$

by sending the class of a based continuous map $f \colon S^{V^G \oplus \mathbb{R}^{m+k}} \longrightarrow X(V \oplus \mathbb{R}^m)^G$ to the class of the same map, but now viewed as a map

$$f \;:\; S^{\gamma^*(V)^K \oplus \mathbb{R}^{m+k}} \;\longrightarrow\; X(\gamma^*(V) \oplus \mathbb{R}^m)^K \ .$$

Much like in [38, Construction 3.1.15], we exploit here that elements of $\Phi^K_k(X)$ can be unambiguously be represented by K-maps $S^{W^K \oplus \mathbb{R}^{m+k}} \longrightarrow X(W)^K$, for any K-representation W, not necessarily a subrepresentation of the chosen complete universe, see [38, Construction 3.1.13]. The inflation homomorphisms γ^* are clearly contravariantly functorial for composition of continuous epimorphisms. The geometric fixed point homomorphisms (A.14) are compatible with inflation in the sense that the following square commutes, by direct inspection at the level of representatives:

$$\begin{tikzcd} \pi^G_k(R) \arrow[r, "\gamma^*"] \arrow[d, "\Phi^G"'] & \pi^K_k(R) \arrow[d, "\Phi^K"] \\ \Phi^G_k(R) \arrow[r, "\gamma^*"'] & \Phi^K_k(R) \end{tikzcd} \tag{B.9}$$

The inflation homomorphism γ^* is a monoidal natural transformation between two lax symmetric monoidal functors from orthogonal spectra to abelian groups that both invert global equivalences. So by monoidal localization, γ^* descends uniquely to a monoidal natural transformation $\gamma^* \colon \bar{\Phi}^G_* \longrightarrow \bar{\Phi}^K_*$ between the descended functors $\mathcal{S}p^\wedge_{\mathrm{gl}} \longrightarrow \mathcal{G}r\mathcal{A}b^\otimes$ from the ∞-category of global spectra to graded abelian groups; we abuse notation and denote the descended transformation by the same symbol.

Construction B.10. We let $i_{\mathrm{gl}} \colon \mathcal{S}p^\wedge \longrightarrow \mathcal{S}p^\wedge_{\mathrm{gl}}$ be the unique strong symmetric monoidal left adjoint. It is fully faithful by [38, Theorem 4.5.1], and its right adjoint is the forgetful functor $\mathrm{res}^\wedge_{\{1\}} \colon \mathcal{S}p^\wedge_{\mathrm{gl}} \longrightarrow \mathcal{S}p^\wedge$. Then the composite

$$\mathcal{S}p^\wedge \;\xrightarrow{i_{\mathrm{gl}}}\; \mathcal{S}p^\wedge_{\mathrm{gl}} \;\xrightarrow{\mathrm{res}^\wedge_G}\; \mathcal{S}p^\wedge_G$$

is a strong symmetric monoidal left adjoint; by virtue of $\mathcal{S}p^\wedge$ being initial in $\mathrm{CAlg}(\mathrm{Pr}^L_{\mathrm{st}})$, the composite is uniquely monoidally equivalent to the inflation functor (A.7):

$$\mathrm{res}^\wedge_G \circ i_{\mathrm{gl}} \ \sim \ i_G \ . \tag{B.11}$$

A global spectrum Y is *left induced* if it is in the essential image of the fully faithful functor i_{gl}. In this case, the inflation homomorphism (B.8)

$$p_G^* \ : \ \bar{\pi}_k(Y) = \bar{\Phi}_k^{\{1\}}(Y) \ \longrightarrow \ \bar{\Phi}^G_k(Y)$$

associated to the unique homomorphism $p_G\colon G \longrightarrow \{1\}$ is an isomorphism, see [38, Example 4.5.10]. In particular, for every non-equivariant spectrum X in $\mathcal{S}p$, the inflation homomorphism

$$p_G^* \ : \ \bar{\pi}_*(X) \cong_{(\mathrm{B.11})} \bar{\Phi}_*^{\{1\}}(i_{\mathrm{gl}}(X)) \ \longrightarrow \ \bar{\Phi}^G_*(i_{\mathrm{gl}}(X)) \ \cong_{(\mathrm{B.11})} \ \bar{\Phi}^G_*(i_G(X)) \tag{B.12}$$

is a monoidal natural isomorphism between lax symmetric monoidal functors $\mathcal{S}p^\wedge \longrightarrow \mathcal{G}r\mathcal{A}b^\otimes$.

Proposition B.13. *There is a unique monoidal natural transformation*

$$\tau \ : \ \bar{\pi}_* \circ \Phi^G \ \xrightarrow{\cong} \ \bar{\Phi}^G_* \ : \ \mathcal{S}p^\wedge_G \ \longrightarrow \ \mathcal{G}r\mathcal{A}b^\otimes \tag{B.14}$$

such that the composite

$$\bar{\pi}_* \ \xrightarrow[\cong]{\bar{\pi}_*\circ(\mathrm{A.8})} \ \bar{\pi}_* \circ \Phi^G \circ i_G \ \xrightarrow[\cong]{\tau\circ i_G} \ \bar{\Phi}^G_* \circ i_G$$

equals the monoidal isomorphism p_G^* from (B.12)*. Moreover, τ is a monoidal natural isomorphism.*

Proof. For every orthogonal G-spectrum X, the geometric fixed point functor $\Phi^G_k\colon \mathrm{Sp}^O_G \longrightarrow \mathcal{A}b$ inverts the morphism $X \wedge i\colon X \longrightarrow X \wedge \tilde{E}\mathcal{P}$, simply because the G-fixed point functor $(-)^G$ is strong monoidal for the smash product of based G-spaces, and the map $i^G\colon S^0 = (S^0)^G \longrightarrow (\tilde{E}\mathcal{P})^G$ is a homeomorphism. So the descended functor $\bar{\Phi}^G_k\colon \mathcal{S}p_G \longrightarrow \mathcal{A}b$ inverts geometric fixed point equivalences. The functor $\bar{\pi}_* \circ \Phi^G$ also inverts geometric fixed point equivalences, so both $\bar{\Phi}^G_*$ and $\bar{\pi}_* \circ \Phi^G$ are objects of the ∞-category $\mathrm{Fun}^{\tilde{E}\mathcal{P}}_{\mathrm{lax}}(\mathcal{S}p^\wedge_G, \mathcal{G}r\mathcal{A}b^\otimes)$. Theorem A.9 shows that the functor

$$\mathrm{Fun}_{\mathrm{lax}}(i_G, \mathcal{G}r\mathcal{A}b^\otimes)\colon \ \mathrm{Fun}^{\tilde{E}\mathcal{P}}_{\mathrm{lax}}(\mathcal{S}p^\wedge_G, \mathcal{G}r\mathcal{A}b^\otimes) \ \longrightarrow \ \mathrm{Fun}_{\mathrm{lax}}(\Phi^G, \mathcal{G}r\mathcal{A}b^\otimes)$$

is an equivalence of ∞-categories, which proves the claim. □

An important consequence of the monoidal isomorphism (B.14) is that for homotopy G-ring spectra R, the two ways of assigning $\mathbb{Z}$-graded geometric fixed point rings are naturally isomorphic. On the one hand, the composite

$$\Phi^G(R) \otimes \Phi^G(R) \ \xrightarrow{\sim} \ \Phi^G(R \otimes R) \ \xrightarrow{\Phi^G(\mu)} \ \Phi^G(R)$$

turns a homotopy G-ring spectrum R into a non-equivariant homotopy ring spectrum, where the first map is the strong monoidal structure on the geometric fixed point functor. This homotopy ring spectrum has a graded homotopy ring $\bar{\pi}_*(\Phi^G(R))$. On the other hand, the lax symmetric monoidal structure on the functor $\bar{\Phi}^G_*\colon \mathcal{S}p_G \longrightarrow \mathcal{G}r\mathcal{A}b$ that descended from the point-set level pairing (A.12) provides the $\mathbb{Z}$-graded abelian group $\bar{\Phi}^G_*(R)$ with a graded ring structure via the composite

$$\bar{\Phi}^G_k(R) \times \bar{\Phi}^G_l(R) \ \xrightarrow{\times} \ \bar{\Phi}^G_{k+l}(R \otimes R) \ \xrightarrow{\bar{\Phi}^G_{k+l}(\mu)} \ \bar{\Phi}^G_{k+l}(R) \ . \tag{B.15}$$

Corollary B.16. *Let R be a homotopy G-ring spectrum. The value of the monoidal natural isomorphism* (B.14) *at the underlying G-spectrum of R is an isomorphism of graded rings from $\bar{\pi}_*(\Phi^G(R))$ to $\bar{\Phi}^G_*(R)$.*

Our next task is to prove two rectification results for 'global modules' over an orthogonal ring spectrum. The first one, Theorem B.17, is essentially an instance of Lurie's rectification theorem [30, Theorem 4.3.3.17] for modules over an algebra in a monoidal model category. It says that for every orthogonal ring spectrum A, the relative category of left A-module spectra, with respect to global equivalence, presents the ∞-category $\mathrm{LMod}_A(\mathcal{S}p^{\wedge}_{\mathrm{gl}})$ of left modules over the global ring spectrum represented by A. As we explain in the proof, Lurie's result cannot be applied directly, and some minor adjustments need to be made.

The second rectification result, Theorem B.20, applies to *commutative* orthogonal ring spectra, and lifts this result to an equivalence of symmetric monoidal ∞-categories. In other words, for every orthogonal ring spectrum A, the symmetric monoidal model category of A-module spectra presents the ∞-category $\mathrm{Mod}^{\otimes}_A(\mathcal{S}p^{\wedge}_{\mathrm{gl}})$ of modules over the commutative global ring spectrum represented by A. The entire discussion works relative to a multiplicative global family, so we will argue in that generality.

For any algebra T in a monoidal ∞-category $\mathcal{C}$, Lurie defines an ∞-category $\mathrm{LMod}_T(\mathcal{C})$ of left T-modules in [30, Example 4.2.1.8]. The construction is functorial for lax symmetric monoidal functors $F^{\otimes}\colon \mathcal{C} \longrightarrow \mathcal{D}$, which induce a functor

$$\mathrm{LMod}(F^{\otimes}) \;:\; \mathrm{LMod}_T(\mathcal{C}) \;\longrightarrow\; \mathrm{LMod}_{FT}(\mathcal{D}) \ ,$$

where $FT = \mathrm{Alg}(F^{\otimes})(T)$.

We let $\mathcal{F}$ be a multiplicative global family, i.e., a class of compact Lie groups that is closed under isomorphism, subgroups, quotient groups, and products. Orthogonal ring spectra are precisely the algebras in the symmetric monoidal 1-category $\mathrm{Sp}^{O,\wedge}$ of orthogonal spectra under smash product. For every orthogonal ring spectrum A, the $\mathcal{F}$-global model structure on the category of orthogonal spectra lifts to a model structure on the category $\mathrm{LMod}_A(\mathrm{Sp}^{O,\wedge})$ of left A-modules, by [38, Corollary 4.3.29 (i)]. The monoidal localization functor $\gamma^{\wedge}_{\mathcal{F}}\colon \mathrm{Sp}^{O,\wedge} \longrightarrow \mathcal{S}p^{\wedge}_{\mathcal{F}}$ induces a functor on module categories

$$\mathrm{LMod}(\gamma^{\wedge}_{\mathcal{F}}) \;:\; \mathrm{LMod}_A(\mathrm{Sp}^{O,\wedge}) \;\longrightarrow\; \mathrm{LMod}_{\gamma(A)}(\mathcal{S}p^{\wedge}_{\mathcal{F}}) \ ,$$

where we abbreviated the $\mathcal{F}$-global ring spectrum represented by A to

$$\gamma(A) \;=\; \mathrm{Alg}(\gamma^{\wedge}_{\mathcal{F}})(A) \ .$$

The following result is essentially an instance of Lurie's rectification theorem for modules [30, Theorem 4.3.3.17]. However, as we explain in the proof, Lurie's result cannot be applied directly, and some minor adjustments need to be made. Still, we make no particular claim to originality.

Theorem B.17. *For every multiplicative global family $\mathcal{F}$ and every orthogonal ring spectrum A, the functor*

$$\mathrm{LMod}(\gamma^{\wedge}_{\mathcal{F}}) \;:\; \mathrm{LMod}_A(\mathrm{Sp}^{O,\wedge}) \;\longrightarrow\; \mathrm{LMod}_{\gamma(A)}(\mathcal{S}p^{\wedge}_{\mathcal{F}})$$

presents the ∞-category $\mathrm{LMod}_{\gamma(A)}(\mathcal{S}p^{\wedge}_{\mathcal{F}})$ as a localization of the 1-category $\mathrm{LMod}_A(\mathrm{Sp}^{O,\wedge})$ by the class of $\mathcal{F}$-equivalences.

Proof. As already mentioned, this is essentially Lurie's rectification theorem for modules [30, Theorem 4.3.3.17], with a minor caveat: Lurie works with a *combinatorial* monoidal model category in the sense of [28, Definition A.2.6.1]. We are *not* in that context, because the category of orthogonal spectra is based on topological spaces, and hence not presentable as a 1-category. While Lurie's theorem does not apply literally, its proof works with one modification in the first step, as follows. We let

$$\gamma_A \;:\; \mathrm{LMod}_A(\mathrm{Sp}^{O,\wedge}) \;\longrightarrow\; \mathrm{LMod}^{\infty}_A$$

be a localization by the class of A-module morphisms that are $\mathcal{F}$-equivalence of underlying orthogonal spectra. Since the functor $\mathrm{LMod}(\gamma_{\mathcal{F}}^{\wedge})$ takes $\mathcal{F}$-equivalences to equivalences, there is a unique functor

$$\theta \ : \ \mathrm{LMod}_A^{\infty} \ \longrightarrow \ \mathrm{LMod}_{\gamma(A)}(\mathcal{S}p_{\mathcal{F}}^{\wedge})$$

equipped with an equivalence $\theta \circ \gamma_A \sim \mathrm{LMod}(\gamma_{\mathcal{F}}^{\wedge})$. We claim that the following diagram commutes in $\mathcal{C}\mathrm{at}_{\infty}$:

(B.18)
$$\begin{tikzcd} \mathrm{LMod}_A^{\infty} \arrow[rr, "\theta"] \arrow[dr, "G"'] & & \mathrm{LMod}_{\gamma(A)}(\mathcal{S}p_{\mathcal{F}}^{\wedge}) \arrow[dl, "G'"] \\ & \mathcal{S}p_{\mathcal{F}} & \end{tikzcd}$$

Here G' is the forgetful functor, and the functor G is descended, by the universal property of localizations, from the composite functor

$$\mathrm{LMod}_A(\mathrm{Sp}^{O,\wedge}) \ \xrightarrow{\text{forget}} \ \mathrm{Sp}^O \ \xrightarrow{\gamma_{\mathcal{F}}} \ \mathcal{S}p_{\mathcal{F}} \ ,$$

which inverts $\mathcal{F}$-equivalences of A-modules. Indeed, after precomposition with the localization functor γ_A, we obtain the diagram:

$$\begin{tikzcd} \mathrm{LMod}_A(\mathrm{Sp}^{O,\wedge}) \arrow[r, "\mathrm{LMod}(\gamma_{\mathcal{F}}^{\wedge})"] \arrow[d, "\text{forget}"'] & \mathrm{LMod}_{\gamma(A)}(\mathcal{S}p_{\mathcal{F}}^{\wedge}) \arrow[d, "G'"] \\ \mathrm{Sp}^O \arrow[r, "\gamma_{\mathcal{F}}"'] & \mathcal{S}p_{\mathcal{F}} \end{tikzcd}$$

This diagram commutes by naturality of the LMod-construction. The universal property of localizations then uniquely descend the homotopy witnessing the commutativity to a witness for the equivalence $G' \circ \theta \sim G$.

As in the proof of [30, Theorem 4.3.3.17], we now verify in five steps that the diagram (B.18) satisfies the hypotheses of [30, Corollary 4.7.3.16]. It is in step (a) of these five steps that Lurie uses the hypothesis 'combinatorial' to show that his category $N({}_A\mathrm{BMod}_B(\mathbf{A})^c)[W'^{-1}]$ is presentable. In our situation, we need to show instead that the ∞-categorical localization LMod_A^{∞} is presentable, which we justify as follows. The $\mathcal{F}$-global model structure on $\mathrm{LMod}_A(\mathrm{Sp}^{O,\wedge})$ established in [38, Corollary 4.3.29] is stable, so its homotopy category $h(\mathrm{LMod}_A^{\infty})$ naturally comes to us as a triangulated category. The free and forgetful functors

$$\mathrm{Sp}^O \ \underset{\text{forget}}{\overset{A\wedge -}{\rightleftarrows}} \mathrm{LMod}_A(\mathrm{Sp}^{O,\wedge})$$

are a Quillen adjoint functor pair for the $\mathcal{F}$-global model structures on both sides, so they induce an adjoint pair of exact functor of triangulated categories

$$h(\mathcal{S}p_{\mathcal{F}}) \ \underset{hG}{\overset{A\wedge^L -}{\rightleftarrows}} h(\mathrm{LMod}_A^{\infty})$$

between the homotopy categories of the localizations. For every compact Lie group G in the family $\mathcal{F}$, the unreduced suspension spectrum $\Sigma_+^{\infty} B_{\mathrm{gl}} G$ represent the functor π_0^G on the global stable homotopy category $h(\mathcal{S}p_{\mathcal{F}})$, see [38, Theorem 4.4.3]. By adjointness, the objects $A \wedge^L \Sigma_+^{\infty} B_{\mathrm{gl}} G$ thus represent the functor π_0^G on the category $h(\mathrm{LMod}_A^{\infty})$. Since $\mathbb{Z}$-equivariant homotopy groups take wedges of orthogonal spectra, and hence coproducts in $h(\mathrm{LMod}_A^{\infty})$ to direct sums, this proves that the object $A \wedge^L \Sigma_+^{\infty} B_{\mathrm{gl}} G$ form a set of compact generators of the triangulated category $h(\mathrm{LMod}_A^{\infty})$, as G ranges over representatives of the isomorphism classes of compact Lie groups in $\mathcal{F}$. Since the triangulated category $h(\mathrm{LMod}_A^\infty)$ is compactly generated, the stable ∞-category LMod_A^∞ is presentable. The presentability of the ∞-category $\mathrm{LMod}_{\gamma(A)}(\mathcal{S}p_{\mathcal{F}}^\wedge)$ and the proof steps (b), (c) and (d) then work exactly as in the proof of [30, Theorem 4.3.3.17].

In the final proof step (e) we must show that a specific natural morphism $\omega\colon G'\circ F' \longrightarrow G\circ F$ is an equivalence. Here F' and F are left adjoints to G' and G, respectively, with adjunction unit $\eta\colon \mathrm{Id}\longrightarrow G\circ F$ and adjunction counit $\epsilon'\colon F'\circ G'\longrightarrow \mathrm{Id}$, and the ω is the composite

$$\begin{aligned} &G'\circ F' \xrightarrow{G'\circ F'\circ\eta} G'\circ F'\circ G\circ F \\ &\xrightarrow[\sim]{G'\circ F'\circ(\mathrm{B.18})\circ F} G'\circ F'\circ G'\circ\theta\circ F \xrightarrow{G'\circ\epsilon'\circ\theta\circ F} G'\circ\theta\circ F \xrightarrow[\sim]{(\mathrm{B.18})\circ F} G\circ F\ . \end{aligned}$$

Unwinding the definitions, we must show that for every $\mathcal{F}$-global spectrum X the natural map

$$G(A\wedge^L X) \ \longrightarrow\ G'(F'(X))\sim \gamma(A)_{\langle 1\rangle}\otimes X$$

is an equivalence in $\mathcal{S}p_{\mathcal{F}}$. Every orthogonal spectrum is globally equivalent to a cofibrant object in the $\mathcal{F}$-global model structure, and these are in particular flat orthogonal spectra in the sense of [38, Definition 3.4.2]. So we can assume without loss of generality that $X=\gamma_{\mathcal{F}}(C)$ is represented by a flat orthogonal spectrum C. Then $A\wedge^L X$ is represented by the smash product $A\wedge C$ in the 1-category $\mathrm{Sp}^{O,\wedge}$. Smashing with a flat orthogonal spectrum preserves $\mathcal{F}$-global equivalences by [38, Theorem 4.3.27]; so $A\wedge C$ also represents the derived smash product $\gamma(A)_{\langle 1\rangle}\otimes\gamma_{\mathcal{F}}(C)=\gamma(A)_{\langle 1\rangle}\otimes X$. This completes the proof. □

Now we lift the previous rectification result to a symmetric monoidal rectification for commutative orthogonal ring spectra. For any commutative algebra T in a symmetric monoidal ∞-category $\mathcal{C}$, Lurie defines an ∞-operad $\mathrm{Mod}_T^\otimes(\mathcal{C})$ in [30, Definition 4.5.1.1]; Lurie uses the notation $\mathrm{Mod}_T(\mathcal{C})^\otimes$. He shows in [30, Theorem 4.5.2.1] that if the underlying category $\mathcal{C}_{\langle 1\rangle}$ admits geometric realization of simplicial objects, and if the tensor product functor $\otimes\colon\mathcal{C}_{\langle 1\rangle}\times\mathcal{C}_{\langle 1\rangle}\longrightarrow\mathcal{C}_{\langle 1\rangle}$ preserves geometric realization of simplicial objects in each variable separately, then $\mathrm{Mod}_T^\otimes(\mathcal{C})$ is in fact a symmetric monoidal ∞-category. We will make this a standing assumption throughout the following discussion. We will apply this when $\mathcal{C}_{\langle 1\rangle}$ admits arbitrary colimits, and tensor product functor $\otimes\colon\mathcal{C}_{\langle 1\rangle}\times\mathcal{C}_{\langle 1\rangle}\longrightarrow\mathcal{C}_{\langle 1\rangle}$ preserves all colimits in each variable separately, so the standing assumption holds.

We will need the following additional data of the construction:

- The construction is functorial for lax symmetric monoidal functors $F^\otimes\colon\mathcal{C}\longrightarrow\mathcal{D}$, which induce a lax symmetric monoidal functor
$$\mathrm{Mod}(T|F^\otimes)\ :\ \mathrm{Mod}_T^\otimes(\mathcal{C})\ \longrightarrow\ \mathrm{Mod}_{FT}^\otimes(\mathcal{D})\ ,$$
where $FT=\mathrm{CAlg}(F^\otimes)(T)$.
- The construction comes with a monoidal adjunction
$$T\otimes -\colon\mathcal{C} \rightleftarrows \mathrm{Mod}_T(\mathcal{C})\colon u_T \tag{B.19}$$
i.e., a relative adjunction over Fin_* in the sense of [30, Definition 7.3.22], such that the left adjoint $T\otimes -$ is strong symmetric monoidal, and the right adjoint 'forgetful' functor u is lax symmetric monoidal. One way to obtain this monoidal adjunction is to appeal to [30, Theorem 4.5.3.1] for the unique morphism $\mathbb{1}\longrightarrow T$ in $\mathrm{CAlg}(\mathcal{C})$ from an initial commutative algebra, and then exploit [30, Proposition 3.4.2.1] for the commutative operad to identify $\mathrm{Mod}_{\mathbb{1}}^\otimes(\mathcal{C})$ with $\mathcal{C}$.

And we will need the following properties of the previous data: If T is an initial object of $\mathrm{CAlg}(\mathcal{C})$— or equivalently, if the unit morphism $1_{\mathcal{C}}\longrightarrow T$ is an equivalence in $\mathcal{C}$—then the adjoint functors (B.19) are adjoint equivalences of symmetric monoidal ∞-categories; this is the special case of [30, Proposition 3.4.2.1] for the commutative operad.

In the next rectification result we consider a commutative orthogonal ring spectrum A. We continue to let $\mathcal{F}$ be a multiplicative global family. By [38, Corollary 4.3.29 (i)], the $\mathcal{F}$-global model structure on the category of orthogonal spectra lifts to a model structure on the category Mod_A of left A-modules. And if A is commutative, then by [38, Corollary 4.3.29 (ii)], this $\mathcal{F}$-global model structure is a symmetric monoidal model structure with respect to the relative smash product $\wedge_A$ of A-modules.

Commutative orthogonal ring spectra are precisely the commutative algebras in the symmetric monoidal 1-category $\mathrm{Sp}^{O,\wedge}$ of orthogonal spectra under smash product. So for every commutative orthogonal ring spectrum A, the lax symmetric monoidal localization functor $\gamma_{\mathcal{F}}^{\wedge}\colon \mathrm{Sp}^{O,\wedge} \longrightarrow \mathcal{S}p_{\mathcal{F}}^{\wedge}$ induces a lax symmetric monoidal functor on module categories

$$\mathrm{Mod}(A|\gamma_{\mathcal{F}}^{\wedge}) \ : \ \mathrm{Mod}_A^{\wedge} \ \longrightarrow \ \mathrm{Mod}^{\otimes}_{\gamma(A)}(\mathcal{S}p_{\mathcal{F}}^{\wedge}) \ ,$$

where we abbreviated $\mathrm{Mod}^{\otimes}_A(\mathrm{Sp}^{O,\wedge})$ to $\mathrm{Mod}_A^{\wedge}$, and the commutative $\mathcal{F}$-global ring spectrum represented by A to

$$A_{\mathcal{F}} \ = \ \mathrm{CAlg}(\gamma_{\mathcal{F}}^{\wedge})(A) \ .$$

Theorem B.20. *For every multiplicative global family $\mathcal{F}$ and every commutative orthogonal ring spectrum A, the lax symmetric monoidal functor*

$$\mathrm{Mod}(A|\gamma_{\mathcal{F}}^{\wedge}) \ : \ \mathrm{Mod}_A^{\wedge} \ \longrightarrow \ \mathrm{Mod}^{\wedge}_{A_{\mathcal{F}}}(\mathcal{S}p_{\mathcal{F}}^{\wedge})$$

is a monoidal localization at the class of $\mathcal{F}$-equivalences of A-modules.

Proof. We start with the special case where the underlying orthogonal spectrum of A is flat. We let

$$\gamma_A^{\wedge} \ : \ \mathrm{Mod}_A^{\wedge} \ \longrightarrow \ (\mathrm{Mod}_A^{\wedge})^{\infty}$$

be a monoidal localization of $\mathcal{F}$-global model structure, which exists by [37, Theorem A.7]. Now we show that $F^{\otimes}$ is strong (and not just lax) symmetric monoidal. The free A-module functor and the forgetful functor (B.19) form a Quillen adjoint functor pair

$$A \wedge -\colon \mathrm{Sp}^{O,\wedge} \ \rightleftarrows \ \mathrm{Mod}_A^{\wedge}\colon u_A$$

for the $\mathcal{F}$-global model structures. Moreover, both functors preserve $\mathcal{F}$-global equivalence, the right adjoint u by definition, and the left adjoint $A\wedge -$ by [38, Theorem 4.3.27] because A is flat. Moreover, the left adjoint comes with a strong symmetric monoidal structure, the right adjoint with a lax one, making them monoidal adjoints. The entire monoidal adjunction data thus descends to the localizations, and we obtain a monoidal adjoint functor pair participating in the following diagram in $\mathrm{SymMon}^{\mathrm{lax}}$:

$$\text{(B.21)} \qquad \begin{array}{ccc} \mathrm{Sp}^{O,\wedge} & \underset{u_A}{\overset{A\wedge -}{\rightleftarrows}} & \mathrm{Mod}_A^{\wedge} \\ {\scriptstyle \gamma_{\mathcal{F}}^{\wedge}}\downarrow & & \downarrow{\scriptstyle \gamma_A^{\wedge}} \\ \mathcal{S}p_{\mathcal{F}}^{\wedge} & \underset{u_A^{\infty}}{\overset{L^{\wedge}}{\rightleftarrows}} & (\mathrm{Mod}_A^{\wedge})^{\infty} \end{array}$$

Here the square involving the two vertical left adjoints commutes; and the square involving the two vertical right adjoints commutes; and the descended left adjoint $L^{\wedge}$ is again strong (as opposed to only lax) symmetric monoidal.

We write $A^{\infty} = \mathrm{CAlg}(\gamma_A^{\wedge})(A)$ for the commutative algebra in $(\mathrm{Mod}_A^{\wedge})^{\infty}$ represented by the commutative orthogonal ring spectrum A. Applying $\mathrm{CAlg}(-)$ to the commutative diagram (B.21) provides an equivalence

$$\mathrm{CAlg}(u_A^{\infty})(A^{\infty}) \ \sim \ \mathrm{CAlg}(\gamma_{\mathcal{F}}^{\wedge})(A) \ = \ A_{\mathcal{F}}$$

in $\mathrm{CAlg}(\mathcal{S}p^\wedge_{\mathcal{F}})$ that we use to identify these two objects. The unit of the descended monoidal adjunction is a monoidal natural transformation

$$\eta^\wedge \ :\ \mathrm{Id}_{\mathcal{S}p^\wedge_{\mathcal{F}}} \ \longrightarrow\ u^\infty_A \circ L^\wedge \ :\ \mathcal{S}p^\wedge_{\mathcal{F}} \ \longrightarrow\ \mathcal{S}p^\wedge_{\mathcal{F}}\ .$$

This induces a natural transformation of functors $\mathcal{S}p^\wedge_{\mathcal{F}} \longrightarrow \mathcal{S}p^\wedge_{\mathcal{F}}$

$$\begin{aligned} A_{\mathcal{F}} \otimes Y \ \sim\ (u^\infty_A \circ L^\wedge)(\mathbb{S}) \otimes Y \ &\xrightarrow{\mathrm{Id}\otimes\eta} (u^\infty_A \circ L^\wedge)(\mathbb{S}) \otimes (u^\infty_A \circ L^\wedge)(Y) \\ &\longrightarrow (u^\infty_A \circ L^\wedge)(\mathbb{S}\otimes Y) \ \sim\ (u^\infty_A \circ L^\wedge)(Y)\ , \end{aligned} \tag{B.22}$$

where the second, unnamed, morphism is the lax monoidal structure on the composite functor, and the final one is induced by the left unit equivalence $\mathbb{S} \otimes Y \sim Y$. We claim that this composite is an equivalence. The restriction $\gamma^\wedge_{\mathcal{F},\mathrm{cof}} \colon \mathrm{Sp}^{O,\wedge}_{\mathrm{cof}} \longrightarrow \mathcal{S}p^\wedge_{\mathcal{F}}$ of the monoidal localization to cofibrant (i.e., flat) orthogonal spectra is strong symmetric monoidal and essentially surjective on objects. So suffices to show the claim after precomposing the transformation (B.22) with $\gamma^\wedge_{\mathcal{F}}$, restricted to cofibrant orthogonal spectra. Since $u^\infty_A \circ L^\wedge$ is descended from the homotopical functor $u_A \circ (A \wedge -) \colon \mathrm{Sp}^{O,\wedge} \longrightarrow \mathrm{Sp}^{O,\wedge}$, we have $(u^\infty_A \circ L^\wedge)(\gamma_{\mathcal{F}}(X)) = \gamma_{\mathcal{F}}(A \wedge X)$. And for $Y = \gamma_{\mathcal{F}}(X)$ with X in $\mathrm{Sp}^O_{\mathrm{cof}}$, the lax monoidal structure of $\gamma^\wedge_{\mathcal{F}}$ provides a morphism of $\mathcal{F}$-global spectra

$$A_{\mathcal{F}} \otimes \gamma_{\mathcal{F}}(X) = \gamma_{\mathcal{F}}(A) \otimes \gamma_{\mathcal{F}}(X) \ \longrightarrow\ \gamma_{\mathcal{F}}(A \wedge X) = (u^\infty_A \circ L^\wedge)(\gamma_{\mathcal{F}}(X))$$

that is an equivalence because both A and X are cofibrant. So for $Y = \gamma_{\mathcal{F}}(X)$ for cofibrant X, the transformation (B.22) specializes to an equivalence. This concludes the proof that (B.22) is a monoidal equivalence.

We define a lax symmetric monoidal functor $F^\otimes \colon (\mathrm{Mod}^\wedge_A)^\infty \longrightarrow \mathrm{Mod}_{A_{\mathcal{F}}}(\mathcal{S}p^\wedge_{\mathcal{F}})$ as the composite

$$(\mathrm{Mod}^\wedge_A)^\infty \ \underset{\sim}{\xleftarrow{u_{A^\infty}}}\ \mathrm{Mod}_{A^\infty}((\mathrm{Mod}^\wedge_A)^\infty) \ \xrightarrow{\mathrm{Mod}(A^\infty|u^\infty_A)}\ \mathrm{Mod}_{A_{\mathcal{F}}}(\mathcal{S}p^\wedge_{\mathcal{F}})\ .$$

We claim that $F^\otimes$ is an equivalence of symmetric monoidal ∞-categories. We show first that the underlying functor $F^\otimes_{\langle 1\rangle} \colon (\mathrm{Mod}^\wedge_A)^\infty_{\langle 1\rangle} \longrightarrow \mathrm{Mod}^\otimes_{A_{\mathcal{F}}}(\mathcal{S}p^\wedge_{\mathcal{F}})_{\langle 1\rangle}$ is an equivalence of ∞-categories. To see this, we contemplate the following larger diagram in $\mathrm{SymMon}^{\mathrm{lax}}$:

$$\begin{tikzcd}
\mathrm{Mod}^\wedge_A \arrow[rr, "\gamma^\wedge_A"] & & (\mathrm{Mod}^\wedge_A)^\infty \arrow[dd, bend left=60, "F^\otimes"] \\
\mathrm{Mod}^\otimes_A(\mathrm{Mod}^\wedge_A) \arrow[u, "u_A", "\cong"'] \arrow[rr, "\mathrm{Mod}(A|\gamma^\wedge_A)"] \arrow[d, "\mathrm{Mod}(A|u_A)"] & & \mathrm{Mod}^\otimes_{A^\infty}((\mathrm{Mod}^\wedge_A)^\infty) \arrow[u, "u_{A^\infty}"', "\sim"] \arrow[d, "\mathrm{Mod}(A^\infty|u^\infty_A)"'] \\
\mathrm{Mod}^\wedge_A(\mathrm{Sp}^{O,\wedge}) \arrow[uu, equal, bend left=60] \arrow[rr, "\mathrm{Mod}(A|\gamma^\wedge_{\mathcal{F}})"'] & & \mathrm{Mod}^\otimes_{A_{\mathcal{F}}}(\mathcal{S}p^\wedge_{\mathcal{F}})
\end{tikzcd}$$

The upper rectangle commutes by naturality of the forgetful functors; the homotopy witnessing the commutativity of the lower rectangle is provided by applying $\mathrm{Mod}(A|-)$ to the diagram (B.21). The commutative outer part of this diagram provides a monoidal equivalence of lax symmetric monoidal functors

$$F^\otimes \circ \gamma^\wedge_A \ \sim\ \mathrm{Mod}(A|\gamma^\wedge_{\mathcal{F}}) \ :\ \mathrm{Mod}^\wedge_A \ \longrightarrow\ \mathrm{Mod}^\otimes_{A_{\mathcal{F}}}(\mathcal{S}p^\wedge_{\mathcal{F}})\ . \tag{B.23}$$

At the level of underlying ∞-categories, this provides an equivalence of functors

$$F^\otimes_{\langle 1\rangle} \circ \gamma_A \ \sim\ \mathrm{Mod}(A|\gamma^\wedge_{\mathcal{F}})_{\langle 1\rangle} \ :\ \mathrm{Mod}_A \ \longrightarrow\ \mathrm{Mod}_{A_{\mathcal{F}}}(\mathcal{S}p^\wedge_{\mathcal{F}})\ . \tag{B.24}$$

Corollary 4.5.1.6 of [30] exhibits a natural equivalence $\mathrm{Mod}^\otimes_T(\mathcal{C})_{\langle 1\rangle} \sim \mathrm{LMod}_T(\mathcal{C})$ between the underlying ∞-category of $\mathrm{Mod}^\otimes_T(\mathcal{C})$ and the ∞-category $\mathrm{LMod}_{uT}(\mathcal{C})$ of left modules over the underlying associative algebra of the commutative algebra T. We apply this to $\mathcal{C} = \mathcal{S}p^{\wedge}_{\mathcal{F}}$ and $T = A_{\mathcal{F}}$; when combined with Theorem B.17, we deduce that the functor $\mathrm{Mod}^{\otimes}(A|\gamma^{\wedge}_{\mathcal{F}})_{\langle 1\rangle}$ presents $\mathrm{Mod}^{\otimes}_{A_{\mathcal{F}}}(\mathcal{S}p^{\wedge}_{\mathcal{F}})_{\langle 1\rangle}$ as a localization of the 1-category Mod_A at the class of $\mathcal{F}$-equivalences. The underlying functor $\gamma_A = (\gamma^{\wedge}_A)_{\langle 1\rangle}\colon \mathrm{Mod}_A \longrightarrow (\mathrm{Mod}^{\wedge}_A)^{\infty}_{\langle 1\rangle}$ is also a localization at the class of $\mathcal{F}$-equivalences. The equivalence (B.24) and the fact that γ_A and $\mathrm{Mod}(A|\gamma^{\wedge}_{\mathcal{F}})_{\langle 1\rangle}$ are localizations at the same class of morphisms then shows that $F^{\otimes}_{\langle 1\rangle}$ is an equivalence of ∞-categories.

Now we show that $F^{\otimes}$ is strong (and not just lax) symmetric monoidal. Since the underlying functor $F^{\otimes}_{\langle 1\rangle}$ is an equivalence of ∞-categories, it preserves colimits. The underlying ∞-category of $(\mathrm{Mod}^{\wedge}_A)^{\infty}$ is generated under colimits by the 'free' objects, i.e., the essential image of the functor $L^{\wedge}_{\langle 1\rangle}\colon \mathcal{S}p_{\mathcal{F}} \longrightarrow (\mathrm{Mod}^{\wedge}_A)^{\infty}_{\langle 1\rangle}$. Moreover, the monoidal product of $(\mathrm{Mod}^{\wedge}_A)^{\infty}$ was derived from a symmetric monoidal model category, so it preserves colimits in each variable separately. So in order to show that the lax functor $F^{\otimes}$ is strong, it suffices to show that the composite $F^{\otimes} \circ L^{\wedge}\colon \mathcal{S}p^{\wedge}_{\mathcal{F}} \longrightarrow \mathrm{Mod}^{\otimes}_{A_{\mathcal{F}}}(\mathcal{S}p^{\wedge}_{\mathcal{F}})$ is strong (as opposed to only lax).

The following diagram commutes in $\mathrm{SymMon}^{\mathrm{lax}}$ by naturality of the forgetful functor $u_T\colon \mathrm{Mod}^{\otimes}_T(\mathcal{C}^{\otimes}) \longrightarrow \mathcal{C}^{\otimes}$:

$$\begin{tikzcd}
\mathcal{S}p^{\wedge}_{\mathcal{F}} \arrow[rr, "L^{\wedge}"] & & (\mathrm{Mod}^{\wedge}_A)^{\infty} \arrow[dd, bend left=60, "F^{\otimes}"] \\
\mathrm{Mod}^{\wedge}_{\mathbb{S}}(\mathcal{S}p^{\wedge}_{\mathcal{F}}) \arrow[u, "u_{\mathbb{S}}", "\sim"'] \arrow[rr, "\mathrm{Mod}(\mathbb{S}|L^{\wedge})"] \arrow[drr, bend right=10, "\mathrm{Mod}(\mathbb{S}|A_{\mathcal{F}}\otimes -)"'] & & \mathrm{Mod}^{\otimes}_{A^{\infty}}((\mathrm{Mod}^{\wedge}_A)^{\infty}) \arrow[u, "u_{A^{\infty}}", "\sim"'] \arrow[d, "\mathrm{Mod}(A^{\infty}|u^{\infty}_A)"'] \\
& & \mathrm{Mod}^{\otimes}_{A_{\mathcal{F}}}(\mathcal{S}p^{\wedge}_{\mathcal{F}})
\end{tikzcd}$$

The homotopy that makes the lower triangle commute is obtained by applying $\mathrm{Mod}(\mathbb{S}|-)$ to the monoidal equivalence (B.22). The counter clockwise composite $\mathcal{S}p^{\wedge}_{\mathcal{F}} \longrightarrow \mathrm{Mod}_{A_{\mathcal{F}}}(\mathcal{S}p^{\wedge}_{\mathcal{F}})$ is the monoidal left adjoint $A_{\mathcal{F}} \otimes -\colon \mathcal{S}p^{\wedge}_{\mathcal{F}} \longrightarrow \mathrm{Mod}_{A_{\mathcal{F}}}(\mathcal{S}p^{\wedge}_{\mathcal{F}})$ to the forgetful functor $u_{A_{\mathcal{F}}}$, and hence strong (as opposed to only lax) symmetric monoidal. Hence the composite $F^{\otimes} \circ L^{\wedge}$ is strong symmetric monoidal.

This concludes the proof that the functor $F^{\otimes}$ is strong symmetric monoidal. Since the underlying functor $F^{\otimes}_{\langle 1\rangle}$ is an equivalence of ∞-categories, $F^{\otimes}$ is an equivalence of symmetric monoidal ∞-categories. The monoidal natural equivalence $F^{\otimes} \circ \gamma^{\wedge}_A \sim \mathrm{Mod}(A|\gamma^{\wedge}_{\mathcal{F}})$ from (B.23) and the fact that $\gamma^{\wedge}_A$ is a monoidal localization completes the proof that the lax symmetric monoidal functor $\mathrm{Mod}(A|\gamma^{\wedge}_{\mathcal{F}})$ is a monoidal localization. This completes the proof in the special case where the underlying orthogonal spectrum of A is flat.

In the general case we choose a cofibrant replacement $f\colon A^{\flat} \longrightarrow A$ in the global model structure on the category of commutative orthogonal ring spectra established in [38, Theorem 5.4.3 (i)]. In other words, the commutative orthogonal ring spectrum $A^{\flat}$ is cofibrant, and the morphism f is a global equivalence of underlying orthogonal spectra. Because the sphere spectrum $\mathbb{S}$ is flat as an orthogonal spectrum, [38, Theorem 5.4.3 (ii)] shows that the unit morphism $\mathbb{S} \longrightarrow A^{\flat}$ is a flat cofibration of orthogonal spectra. In particular, the underlying orthogonal spectrum of $A^{\flat}$ is flat. Scalar extension along the morphism f

$$f_* \ : \ \mathrm{Mod}^{\wedge}_{A^{\flat}} \ \longrightarrow \ \mathrm{Mod}^{\wedge}_A \ , \quad M \ \longmapsto \ A \wedge_{A^{\flat}} M \ .$$

is a strong symmetric monoidal functor, and left Quillen equivalence. Indeed, the right adjoint $f^*\colon \mathrm{Mod}^{\wedge}_A \longrightarrow \mathrm{Mod}^{\wedge}_{A^{\flat}}$ preserves $\mathcal{F}$-equivalences and $\mathcal{F}$-global fibrations by design, since these are defined on underlying orthogonal spectra. So f^* is a fully homotopical right Quillen functor. Moreover, for every cofibrant $A^{\flat}$-module M, the adjunction unit $M \longrightarrow f^*(A \wedge_{A^{\flat}} M)$ is given on underlying orthogonal spectra by the morphism

$$f \wedge_{A^\flat} M \;:\; M \cong A^\flat \wedge_{A^\flat} M \;\longrightarrow\; A \wedge_{A^\flat} M \;,$$

which is a global equivalence by [38, Proposition 4.3.30]. So the adjoint functor pair (f_*, f^*) is a Quillen equivalence, as claimed.

The functor $\mathrm{CAlg}(\gamma^\wedge_{\mathcal{F}})\colon \mathrm{CAlg}(\mathrm{Sp}^{O,\wedge}) \longrightarrow \mathrm{CAlg}(\mathcal{S}p^\wedge_{\mathcal{F}})$ takes the $\mathcal{F}$-equivalence of commutative orthogonal ring spectra $f\colon A^\flat \longrightarrow A$ to an equivalence $f_{\mathcal{F}} = \mathrm{CAlg}(\gamma^\wedge_{\mathcal{F}})(f)\colon A^\flat_{\mathcal{F}} \longrightarrow A_{\mathcal{F}}$ in $\mathrm{CAlg}(\mathcal{S}p^\wedge_{\mathcal{F}})$. The diagram of associated restriction functors then commutes in $\mathrm{SymMon}^{\mathrm{lax}}$:

$$\begin{tikzcd}
\mathrm{Mod}^\wedge_A \arrow[r, "f^*"] \arrow[d, "\mathrm{Mod}(A|\gamma^\wedge_{\mathcal{F}})"'] & \mathrm{Mod}^\wedge_{A^\flat} \arrow[d, "\mathrm{Mod}(A^\flat|\gamma^\wedge_{\mathcal{F}})"] \\
\mathrm{Mod}^\wedge_{A_{\mathcal{F}}}(\mathcal{S}p^\wedge_{\mathcal{F}}) \arrow[r, "\sim", "f^*_{\mathcal{F}}"'] & \mathrm{Mod}^\wedge_{A^\flat_{\mathcal{F}}}(\mathcal{S}p^\wedge_{\mathcal{F}})
\end{tikzcd}$$

The functor $\mathrm{Mod}(A^\flat|\gamma^\wedge_{\mathcal{F}})$ is monoidal localization by the above, because $A^\flat$ is flat as an orthogonal spectrum. Since f_* is a strong symmetric monoidal left Quillen equivalence, the clockwise composite in the above diagram is then also a monoidal localization. Since $f_{\mathcal{F}}\colon A^\flat_{\mathcal{F}} \longrightarrow A_{\mathcal{F}}$ is an equivalence, the lower vertical restriction functor is an equivalence of symmetric monoidal ∞-categories. So also the left vertical functor $\mathrm{Mod}(A|\gamma^\wedge_{\mathcal{F}})$ is a monoidal localization. This concludes the proof. $\square$

In this remainder of this appendix we study the more refined and highly structured version of commutative global ring spectra that we call 'ultra-commutative'.

Definition B.25 (Ultra-commutative ring spectra)**.** The ∞*-category of ultra-commutative ring spectra* $\mathrm{URing}_{\mathrm{gl}}$ is the localization of the 1-category of commutative orthogonal ring spectra at the class of those morphisms that are global equivalences of underlying orthogonal spectra.

Construction B.26. We write

$$\gamma^{uc}_{\mathrm{gl}} \;:\; \mathrm{CAlg}(\mathrm{Sp}^{O,\wedge}) \;\longrightarrow\; \mathrm{URing}_{\mathrm{gl}}$$

for the localization functor that is part of the definition of the ∞-category of ultra-commutative ring spectra. We shall consistently abuse notation and denote a commutative orthogonal ring spectrum and the ultra-commutative ring spectra that it represents, i.e., the image under the localization functor γ^{uc}, by the same symbol.

We define another 'forgetful' functor

$$u_{\mathrm{gl}} \;:\; \mathrm{URing}_{\mathrm{gl}} \;\longrightarrow\; \mathrm{CAlg}(\mathcal{S}p^\wedge_{\mathrm{gl}}) \tag{B.27}$$

from ultra-commutative ring spectra to commutative global ring spectra. The monoidal localization $\gamma^\wedge_{\mathrm{gl}} : \mathrm{Sp}^{O,\wedge} \longrightarrow \mathcal{S}p^\wedge_{\mathrm{gl}}$ induces a functor

$$\mathrm{CAlg}(\gamma^\wedge_{\mathrm{gl}}) \;:\; \mathrm{CAlg}(\mathrm{Sp}^{O,\wedge}) \;\longrightarrow\; \mathrm{CAlg}(\mathcal{S}p^\wedge_{\mathrm{gl}})$$

between the respective categories of commutative algebras, i.e., from the 1-category of commutative orthogonal ring spectra to the ∞-category of commutative global ring spectra. Since the functor γ^{uc} inverts global equivalences, the functor $\mathrm{CAlg}(\gamma^\wedge_{\mathrm{gl}})$ sends global equivalence of commutative orthogonal ring spectra to equivalences in $\mathrm{CAlg}(\mathcal{S}p^\wedge_{\mathrm{gl}})$. So the universal property of localizations provide an essentially unique functor (B.27) equipped with a natural equivalence

$$u_{\mathrm{gl}} \circ \gamma^{uc}_{\mathrm{gl}} \;\sim\; \mathrm{CAlg}(\gamma^\wedge_{\mathrm{gl}}) \;.$$

The forgetful functor (B.27) is *not* an equivalence; rather, it exhibits the ∞-category of ultra-commutative ring spectra as monadic and comonadic over the ∞-category of commutative global ring spectra, see Theorem B.28 (iii) below. This suggests thinking of an ultra-commutative ring spectrum as extra structure on a commutative global ring spectra. The difference between ultra-commutative and commutative global ring spectra can already been seen at the level of natural operations: the 0th equivariant homotopy groups of a commutative global ring spectrum form a global Green functor [38, Definition 5.1.3]. For ultra-commutative ring spectra, this Green functor comes with additional multiplicative operations that can equivalently be encoded as power operations or norm maps; depending on the perspective, these operations enhance the global Green functor to a global power functor [38, Theorem 5.1.11], or a global Tambara functor, see [38, Remark 5.1.7]. At the level of homotopy operations, the comonadicity is made explicit in [38, Theorem 5.2.13], and the monadicity is sketched in [38, Remark 5.2.20].

Theorem B.28.
(i) *The functor* $\mathrm{ev}_{\langle 1\rangle}\circ u_{\mathrm{gl}}\colon \mathrm{URing}_{\mathrm{gl}} \longrightarrow \mathcal{S}p_{\mathrm{gl}}$ *is monadic and preserves sifted colimits.*
(ii) *The* ∞*-category* $\mathrm{URing}_{\mathrm{gl}}$ *of ultra-commutative ring spectra is presentable.*
(iii) *The functor* $u_{\mathrm{gl}}\colon \mathrm{URing}_{\mathrm{gl}} \longrightarrow \mathrm{CAlg}(\mathcal{S}p_{\mathrm{gl}}^{\wedge})$ *is both monadic and comonadic.*

Proof. We start by recording that the ∞-category $\mathrm{URing}_{\mathrm{gl}}$ is complete and cocomplete, because it is the localization of a model structure established in [38, Theorem 5.3.4], see for example [3, Theorem 2.5.9].

We recall that $\gamma_{\mathrm{gl}}^{\wedge}\colon \mathrm{Sp}^{O,\wedge} \longrightarrow \mathcal{S}p_{\mathrm{gl}}^{\wedge}$ denotes the monoidal localization functor from orthogonal spectra to global spectra. To attack the remaining items, we show that the functor

$$\mathrm{CAlg}(\gamma_{\mathrm{gl}}^{\wedge})\colon \mathrm{CAlg}(\mathrm{Sp}^{O,\wedge}) \longrightarrow \mathrm{CAlg}(\mathcal{S}p_{\mathrm{gl}}^{\wedge})$$

preserves two particular kinds of colimits:

(1) The functor $\mathrm{CAlg}(\gamma_{\mathrm{gl}}^{\wedge})$ takes pushouts along cofibrations between cofibrant commutative orthogonal ring spectra to pushouts in $\mathrm{CAlg}(\mathcal{S}p_{\mathrm{gl}}^{\wedge})$.
(2) The functor $\mathrm{CAlg}(\gamma_{\mathrm{gl}}^{\wedge})$ takes coproducts between cofibrant commutative orthogonal ring spectra to coproducts.

Granted this for the moment, we now deduce from (1) and (2) the fact that

(3) the functor $u\colon \mathrm{URing}_{\mathrm{gl}} \longrightarrow \mathrm{CAlg}(\mathcal{S}p_{\mathrm{gl}}^{\wedge})$ preserves all colimits.

To prove (3), we exploit that the cofibrations and weak equivalence of the global model category structure from [38, Theorem 5.3.4] make that category $\mathrm{CAlg}(\mathrm{Sp}^{O,\wedge})$ of commutative orthogonal ring spectra into a *category with weak equivalences and cofibrations* in the sense of Cisinski [10, Definition 7.4.12] that is moreover *homotopy cocomplete* in the sense of [10, Definition 7.7.2]. In this language, property (1) means that the functor $\mathrm{CAlg}(\gamma_{\mathrm{gl}}^{\wedge})$ is *right exact* in the sense of [10, Definition 7.5.2], with respect to the degenerate cofibration structure on the ∞-category $\mathrm{CAlg}(\mathcal{S}p_{\mathrm{gl}}^{\wedge})$, i.e., where all morphisms are cofibrations, and the weak equivalences are the equivalences. And property (2) means that the functor $\mathrm{CAlg}(\gamma_{\mathrm{gl}}^{\wedge})$ is *homotopy cocontinuous* in the sense of [10, Definition 7.7.2]. Since the weak equivalences in the degenerate cofibration structure on $\mathrm{CAlg}(\mathcal{S}p_{\mathrm{gl}}^{\wedge})$ are already invertible, $\mathrm{CAlg}(\mathcal{S}p_{\mathrm{gl}}^{\wedge})$ is its own localization, and the left derived functor of $\mathrm{CAlg}(\gamma_{\mathrm{gl}}^{\wedge})$ is nothing but the factorization $u\colon \mathrm{URing}_{\mathrm{gl}} \longrightarrow \mathrm{CAlg}(\mathcal{S}p_{\mathrm{gl}}^{\wedge})$ through the localization of the source. The dual of [10, Corollary 7.7.5] thus shows that u preserves all colimits.

Now we show properties (1) and (2). For (1) we consider a commutative algebra T in a symmetric monoidal ∞-category $\mathcal{D}$. The construction of $\mathrm{Mod}_T(\mathcal{D})^{\otimes}$ comes with a lax symmetric monoidal forgetful functor

$$u^* \;:\; \mathrm{Mod}_T^{\otimes}(\mathcal{D}) \;\longrightarrow\; \mathcal{D}\ .$$

The induced functor of commutative algebras

$$\mathrm{CAlg}(u^*) \ : \ \mathrm{CAlg}(\mathrm{Mod}_T^{\otimes}(\mathcal{D})) \ \longrightarrow \ \mathrm{CAlg}(\mathcal{D})$$

takes the initial object of the source to T; so it lifts to a functor

$$\mathrm{CAlg}(\mathrm{Mod}_T(\mathcal{D})^{\otimes}) \ \xrightarrow{\ \sim\ } \ T\backslash \mathrm{CAlg}(\mathcal{D}) \ , \tag{B.29}$$

and this functor is an equivalence, see [30, Corollary 3.4.1.5] for $\mathcal{O} = \mathcal{C}omm$, and taking the fiber over T.

Next we consider a commutative orthogonal ring spectrum A. A general property of any monoidal localization $\gamma_{\mathcal{M}}^{\otimes} \colon N(\mathcal{M}^{\otimes}) \longrightarrow \mathcal{M}_{\infty}^{\otimes}$ of a symmetric monoidal model category is that the restriction to $N(\mathcal{M}_{\mathrm{cof}}^{\otimes})$ is strong (as opposed to lax) symmetric monoidal. For the monoidal localization $\mathrm{Mod}(A|\gamma_{\mathrm{gl}}^{\wedge})$ provided by Theorem B.20, for the global family of all compact Lie groups, this becomes the statement that the lax symmetric monoidal composite

$$\mathrm{Mod}_{\mathrm{cof}}(\gamma_{\mathrm{gl}}^{\wedge}) \ : \ \ \mathrm{Mod}_{A,\mathrm{cof}}^{\wedge} \xrightarrow{\mathrm{incl}} \mathrm{Mod}_A^{\wedge} \xrightarrow{\mathrm{Mod}(A|\gamma_{\mathrm{gl}}^{\wedge})} \mathrm{Mod}_{\gamma(A)}^{\wedge}(\mathcal{S}p_{\mathrm{gl}}^{\wedge})$$

is strong symmetric monoidal. Since finite coproducts in categories of commutative algebras are given by the underlying monoidal product, the induced functor on categories of commutative algebras preserves finite coproducts. This functor features in the composite:

$$\mathrm{CAlg}(\mathrm{Mod}_{A,\mathrm{cof}}^{\wedge}) \ \xrightarrow{\mathrm{CAlg}(\mathrm{Mod}(A|\gamma_{\mathrm{gl}}^{\wedge}))} \ \mathrm{CAlg}(\mathrm{Mod}_{\gamma(A)}^{\wedge}(\mathcal{S}p_{\mathrm{gl}}^{\wedge})) \ \xrightarrow[\sim]{(\mathrm{B.29})} \gamma(A)\backslash \mathrm{CAlg}(\mathcal{S}p_{\mathrm{gl}}^{\wedge}) \ . \tag{B.30}$$

The first functor in this composite preserves finite coproducts, and the second one is an equivalence as a special case of (B.29). So also the composite (B.30) preserves finite coproducts.

Now we can wrap up and prove (1). We consider a pushout square in $\mathrm{CAlg}(\mathrm{Sp}^{O,\wedge})$

$$\begin{tikzcd} A \arrow[r, "j"] \arrow[d, "i"'] & C \arrow[d, "k"] \\ B \arrow[r, "h"'] & D \end{tikzcd} \tag{B.31}$$

in which all four commutative orthogonal ring spectra are cofibrant, and all morphisms are cofibrations, in the global model structure of $\mathrm{CAlg}(\mathrm{Sp}^{O,\wedge})$ established in [38, Theorem 5.3.4]. We write $l = hi = kh \colon A \longrightarrow D$ for the composite. By [38, Theorem 5.4.3 (ii a)], the underlying morphisms of A-modules of i, j and l are cofibrations in the global model structure on A-modules from [38, Corollary 4.3.29 (i)]. Since A is cofibrant as a module over itself, we conclude that all of A, B, C and D have cofibrant underlying A-modules. So we can—and will—view the pushout square (B.31) as a pushout square in $\mathrm{CAlg}(\mathrm{Mod}_{A,\mathrm{cof}}^{\wedge})$. Since A is initial in $\mathrm{CAlg}(\mathrm{Mod}_{A,\mathrm{cof}}^{\wedge})$, this means that the morphisms $h \colon B \longrightarrow D$ and $k \colon C \longrightarrow D$ witness $l \colon A \longrightarrow D$ as the coproduct of $i \colon A \longrightarrow B$ and $j \colon A \longrightarrow C$ in the category $\mathrm{CAlg}(\mathrm{Mod}_{A,\mathrm{cof}}^{\wedge})$. Since the composite (B.30) preserves binary coproducts, we conclude that $\gamma(h)$ and $\gamma(k)$ express $\gamma(l) \colon \gamma(A) \longrightarrow \gamma(D)$ as the coproduct of $\gamma(i) \colon \gamma(A) \longrightarrow \gamma(B)$ and $\gamma(j) \colon \gamma(A) \longrightarrow \gamma(C)$ in the slice category $\gamma(A)\backslash \mathrm{CAlg}(\mathcal{S}p_{\mathrm{gl}}^{\wedge})$. This precisely means that γ takes the pushout square (B.31) to a pushout in the ∞-category $\mathrm{CAlg}(\mathcal{S}p_{\mathrm{gl}}^{\wedge})$, and we have shown claim (1).

Now we prove claim (2) by exploiting that infinite coproducts are specific filtered colimits of finite coproducts. The initial object in both $\mathrm{CAlg}(\mathrm{Sp}^{O,\wedge})$ and $\mathrm{CAlg}(\mathcal{S}p_{\mathrm{gl}}^{\wedge})$ is given by the monoidal unit, the sphere spectrum, with its unique commutative algebra structure. So the functor $\mathrm{CAlg}(\gamma_{\mathrm{gl}}^{\wedge})$ preserves initial objects. Since $\mathrm{CAlg}(\gamma_{\mathrm{gl}}^{\wedge})$ preserves initial objects and pushouts along cofibrations between cofibrant objects, it also preserves all finite coproducts of cofibrant objects.

Now we let $\{R_i\}_{i\in I}$ be an I-indexed family of cofibrant objects in the global model structure of $\mathrm{CAlg}(\mathrm{Sp}^{O,\wedge})$. We write P for the filtered poset, under inclusion, of finite subsets of I. We define a functor $\Phi\colon P \longrightarrow \mathrm{CAlg}(\mathrm{Sp}^{O,\wedge})$ by sending a finite subset S of I to the coproduct $\bigwedge_{i\in S} R_i$, and an inclusion $S \subset T$ to the canonical morphism $\bigwedge_S R_i \longrightarrow \bigwedge_T R_i$.

We claim that the underlying Sp^O-valued functor of Φ, i.e., the composite

$$P \xrightarrow{\ \Phi\ } \mathrm{CAlg}(\mathrm{Sp}^{O,\wedge}) \xrightarrow{\mathrm{ev}_{\langle 1\rangle}} \mathrm{Sp}^O$$

is Reedy cofibrant, i.e., for every nonempty finite subset S of I, the latching morphism

$$l_S \ :\ \mathrm{colim}_{C\subsetneq S}\,\Phi(C) \ \longrightarrow\ \Phi(S)$$

is a flat cofibration of orthogonal spectra; here the colimit in the source of l_S is formed over the poset of proper subsets of S. We prove this by induction over the cardinality of S. For $S=\emptyset$ there are no proper subsets, so the latching morphism is the unique morphism from the initial object to $\Phi(\emptyset)=\mathbb{1}$. This is a cofibration by hypothesis. If S is not empty, we choose an $i\in S$ and we set $T=S\setminus\{i\}$. Because the monoidal product preserves colimits in each variable, the latching object can be presented the following pushout:

$$\begin{tikzcd}
\mathbb{S}\wedge(\mathrm{colim}_{C\subsetneq T}\,\Phi(C)) \arrow[r, "\mathbb{S}\wedge l_T"] \arrow[d, "\mathrm{unit}\wedge\mathrm{Id}"'] & \mathbb{S}\wedge\Phi(S) \arrow[d] \\
R_i\wedge(\mathrm{colim}_{C\subsetneq T}\,\Phi(C)) \arrow[r] & \mathrm{colim}_{C\subsetneq S}\,\Phi(C)
\end{tikzcd}$$

Under this interpretation, the latching morphism $l_S\colon\ \mathrm{colim}_{C\subsetneq S}\,\Phi(C)\longrightarrow\Phi(S)=R_i\wedge\Phi(T)$ becomes the pushout product of the unit morphism $\mathbb{S}\longrightarrow R_i$ and the latching morphism $l_T\colon\ \mathrm{colim}_{C\subsetneq T}\,\Phi(C)\longrightarrow\Phi(T)$. The former is flat cofibration by [38, Theorem 5.4.3 (ii c)], and l_T is a flat cofibration by induction. So the pushout product property lets us conclude that l_S is a flat cofibration. This concludes the proof that the underlying Sp^O-valued functor of Φ is Reedy cofibrant.

Now we let $\bar\Phi\colon P^{\triangleright}\longrightarrow\mathrm{CAlg}(\mathrm{Sp}^{O,\wedge})$ be a colimit of Φ. Then the value of $\bar\Phi$ at the cone point is in particular a coproduct in the 1-category $\mathrm{CAlg}(\mathrm{Sp}^{O,\wedge})$ of the family $\{R_i\}_{i\in I}$. We contemplate the following commutative diagram of ∞-categories

$$\begin{tikzcd}
P^{\triangleright} \arrow[r, "\bar\Phi"] \arrow[dr, bend right, "\mathrm{CAlg}(\gamma_{\mathrm{gl}}^{\wedge})\circ\bar\Phi"'] & \mathrm{CAlg}(\mathrm{Sp}^{O,\wedge}) \arrow[r, "\mathrm{ev}_{\langle 1\rangle}"] \arrow[d, "\mathrm{CAlg}(\gamma_{\mathrm{gl}}^{\wedge})"] & \mathrm{Sp}^O \arrow[d, "\gamma_{\mathrm{gl}}"] \\
& \mathrm{CAlg}(\mathcal{S}p_{\mathrm{gl}}^{\wedge}) \arrow[r, "\mathrm{ev}_{\langle 1\rangle}"'] & \mathcal{S}p_{\mathrm{gl}}^{\wedge}
\end{tikzcd}$$

The 1-categorical forgetful functor $\mathrm{ev}_{\langle 1\rangle}\colon\ \mathrm{CAlg}(\mathrm{Sp}^{O,\wedge})\longrightarrow\mathrm{Sp}^O$ preserves filtered colimits, so the composite $\mathrm{ev}_{\langle 1\rangle}\circ\Phi$ is a colimit in the 1-category of orthogonal spectra. As a colimit of a Reedy cofibrant diagram, it represents the colimit in the associated ∞-category, i.e., the composite $\gamma_{\mathrm{gl}}\circ\mathrm{ev}_{\langle 1\rangle}\circ\bar\Phi$ is a colimit in the ∞-category $\mathcal{S}p_{\mathrm{gl}}^{\wedge}$ of global spectra. The functor $\mathrm{ev}_{\langle 1\rangle}\colon\ \mathrm{CAlg}(\mathcal{S}p_{\mathrm{gl}}^{\wedge})\longrightarrow\mathcal{S}p_{\mathrm{gl}}$ creates filtered colimits by [30, Proposition 3.2.3.1]. So $\mathrm{CAlg}(\gamma_{\mathrm{gl}}^{\wedge})\circ\bar\Phi$ is a colimit in $\mathrm{CAlg}(\mathcal{S}p_{\mathrm{gl}}^{\wedge})$. We have already shown that $\mathrm{CAlg}(\gamma_{\mathrm{gl}}^{\wedge})$ preserves finite coproducts, so for every finite subset S of I, the object $\mathrm{CAlg}(\gamma_{\mathrm{gl}}^{\wedge})(\Phi(S))$ is a coproduct in $\mathrm{CAlg}(\mathcal{S}p_{\mathrm{gl}}^{\wedge})$ of $\{\mathrm{CAlg}(\gamma_{\mathrm{gl}}^{\wedge})(R_i)\}_{i\in S}$. Hence the value of $\mathrm{CAlg}(\gamma_{\mathrm{gl}}^{\wedge})\circ\bar\Phi$ at the cone point is the P-filtered colimit of the finite coproducts, and hence the coproduct of the family $\{\mathrm{CAlg}(\gamma_{\mathrm{gl}}^{\wedge})(R_i)\}_{i\in I}$.

Now we show part (i). The weak equivalences in any model category are saturated, i.e., a morphism is a weak equivalence if and only it becomes an equivalence in the localization. So the functor $\mathrm{ev}_{\langle 1\rangle}\circ u_{\mathrm{gl}}\colon\ \mathrm{URing}_{\mathrm{gl}}\longrightarrow\mathcal{S}p_{\mathrm{gl}}^{\wedge}$ is conservative. Since this composite is descended from a fully homotopical right Quillen functor of model categories, it has a left adjoint by [36, Theorem 2.1]. We showed above that the functor $u_{\mathrm{gl}}\colon\ \mathrm{URing}_{\mathrm{gl}}\longrightarrow\mathrm{CAlg}(\mathcal{S}p_{\mathrm{gl}}^{\wedge})$ preserves all colimits. The forgetful functor

$\mathrm{ev}_{\langle 1\rangle}\colon \mathrm{CAlg}(\mathcal{S}p^{\wedge}_{\mathrm{gl}}) \longrightarrow \mathcal{S}p_{\mathrm{gl}}$ preserves sifted colimits by [30, Proposition 3.2.3.1]. So the composite $\mathrm{ev}_{\langle 1\rangle} \circ u_{\mathrm{gl}}\colon \mathrm{URing}_{\mathrm{gl}} \longrightarrow \mathcal{S}p_{\mathrm{gl}}$ preserves sifted colimits. In particular, $\mathrm{ev}_{\langle 1\rangle} \circ u_{\mathrm{gl}}$ preserves geometric realization, i.e., colimits of simplicial objects. Since $\mathrm{ev}_{\langle 1\rangle} \circ u_{\mathrm{gl}}$ is conservative, has a left adjoint and preserves geometric realization, it is monadic by the Lurie–Barr–Beck theorem [30, Theorem 4.7.0.3].

(ii) We exploit that the ∞-category $\mathcal{S}p_{\mathrm{gl}}$ of global spectra is compactly generated, hence κ-presentable, for some regular cardinal κ. The functor $\mathrm{ev}_{\langle 1\rangle} \circ u\colon \mathrm{URing}_{\mathrm{gl}} \longrightarrow \mathcal{S}p_{\mathrm{gl}}$ is monadic by (i). So there is a monad T on the presentable ∞-category $\mathcal{S}p_{\mathrm{gl}}$ such that $\mathrm{URing}_{\mathrm{gl}}$ is equivalent to $\mathrm{Alg}_T(\mathcal{S}p_{\mathrm{gl}})$. Since the functor $\mathrm{ev}_{\langle 1\rangle} \circ u\colon \mathrm{URing}_{\mathrm{gl}} \longrightarrow \mathcal{S}p_{\mathrm{gl}}$ preserves sifted colimits, the underlying endofunctor $T\colon \mathcal{S}p_{\mathrm{gl}} \longrightarrow \mathcal{S}p_{\mathrm{gl}}$ of the monad preserves sifted colimits. So the ∞-category of algebras over T, and hence also $\mathrm{URing}_{\mathrm{gl}}$, is κ-presentable by [20, Corollary 6.8].

(iii) The functor $\mathrm{ev}_{\langle 1\rangle} \circ u_{\mathrm{gl}}\colon \mathrm{URing}_{\mathrm{gl}} \longrightarrow \mathcal{S}p_{\mathrm{gl}}$ is conservative, hence so is the functor $u_{\mathrm{gl}}\colon \mathrm{URing}_{\mathrm{gl}} \longrightarrow \mathrm{CAlg}(\mathcal{S}p^{\wedge}_{\mathrm{gl}})$. We argued above that the functor u_{gl} preserves all colimits; so it is in particular accessible. The functor $\mathrm{ev}_{\langle 1\rangle} \circ u_{\mathrm{gl}}\colon \mathrm{URing}_{\mathrm{gl}} \longrightarrow \mathcal{S}p_{\mathrm{gl}}$ has a left adjoint, so it preserves all limits. Since $\mathrm{ev}_{\langle 1\rangle}\colon \mathrm{CAlg}(\mathrm{Sp}^{\wedge}_{\mathrm{gl}}) \longrightarrow \mathcal{S}p_{\mathrm{gl}}$ creates all limits by [30, Corollary 3.2.2.5], we deduce that functor u_{gl} preserves all limits. The ∞-category $\mathrm{CAlg}(\mathcal{S}p^{\wedge}_{\mathrm{gl}})$ is presentable by [30, Corollary 3.2.3.5]. Since $\mathrm{URing}_{\mathrm{gl}}$ is also presentable by (iii), and u_{gl} preserves all limits and colimits, it has both a left adjoint and a right adjoint by Lurie's adjoint functor theorem [28, Corollary 5.5.2.9].

Since u_{gl} is conservative, has a left adjoint, and preserves all colimits, u_{gl} is monadic by the Lurie–Barr–Beck theorem [30, Theorem 4.7.0.3]. Since the functor $u^{\mathrm{op}}_{\mathrm{gl}}\colon (\mathrm{URing}_{\mathrm{gl}})^{\mathrm{op}} \longrightarrow (\mathrm{CAlg}(\mathcal{S}p^{\wedge}_{\mathrm{gl}}))^{\mathrm{op}}$ is conservative, has a left adjoint and preserves all colimits, $u^{\mathrm{op}}_{\mathrm{gl}}$ is monadic by the same theorem; thus u_{gl} is comonadic. □

The proof strategy leading to Theorem B.28 also works in the unstable global setting, mutatis mutandis. A commutative algebra in orthogonal spaces under the box product is called an *ultra-commutative monoid* in [38, Chapter 2]. By the defining property convolution products, the structure is equivalent to that of a continuous lax symmetric monoidal functor from the linear isometries category (under orthogonal direct sum) to spaces (under cartesian product). The ∞-category of ultra-commutative monoids is the localization at the class of morphisms that are global equivalences, in the sense of [38, Definition 1.1.2], of underlying orthogonal spaces. Essentially the same proof as in Theorem B.28 shows that the ∞-category of ultra-commutative monoids is presentable, and bimonadic over the ∞-category $\mathrm{CAlg}(spc^{\times}_{\mathrm{gl}})$ of commutative global monoids, i.e., commutative algebras in the cartesian monoidal ∞-category of global spaces. Since we do not need this bimonadicity in the present paper, we do not go into any details.

Appendix C. Global classes

A *global class* is a non-empty class of compact Lie groups that is closed under isomorphism and passage to closed subgroups. Every global family in the sense of [38, Definition 1.4.1] is a global class, but global families are additionally assumed to be closed under quotient groups. The closure property under quotient groups was used in [38, Section 4.3] to establish the $\mathcal{F}$-global stable model structures. In this appendix, we make no use of these model structure, so we develop the theory more generally for global classes.

Besides the maximal global family of all compact Lie groups, important examples of global families are all finite groups, or all abelian compact Lie groups. Another global family relevant for this paper is the global family of all compact Lie groups whose identity component is abelian. In this section we explore global stable homotopy theory relative to a global class from a higher categorical perspective, with particular emphasis on multiplicative features.

Example C.1. An example of a multiplicative global class that is not a global family is the class $\mathcal{F}(2)$ of all compact Lie groups that are isomorphic to a closed subgroup of $U(2)^k$ for some $k \geq 1$. By design, $\mathcal{F}(2)$ is the smallest multiplicative global class that contains the group $U(2)$.

The alternating group on four letters A_4 of order 12 has three 1-dimensional irreducible representations, and one irreducible representation of dimension 3. So every continuous homomorphism $A_4 \longrightarrow U(2)$ annihilates the commutator subgroup of A_4. Since A_4 is nonabelian, it does not embed into $U(2)^k$ for any $k \geq 0$, and hence does not belong to the class $\mathcal{F}(2)$.

The group A_4 embeds into $SO(3)$ as the orientation preserving linear isometries of the regular tetrahedron. The binary tetrahedral group $\tilde{A}_4$ of order 24 is the preimage of A_4 under the epimorphism $SU(2) \longrightarrow SO(3)$. As a subgroup of $SU(2)$, the binary tetrahedral group belongs to the class $\mathcal{F}(2)$, but its quotient A_4 does not. So the global class $\mathcal{F}(2)$ is not closed under quotients, and hence not a global family in the sense of [38].

Let $\mathcal{F}$ be a global class of compact Lie groups. A morphism of orthogonal spectra $f\colon X \longrightarrow Y$ is an *$\mathcal{F}$-equivalence* if the map $\pi_k^G(f)\colon \pi_k^G(X) \longrightarrow \pi_k^G(Y)$ is an isomorphism for all $k \in \mathbb{Z}$ and all compact Lie groups G in the class $\mathcal{F}$. Since every global equivalence is in particular an $\mathcal{F}$-equivalence, the $\mathcal{F}$-equivalences descend to a well-defined class of morphisms in the ∞-category $\mathcal{S}p_{\mathrm{gl}}$ of global spectra. Then by design, a morphism f in $\mathcal{S}p_{\mathrm{gl}}$ is an $\mathcal{F}$-equivalence if and only if for every compact Lie group G in $\mathcal{F}$, the morphism of genuine G-spectra $\mathrm{res}_G\, f$ is an equivalence in $\mathcal{S}p_G$.

In the following we write

$$\mathcal{S}p_{\mathcal{F}} \ = \ \langle \Sigma^\infty_+ B_{\mathrm{gl}} G \colon G \in \mathcal{F} \rangle$$

for the localizing subcategory of the stable ∞-category $\mathcal{S}p_{\mathrm{gl}}$ of global spectra generated by the global spectra $\Sigma^\infty_+ B_{\mathrm{gl}} G$ for all $G \in \mathcal{F}$. We show in part (iv) of the following theorem $\mathcal{S}p_{\mathcal{F}}$ coincides with the essential image of the left adjoint to the Bousfield localization of $\mathcal{S}p_{\mathrm{gl}}$ by the class of $\mathcal{F}$-equivalences. So a global spectrum belongs to $\mathcal{S}p_{\mathcal{F}}$ if and only if is left induced from $\mathcal{F}$ in the sense of [38, Definition 4.5.6]. We will therefore refer to the global spectra in $\mathcal{S}p_{\mathcal{F}}$ as *$\mathcal{F}$-left induced.*

We call a functor $f\colon \mathcal{C} \longrightarrow \mathcal{D}$ of ∞-categories a *Bousfield localization* if it has a fully faithful right adjoint. Dually, f is a *Bousfield colocalization* if it has a fully faithful left adjoint. An exact functor of stable ∞-categories is a *recollement* if it is simultaneously a Bousfield localization and a Bousfield colocalization, i.e., it has fully faithful adjoints on both sides.

The most important features of the ∞-category of $\mathcal{F}$-global spectra are summarized in the following theorem. For *multiplicative* global classes, the subcategory $\mathcal{S}p_{\mathcal{F}}$ of $\mathcal{F}$-left induced global spectra interacts well with the global smash products, and Theorem C.6 records the additional multiplicative features in this situation. When the multiplicative global class is additionally closed under wreath products, then $\mathcal{S}p_{\mathcal{F}}$ also behaves well with respect to ultra-commutative multiplications, as we explain in Theorem C.10.

Theorem C.2. *Let $\mathcal{F}$ be a global class of compact Lie groups.*

(i) *The ∞-category $\mathcal{S}p_{\mathcal{F}}$ is stable and compactly generated, and hence presentable.*
(ii) *The inclusion $\mathcal{S}p_{\mathcal{F}} \longrightarrow \mathcal{S}p_{\mathrm{gl}}$ has a right adjoint $\rho_{\mathcal{F}}\colon \mathcal{S}p_{\mathrm{gl}} \longrightarrow \mathcal{S}p_{\mathcal{F}}$, and for every global spectrum X, the adjunction counit $\epsilon_X\colon \rho_{\mathcal{F}}(X) \longrightarrow X$ is an $\mathcal{F}$-equivalence.*
(iii) *Any right adjoint $\rho_{\mathcal{F}}\colon \mathcal{S}p_{\mathrm{gl}} \longrightarrow \mathcal{S}p_{\mathcal{F}}$ to the inclusion itself has a fully faithful right adjoint. So $\rho_{\mathcal{F}}$ is a recollement of stable ∞-categories.*
(iv) *The right adjoint $\rho_{\mathcal{F}}\colon \mathcal{S}p_{\mathrm{gl}} \longrightarrow \mathcal{S}p_{\mathcal{F}}$ to the inclusion is a localization of ∞-categories at the class of $\mathcal{F}$-equivalences.*
(v) *For every $\mathcal{F}$-equivalence $f\colon X \longrightarrow Y$ and every global spectrum Z, the morphism $f \wedge Z\colon X \wedge Z \longrightarrow Y \wedge Z$ is an $\mathcal{F}$-equivalence.*

Proof. (i) By design, $\mathcal{S}p_{\mathcal{F}}$ is the full subcategory of $\mathcal{S}p_{\mathrm{gl}}$ generated under colimits by the objects $(\Sigma^\infty_+ B_{\mathrm{gl}} G)[n]$ for all $G \in \mathcal{F}$ and all $n \in \mathbb{N}$. All these generating global spectra are compact objects of $\mathcal{S}p_{\mathrm{gl}}$ by [38, Theorem 4.4.3], so $\mathcal{S}p_{\mathcal{F}}$ is a compactly generated stable ∞-category, and hence presentable.

(ii) By design, $\mathcal{S}p_{\mathcal{F}}$ is closed under colimits inside $\mathcal{S}p_{\mathrm{gl}}$. So the inclusion $\mathcal{S}p_{\mathcal{F}} \longrightarrow \mathcal{S}p_{\mathrm{gl}}$ preserves colimits. Since source and target are presentable, the adjoint functor theorem [28, Corollary 5.5.2.9] provides a right adjoint $\rho_{\mathcal{F}} \colon \mathcal{S}p_{\mathrm{gl}} \longrightarrow \mathcal{S}p_{\mathcal{F}}$ to the inclusion. We write $\epsilon_X \colon \rho_{\mathcal{F}}(X) \longrightarrow X$ for the adjunction counit. For all $G \in \mathcal{F}$ and all integers k, the upper horizontal map in the following commutative diagram is the adjunction bijection:

$$\begin{tikzcd}
h(\mathcal{S}p_{\mathrm{gl}})((\Sigma^\infty_+ B_{\mathrm{gl}}G)[k], \rho_{\mathcal{F}}(X)) \arrow[r, "(\epsilon_X)_*", "\cong"'] \arrow[d, "\cong"'] & h(\mathcal{S}p_{\mathrm{gl}})((\Sigma^\infty_+ B_{\mathrm{gl}}G)[n], X) \arrow[d, "\cong"] \\
\pi_k^G(\rho_{\mathcal{F}}(X)) \arrow[r, "(\epsilon_X)_*"'] & \pi_k^G(X)
\end{tikzcd}$$

The two vertical maps are isomorphisms because $\Sigma^\infty_+ B_{\mathrm{gl}}G$ represents the functor π_0^G, compare [38, Theorem 4.4.3 (i)]. So the lower horizontal map is an isomorphism, and thus ϵ_X is an $\mathcal{F}$-equivalence, as claimed.

(iii) As a right adjoint, $\rho_{\mathcal{F}}$ preserves limits. Since source and target are stable, the functor $\rho_{\mathcal{F}}$ preserves finite colimits. We claim that $\rho_{\mathcal{F}}$ also preserves infinite sums. Indeed, for any family $(X_i)_{i\in I}$ of global spectra and all G in $\mathcal{F}$, the following diagram commutes:

$$\begin{tikzcd}
\bigoplus_I \pi_k^G(\rho_{\mathcal{F}}(X_i)) \arrow[r, "\cong"] \arrow[d, "\cong"'] & \pi_k^G(\bigoplus_I \rho_{\mathcal{F}}(X_i)) \arrow[r] & \pi_k^G(\rho_{\mathcal{F}}(\bigoplus_I X_i)) \arrow[d, "\cong"] \\
\bigoplus_I \pi_k^G(X_i) \arrow[rr, "\cong"'] & & \pi_k^G(\bigoplus_I X_i)
\end{tikzcd}$$

The two vertical maps are isomorphisms by (ii). The two horizontal isomorphisms exploit that the functor π_k^G takes coproducts of global spectra to direct sums of abelian groups. Since this holds for all $G \in \mathcal{F}$, the colimit exchange morphism $\bigoplus_I \rho_{\mathcal{F}}(X_i) \longrightarrow \rho_{\mathcal{F}}(\bigoplus_I X_i)$ is an $\mathcal{F}$-equivalence, and hence an isomorphism in $h(\mathcal{S}p_{\mathcal{F}})$. This concludes the proof that $\rho_{\mathcal{F}}$ preserves arbitrary sums. Since $\rho_{\mathcal{F}}$ preserves finite colimits and sums, it preserves all colimits. Since $\mathcal{S}p_{\mathcal{F}}$ and $\mathcal{S}p_{\mathrm{gl}}$ are presentable, the adjoint functor theorem [28, Corollary 5.5.2.9] provides a right adjoint to $\rho_{\mathcal{F}}$. The functor $\rho_{\mathcal{F}}$ thus has adjoints on both sides. In such cases, one adjoint is fully faithful if and only if the other one is. Since the left adjoint to $\rho_{\mathcal{F}}$ is fully faithful, so is its right adjoint.

(iv) The commutative square

$$\begin{tikzcd}
\rho(X) \arrow[r, "\rho(f)"] \arrow[d, "\epsilon_X"'] & \rho(Y) \arrow[d, "\epsilon_Y"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

and the fact, proved in (ii), that the adjunction counits are $\mathcal{F}$-equivalences, shows that a morphism f of global spectra is an $\mathcal{F}$-equivalence if and only if $\rho(f)$ is an $\mathcal{F}$-equivalence. Since $\rho(X)$ and $\rho(Y)$ belong to $\mathcal{S}p_{\mathcal{F}}$, $\rho(f)$ is moreover an $\mathcal{F}$-equivalence if and only if it is a global equivalence. We can then apply [28, Proposition 5.2.7.12] to the functor $L = \mathrm{incl} \circ \rho_{\mathcal{F}} \colon \mathcal{S}p_{\mathrm{gl}} \longrightarrow \mathcal{S}p_{\mathrm{gl}}$, which yields the desired universal property of a localization.

(v) For any group G in $\mathcal{F}$, the functor $\mathrm{res}_G^\wedge \colon \mathcal{S}p_{\mathrm{gl}}^\wedge \longrightarrow \mathcal{S}p_G^\wedge$ is strong symmetric monoidal, so $\mathrm{res}_G(f \wedge Z)$ is equivalent in $\mathcal{S}p_G$ to $\mathrm{res}_G(f) \wedge \mathrm{res}_G(Z)$. The morphism $\mathrm{res}_G(f)$ is an equivalence in $\mathcal{S}p_G$ because $G \in \mathcal{F}$; hence also $\mathrm{res}_G(f \wedge Z)$ is an equivalence. Since this holds for all $G \in \mathcal{F}$, the morphism $f \wedge Z$ is an $\mathcal{F}$-equivalence. $\square$

Example C.3. Let G be a compact Lie group in a global class $\mathcal{F}$. Then the functor $\mathrm{res}_G\colon \mathcal{S}p_{\mathrm{gl}} \longrightarrow \mathcal{S}p_G$ has a left adjoint

$$G\backslash\!\!\backslash - \ :\ \mathcal{S}p_G \ \longrightarrow\ \mathcal{S}p_{\mathrm{gl}}\ .$$

This left adjoint takes the suspension G-spectrum $\Sigma^\infty_+ G/H$ to $\Sigma^\infty_+ B_{\mathrm{gl}}H$, for example because both represent the functor $\pi_0^H\colon h(\mathcal{S}p_{\mathrm{gl}}) \longrightarrow \mathcal{A}b$. Since G belongs to the class $\mathcal{F}$, so does its subgroup H, and thus $G\backslash\!\!\backslash(\Sigma^\infty_+ G/H) \sim \Sigma^\infty_+ B_{\mathrm{gl}}H$ lies in the subcategory $\mathcal{S}p_{\mathcal{F}}$. Since $\mathcal{S}p_G$ is generated, as a localizing category, by the objects $\Sigma^\infty_+ G/H$, the entire image of the left adjoint $G\backslash\!\!\backslash -$ is contained in the subcategory $\mathcal{S}p_{\mathcal{F}}$. For example, if V and W are representations of a group $G \in \mathcal{F}$, then the global Thom spectrum

$$(B_{\mathrm{gl}}G)^{V-W} \ =\ G\backslash\!\!\backslash(\mathbb{S}^{V-W})$$

of a virtual G-representation is $\mathcal{F}$-left induced.

If a global family is *reflexive* in the sense of [38, Definition 4.5.7], then [38, Proposition 4.5.8] provides a criterion for being left induced in terms of geometric fixed points.

We alert the reader that in general, the category $\mathcal{S}p_{\mathcal{F}}$ of $\mathcal{F}$-left induced global spectra is *not* closed inside $\mathcal{S}p_{\mathrm{gl}}$ under the global smash product. This improves if $\mathcal{F}$ is a *multiplicative* class, i.e., closed under products, see Theorem C.6.

Proposition C.4. *For any global class $\mathcal{F}$, the following two conditions are equivalent.*
(a) *The global class $\mathcal{F}$ is* multiplicative*, i.e., for all compact Lie groups G and K in $\mathcal{F}$, the product $G \times K$ also belongs to $\mathcal{F}$.*
(b) *The global class $\mathcal{F}$ is* reflexive*, i.e., the inclusion of the full subcategory spanned by groups in $\mathcal{F}$ into the category of compact Lie groups and continuous homomorphisms has a left adjoint.*

Proof. (a)$\Longrightarrow$(b) We let K be a compact Lie group. We let N be the intersection of the kernels of all continuous homomorphisms from K to some group in $\mathcal{F}$. Then N is a closed normal subgroup of K, and we claim that the quotient homomorphism $K \longrightarrow K/N$ is initial among continuous homomorphisms from K to groups in $\mathcal{F}$.

We must first show that the quotient group K/N indeed belongs to $\mathcal{F}$. The number of closed normal subgroups of any compact Lie group is countable. Hence also the number of closed normal subgroups of K that occur as kernels of continuous homomorphisms to groups in $\mathcal{F}$ is countable. If there are only finitely many such closed normal subgroups $N_1, N_2, \dots, N_k$, we choose continuous homomorphisms $\alpha_i\colon K \longrightarrow G_i$ with kernel N_i and $G_i \in \mathcal{F}$. Then the homomorphism

$$(\alpha_1, \dots, \alpha_k)\ :\ K \ \longrightarrow\ G_1 \times \dots \times G_k$$

has kernel $N_1 \cap \dots \cap N_k = N$, so K/N embeds into $G_1 \times \dots \times G_k$. Since the family is multiplicative, the product $G_1 \times \dots \times G_k$ belongs to $\mathcal{F}$, hence so does K/N.

If there are countably infinitely many such closed normal subgroups $N_1, N_2, \dots$, then by the descending chain property, the sequence of closed subgroups $\bigcap_{i=1,\dots k} N_i$ stabilizes for sufficiently large $k \geq 1$, and hence $N = N_1 \cap \dots \cap N_k$. The argument of the previous paragraph then shows that K/N belongs to the class $\mathcal{F}$.

Now we can complete the argument. If $\beta\colon K \longrightarrow G$ is any continuous homomorphism to a group in $\mathcal{F}$, then N is contained in the kernel of β, so β factors uniquely through the quotient map $K \longrightarrow K/N$.

(b)$\Longrightarrow$(a) The argument for global families in [38, page 437f] can be adapted to global classes, as follows. For $G, K \in \mathcal{F}$, we let $q\colon G \times K \longrightarrow L$ be an initial continuous homomorphism to a group L that belongs to $\mathcal{F}$. Then the projections $p_G\colon G \times K \longrightarrow G$ and $p_K\colon G \times K \longrightarrow K$ factor as $p_G = i \circ q$ and $p_K = j \circ q$ for unique continuous homomorphisms $i\colon L \longrightarrow G$ and $j\colon L \longrightarrow K$. The composite

$$G \times K \ \xrightarrow{\ q\ }\ L \ \xrightarrow{(i,j)}\ G \times K$$

is the identity, so q is an isomorphism from $G \times K$ onto a closed subgroups of L. Since L belongs to $\mathcal{F}$, so does $G \times K$. □

The following proposition generalizes [38, Proposition 4.5.8] from global families to global classes.

Proposition C.5. *Let $\mathcal{F}$ be a multiplicative, and hence reflexive, global class.*

(i) *For every compact Lie group K, every initial continuous homomorphism $p\colon K \longrightarrow uK$ to a group in $\mathcal{F}$ is surjective.*
(ii) *A global spectrum X is $\mathcal{F}$-left induced if and only if for every compact Lie group K and every initial continuous homomorphism $p\colon K \longrightarrow uK$ to a group in $\mathcal{F}$, the inflation map*
$$p^* \ : \ \Phi_*^{uK}(X) \ \longrightarrow \ \Phi_*^K(X)$$
of geometric fixed point homotopy groups is an isomorphism.

Proof. (i) The image $p(K)$ of $p\colon K \longrightarrow uK$ is a closed subgroup of uK, hence also in the global class $\mathcal{F}$. Because p is initial among homomorphisms to groups in $\mathcal{F}$, there is a unique continuous homomorphism $j\colon uK \longrightarrow p(K)$ such that the composite
$$K \ \xrightarrow{\ p\ } \ uK \ \xrightarrow{\ j\ } \ p(K)$$
agrees with p. So $p(K) = uK$, and this p is surjective. Part (ii) is proved in the same way as for global families in [38, Proposition 4.5.8]. □

As we shall now show, for multiplicative global classes, the $\mathcal{F}$-left induced global spectra are monoidally well-behaved.

Theorem C.6. *Let $\mathcal{F}$ be a multiplicative global class.*

(i) *The full subcategory $\mathcal{S}p_{\mathcal{F}}$ of $\mathcal{F}$-left induced global spectra is closed in $\mathcal{S}p_{\mathrm{gl}}$ under the global smash product.*
(ii) *There is a unique symmetric monoidal structure $\mathcal{S}p^{\wedge}_{\mathcal{F}}$ of the category $\mathcal{S}p_{\mathcal{F}}$ equipped with a lift of the inclusion to a strong symmetric monoidal functor $\mathrm{incl}^{\wedge}\colon \mathcal{S}p^{\wedge}_{\mathcal{F}} \longrightarrow \mathcal{S}p^{\wedge}_{\mathrm{gl}}$.*
(iii) *The monoidal structure $\mathcal{S}p^{\wedge}_{\mathcal{F}}$ is presentably symmetric monoidal, and the ∞-category $\mathrm{CAlg}(\mathcal{S}p^{\wedge}_{\mathcal{F}})$ is presentable.*
(iv) *The functor*
$$\mathrm{CAlg}(\mathrm{incl}^{\wedge}) \ : \ \mathrm{CAlg}(\mathcal{S}p^{\wedge}_{\mathcal{F}}) \ \longrightarrow \ \mathrm{CAlg}(\mathcal{S}p^{\wedge}_{\mathrm{gl}})$$
is fully faithful, and its essential image is the subcategory of $\mathrm{CAlg}(\mathcal{S}p^{\wedge}_{\mathrm{gl}})$ spanned by those commutative global ring spectra whose underlying global spectra are $\mathcal{F}$-left induced.
(v) *The pullback square of ∞-categories*
$$\begin{tikzcd} \mathrm{CAlg}(\mathcal{S}p^{\wedge}_{\mathcal{F}}) \arrow[r, "\mathrm{CAlg}(\mathrm{incl}^{\wedge})"] \arrow[d, "\mathrm{ev}_{\langle 1\rangle}"'] & \mathrm{CAlg}(\mathcal{S}p^{\wedge}_{\mathrm{gl}}) \arrow[d, "\mathrm{ev}_{\langle 1\rangle}"] \\ \mathcal{S}p_{\mathcal{F}} \arrow[r, "\mathrm{incl}"'] & \mathcal{S}p_{\mathrm{gl}} \end{tikzcd}$$
is vertically left adjointable and horizontally right adjointable.
(vi) *The lax symmetric monoidal structure induced by $\mathrm{incl}^{\wedge}$ on the right adjoint $\rho_{\mathcal{F}}\colon \mathcal{S}p_{\mathrm{gl}} \longrightarrow \mathcal{S}p_{\mathcal{F}}$ of the inclusion is strong symmetric monoidal.*

Proof. (i) We let G and K be two compact Lie groups from the class $\mathcal{F}$. Then $(\Sigma^\infty_+ B_{\mathrm{gl}}G)\wedge(\Sigma^\infty_+ B_{\mathrm{gl}}K)$ is globally equivalent to $\Sigma^\infty_+ B_{\mathrm{gl}}(G \times K)$, and hence in $\mathcal{S}p_{\mathcal{F}}$ by the multiplicativity hypothesis on $\mathcal{F}$. Since $\mathcal{S}p_{\mathcal{F}}$ is the localizing subcategory of $\mathcal{S}p_{\mathrm{gl}}$ generated by the suspension spectrum $\Sigma^\infty_+ B_{\mathrm{gl}}G$ for all $G \in \mathcal{F}$, and since the global smash product preserves colimits in each variable separately, this proves that $(\mathcal{S}p_{\mathcal{F}}) \wedge (\mathcal{S}p_{\mathcal{F}})$ is contained in $\mathcal{S}p_{\mathcal{F}}$.

(ii) Because the subcategory $\mathcal{S}p_{\mathcal{F}}$ is closed under the global smash product by (i), Proposition 2.2.1.1 and Remark 2.2.1.2 of [30] provide the 'restricted' symmetric monoidal structure on $\mathcal{S}p_{\mathcal{F}}$ along with an extension of the inclusion to a strong symmetric monoidal functor. The induced functor

$$\mathrm{CAlg}(\mathrm{incl}^{\wedge}) \ : \ \mathrm{CAlg}(\mathcal{S}p_{\mathcal{F}}^{\wedge}) \ \longrightarrow \ \mathrm{CAlg}(\mathcal{S}p_{\mathrm{gl}}^{\wedge})$$

is then fully faithful with the essential image as described in (iv).

(iii) The inclusion $\mathcal{S}p_{\mathcal{F}} \longrightarrow \mathcal{S}p_{\mathrm{gl}}$ is a left adjoint by Theorem C.2 (ii). So the inclusion preserves and reflects colimits. Since the inclusion is strong monoidal and the global smash product $\wedge\colon \mathcal{S}p_{\mathrm{gl}} \times \mathcal{S}p_{\mathrm{gl}} \longrightarrow \mathcal{S}p_{\mathrm{gl}}$ preserves colimits in each variable separately, also the restricted global smash product $\wedge\colon \mathcal{S}p_{\mathcal{F}} \times \mathcal{S}p_{\mathcal{F}} \longrightarrow \mathcal{S}p_{\mathcal{F}}$ preserves colimits in each variable separately. So $\mathcal{S}p_{\mathcal{F}}^{\wedge}$ is presentably symmetric monoidal, and its category $\mathrm{CAlg}(\mathcal{S}p_{\mathcal{F}}^{\wedge})$ of commutative algebras is presentable by [30, Corollary 3.2.3.5].

Part (iv) is an application of [30, Remark 2.2.1.5].

(v) The square is horizontally right adjointable by Lemma B.5. Since the vertical functors in the square also left adjoints, horizontal right adjointability is equivalent to vertical left adjointability, compare [12, Lemma A.10]. So the squares is also vertically left adjointable.

(vi) The global sphere spectrum $\mathbb{S}$ is $\{1\}$-left induced, and hence $\mathcal{F}$-left induced for any global class $\mathcal{F}$. So the adjunction counit $\epsilon_{\mathbb{S}}\colon \rho_{\mathcal{F}}(\mathbb{S}) \longrightarrow \mathbb{S}$ is a global equivalence. This adjunction counit is also the unit morphism of the monoidal structure on $\rho_{\mathcal{F}}$, so the lax monoidal structure is unital.

For global spectra X and Y, the monoidal structure morphism $\rho_{\mathcal{F}}(X) \wedge \rho_{\mathcal{F}}(Y) \longrightarrow \rho_{\mathcal{F}}(X \wedge Y)$ participates in the following commutative diagram in $\mathcal{S}p_{\mathrm{gl}}$:

$$\begin{tikzcd}
\rho_{\mathcal{F}}(X) \wedge \rho_{\mathcal{F}}(Y) \arrow[r] \arrow[d, "\rho_{\mathcal{F}}(X)\wedge\epsilon_Y"'] & \rho_{\mathcal{F}}(X \wedge Y) \arrow[d, "\epsilon_{X\wedge Y}"] \\
\rho_{\mathcal{F}}(X) \wedge Y \arrow[r, "\epsilon_X \wedge Y"'] & X \wedge Y
\end{tikzcd}$$

Here $\epsilon\colon \mathrm{incl} \circ \rho_{\mathcal{F}} \longrightarrow \mathrm{Id}_{\mathcal{S}p_{\mathrm{gl}}}$ is the adjunction counit. The two vertical and the lower horizontal morphism are $\mathcal{F}$-equivalences by parts (ii) and (v) of Theorem C.2. So the upper horizontal morphism is an $\mathcal{F}$-equivalence between global spectra in $\mathcal{S}p_{\mathcal{F}}$, and hence a global equivalence. $\square$

In the following we will identify, via the fully faithful functor

$$\mathrm{CAlg}(\mathrm{incl}^{\wedge}) \ : \ \mathrm{CAlg}(\mathcal{S}p_{\mathcal{F}}^{\wedge}) \ \longrightarrow \ \mathrm{CAlg}(\mathcal{S}p_{\mathrm{gl}}^{\wedge}) \ ,$$

the ∞-category $\mathrm{CAlg}(\mathcal{S}p_{\mathcal{F}}^{\wedge})$ with the full subcategory of $\mathrm{CAlg}(\mathcal{S}p_{\mathrm{gl}}^{\wedge})$ spanned by those commutative global ring spectra whose underlying global spectrum is $\mathcal{F}$-left induced. This is legitimate by Theorem C.6 (iv).

Remark C.7 (Commutative global $\mathcal{F}$-approximations)**.** Theorem C.6 makes precise that for multiplicative global classes $\mathcal{F}$, commutative global ring spectra have well-behaved multiplicative approximations by $\mathcal{F}$-left induced objects. Indeed, the fact that the fully faithful functor $\mathrm{CAlg}(\mathrm{incl}^{\wedge})$ has a right adjoint (namely the functor $\mathrm{CAlg}(\rho_{\mathcal{F}}^{\wedge})$) means that the full subcategory of $\mathcal{F}$-left induced commutative global ring spectra is coreflective inside $\mathrm{CAlg}(\mathcal{S}p_{\mathrm{gl}}^{\wedge})$. So for every commutative global ring spectrum R, there exists a morphism $\epsilon\colon \rho_{\mathcal{F}}^{\wedge}(R) \longrightarrow R$ in $\mathrm{CAlg}(\mathcal{S}p_{\mathrm{gl}}^{\wedge})$ that enjoys the following equivalent universal properties:

(a) The pair $(\rho_{\mathcal{F}}^{\wedge}(R), \epsilon)$ represents the restriction of the functor $\mathrm{CAlg}(\mathcal{S}p_{\mathrm{gl}}^{\wedge})(-, R)\colon \mathrm{CAlg}(\mathcal{S}p_{\mathrm{gl}}^{\wedge})^{\mathrm{op}} \longrightarrow \mathcal{S}$ to the subcategory $\mathrm{CAlg}(\mathcal{S}p_{\mathcal{F}}^{\wedge})$. In other words, the underlying global spectrum of $\rho_{\mathcal{F}}^{\wedge}(R)$ is $\mathcal{F}$-left induced, and for every commutative global ring spectrum S whose underlying global spectrum is $\mathcal{F}$-left induced, the map of mapping anima

$$\mathrm{CAlg}(\mathcal{S}p^{\wedge}_{\mathrm{gl}})(S,\epsilon) \ : \ \mathrm{CAlg}(\mathcal{S}p^{\wedge}_{\mathrm{gl}})(S,\rho^{\wedge}_{\mathcal{F}}(R)) \ \longrightarrow \ \mathrm{CAlg}(\mathcal{S}p^{\wedge}_{\mathrm{gl}})(S,R)$$

is an equivalence.

(b) The morphism $\epsilon\colon \rho^{\wedge}_{\mathcal{F}}(R) \longrightarrow R$ is a terminal object in the slice category $\mathrm{CAlg}(\mathcal{S}p^{\wedge}_{\mathcal{F}})_{/R}$.

Said yet another way: whenever the underlying global spectrum of S is $\mathcal{F}$-left induced, then every morphism $f\colon S \longrightarrow R$ of commutative global ring spectra factors uniquely through $\epsilon\colon \rho^{\wedge}_{\mathcal{F}}(R) \longrightarrow R$.

Moreover, the horizontal right adjointability of the square in Theorem C.6 (v) means that the underlying morphism $\epsilon_{\langle 1\rangle}\colon (\rho^{\wedge}_{\mathcal{F}}(R))_{\langle 1\rangle} \longrightarrow R_{\langle 1\rangle}$ is a universal $\mathcal{F}$-approximation of the underlying global spectra, i.e., a terminal object in the slice category $(\mathcal{S}p_{\mathcal{F}})_{/R_{\langle 1\rangle}}$. Informally speaking, not only is there a 'best approximation' of R by an $\mathcal{F}$-left induced object, there is also no ambiguity about whether the term 'best approximation' is to be interpreted in the category $\mathrm{CAlg}(\mathcal{S}p^{\wedge}_{\mathrm{gl}})$ or in the underlying category $\mathcal{S}p_{\mathrm{gl}}$.

In the rest of this section we consider multiplicative global classes that are also closed under wreath products. This condition is somewhat restrictive, and there are not so many natural examples of such classes. As we show in Theorem C.10, the closure under wreath products guarantees that $\mathcal{F}$-left induced spectra 'interact well' with ultra-commutative multiplications. Besides the maximal global family of all compact Lie groups, two relevant examples for us are the following.

Example C.8 (Finite groups)**.** The global family $\mathcal{F}in$ of finite groups is multiplicative and closed under wreath products. If $\mathcal{F}$ is any global class that also closed under wreath products, it contains all symmetric groups, and hence all finite groups. So $\mathcal{F}in$ is minimal among global classes that are closed under wreath products.

Example C.9 (Abelian identity component)**.** We write $\mathcal{A}b^{\circ}$ for the global family of compact Lie groups with abelian identity component; this family is multiplicative and closed under wreath products, and it features in the discussion of the ultra-commutative localization of $\Sigma^{\infty}_{+}\mathbf{P}$ by b_1, see Theorem 6.3. We claim that if a global class $\mathcal{F}$ is also closed under wreath products and contains a compact Lie group of positive dimension, then it contains the global family $\mathcal{A}b^{\circ}$. So among global classes that are closed under wreath products, $\mathcal{A}b^{\circ}$ is next-to-minimal.

To prove the claim we observe that if the global class $\mathcal{F}$ contains a positive dimensional compact Lie group, then it contains the circle group $T = U(1)$. If $\mathcal{F}$ is closed under wreath products, it also contains $\Sigma_n \wr T$ for all $n \geq 1$. We let G be any compact Lie group whose identity component G° is abelian. Then every irreducible G-representation is a direct summand of an induced representation $\mathrm{ind}^{G}_{G^{\circ}}(\chi)$ for some continuous homomorphism $\chi\colon G^{\circ} \longrightarrow T$. So G admits a faithful representation V that is a direct sum of representations of the form $\mathrm{ind}^{G}_{G^{\circ}}(\chi_i)$ for suitable homomorphisms $\chi_i\colon G^{\circ} \longrightarrow T$. If the component group $\pi_0(G) = G/G^{\circ}$ has order n, then the representation $\mathrm{ind}^{G}_{G^{\circ}}(\chi_i)$ is classified by a continuous homomorphism to $\Sigma_n \wr T$. So altogether, the faithful representation V provides an embedding of G into a finite product of copies of $\Sigma_n \wr T$. So G belongs to the global class $\mathcal{F}$.

By Theorem B.28 (iii), the functor $u_{\mathrm{gl}}\colon \mathrm{URing}_{\mathrm{gl}} \longrightarrow \mathrm{CAlg}(\mathcal{S}p^{\wedge}_{\mathrm{gl}})$ is bimonadic; in particular, it is both a left and a right adjoint. We shall write $\lambda_{\mathrm{gl}}\colon \mathrm{CAlg}(\mathcal{S}p^{\wedge}_{\mathrm{gl}}) \longrightarrow \mathrm{URing}_{\mathrm{gl}}$ for a left adjoint to u_{gl}. And by Theorem B.28 (i), the functor $\mathrm{ev}_{\langle 1\rangle} \circ u_{\mathrm{gl}}\colon \mathrm{URing}_{\mathrm{gl}} \longrightarrow \mathcal{S}p_{\mathrm{gl}}$ is monadic, and thus in particular a right adjoint. We write $\mathbb{P}_{\mathrm{gl}}\colon \mathcal{S}p_{\mathrm{gl}} \longrightarrow \mathrm{URing}_{\mathrm{gl}}$ for a left adjoint to $\mathrm{ev}_{\langle 1\rangle} \circ u_{\mathrm{gl}}$, and we refer to it as the *free ultra-commutative ring spectrum functor*.

We let $\mathrm{URing}_{\mathcal{F}}$ denote the full subcategory of $\mathrm{URing}_{\mathrm{gl}}$ spanned by the ultra-commutative ring spectra whose underlying global spectra are $\mathcal{F}$-left induced.

Theorem C.10. *Let $\mathcal{F}$ be a multiplicative global class that is closed under wreath products.*

(i) *The free functor* $\mathbb{P}_{\mathrm{gl}}\colon \mathcal{S}p_{\mathrm{gl}} \longrightarrow \mathrm{URing}_{\mathrm{gl}}$ *takes the subcategory* $\mathcal{S}p_{\mathcal{F}}$ *of* $\mathcal{F}$*-left induced global spectra to the subcategory* $\mathrm{URing}_{\mathcal{F}}$*.*
(ii) *The left adjoint* $\lambda_{\mathrm{gl}}\colon\ \mathrm{CAlg}(\mathcal{S}p_{\mathrm{gl}}^{\wedge}) \longrightarrow \mathrm{URing}_{\mathrm{gl}}$ *to* u_{gl} *takes the subcategory* $\mathrm{CAlg}(\mathcal{S}p_{\mathcal{F}}^{\wedge})$ *to the subcategory* $\mathrm{URing}_{\mathcal{F}}$*.*
(iii) *The restriction* $u_{\mathcal{F}}\colon\ \mathrm{URing}_{\mathcal{F}} \longrightarrow \mathrm{CAlg}(\mathcal{S}p_{\mathcal{F}}^{\wedge})$ *of the functor* u_{gl} *is bimonadic.*
(iv) *The* ∞*-category* $\mathrm{URing}_{\mathcal{F}}$ *is presentable and coreflective in* $\mathrm{URing}_{\mathrm{gl}}$*.*
(v) *The pullback square of* ∞*-categories*

$$\begin{tikzcd}
\mathrm{URing}_{\mathcal{F}} \arrow[r, "\mathrm{incl}"] \arrow[d, "u_{\mathcal{F}}"'] & \mathrm{URing}_{\mathrm{gl}} \arrow[d, "u_{\mathrm{gl}}"] \\
\mathrm{CAlg}(\mathcal{S}p_{\mathcal{F}}^{\wedge}) \arrow[r, "\mathrm{CAlg}(\mathrm{incl}^{\wedge})"'] & \mathrm{CAlg}(\mathcal{S}p_{\mathrm{gl}}^{\wedge})
\end{tikzcd}$$

is vertically left adjointable and horizontally right adjointable.

Proof. For use in several of the items we show first step that $\mathrm{URing}_{\mathcal{F}}$ is closed under colimits in the ambient category $\mathrm{URing}_{\mathrm{gl}}$. We let $F\colon I \longrightarrow \mathrm{URing}_{\mathrm{gl}}$ be a functor with values in $\mathrm{URing}_{\mathcal{F}}$, and we let R be a colimit of F in $\mathrm{URing}_{\mathrm{gl}}$. Since the functor $u_{\mathrm{gl}}\colon\ \mathrm{URing}_{\mathrm{gl}} \longrightarrow \mathrm{CAlg}(\mathcal{S}p_{\mathrm{gl}}^{\wedge})$ preserves colimits by Theorem B.28 (iii), $u_{\mathrm{gl}}(R)$ is a colimit of $u_{\mathrm{gl}}\circ F$ in $\mathrm{CAlg}(\mathcal{S}p_{\mathrm{gl}}^{\wedge})$. Since $\mathcal{F}$ is multiplicative, the full subcategory of $\mathrm{CAlg}(\mathcal{S}p_{\mathcal{F}}^{\wedge})$ is closed under colimits in $\mathrm{CAlg}(\mathcal{S}p_{\mathrm{gl}}^{\wedge})$ by Theorem C.6. So the underlying global spectrum of $u_{\mathrm{gl}}(R)$, which is the same as the underlying global spectrum of R, is $\mathcal{F}$-left induced.

(i) We show that for every group G in $\mathcal{F}$ and every $n \geq 0$, the global spectrum underlying $\mathbb{P}_{\mathrm{gl}}(\Sigma^\infty_+(B_{\mathrm{gl}}G)[-n])$ is $\mathcal{F}$-left induced. The underlying global spectrum is the sum of the global spectra $\mathbb{P}^{[k]}_{\mathrm{gl}}(\Sigma^\infty_+(B_{\mathrm{gl}}G)[-n])$ for $k \geq 0$, it suffices to show that each of these summands is $\mathcal{F}$-left induced. Now

$$\mathbb{P}^{[k]}_{\mathrm{gl}}(\Sigma^\infty_+(B_{\mathrm{gl}}G)[-n]) \ \sim\ (B_{\mathrm{gl}}(\Sigma_k \wr G))^{-\nu_k^n}\ ,$$

the global Thom spectrum associated to the negative of the $(\Sigma_k \wr G)$-representation ν_k^n, the direct sum of n copies of the natural Σ_k-representation on $\mathbb{R}^k$. Since G belongs to $\mathcal{F}$, and $\mathcal{F}$ is closed under wreath products, the group $\Sigma_k \wr G$ belongs to $\mathcal{F}$. So the global Thom spectrum $(B_{\mathrm{gl}}(\Sigma_k \wr G))^{-\nu_k^n}$ is $\mathcal{F}$-left induced by Example C.3. This concludes the proof that the global spectrum underlying $\mathbb{P}_{\mathrm{gl}}(\Sigma^\infty_+(B_{\mathrm{gl}}G)[-n])$ is $\mathcal{F}$-left induced, i.e., $\mathbb{P}_{\mathrm{gl}}(\Sigma^\infty_+(B_{\mathrm{gl}}G)[-n])$ belongs to the subcategory $\mathrm{URing}_{\mathcal{F}}$.

By definition, $\mathcal{S}p_{\mathcal{F}}$ is generated as a localizing subcategory of $\mathcal{S}p_{\mathrm{gl}}$ by the suspension spectra $\Sigma^\infty_+ B_{\mathrm{gl}}G$ for all compact Lie groups G in $\mathcal{F}$. So $\mathcal{S}p_{\mathcal{F}}$ is generated under colimits in $\mathcal{S}p_{\mathrm{gl}}$ by the objects $(\Sigma^\infty_+ B_{\mathrm{gl}}G)[-n]$ for all $G \in \mathcal{F}$ and all $n \geq 0$. As a left adjoint, $\mathbb{P}_{\mathrm{gl}}$ preserves colimits. Since $\mathrm{URing}_{\mathcal{F}}$ is closed under colimits in $\mathrm{URing}_{\mathrm{gl}}$ by the previous paragraph, the functor $\mathbb{P}_{\mathrm{gl}}$ takes all $\mathcal{F}$-left induced global spectra into $\mathrm{URing}_{\mathcal{F}}$.

(ii) By transitivity of adjoints we have $\mathbb{P}_{\mathrm{gl}} = \lambda_{\mathrm{gl}} \circ \mathbb{E}_{\mathrm{gl}}$, where $\mathbb{E}_{\mathrm{gl}}\colon \mathcal{S}p_{\mathrm{gl}} \longrightarrow \mathrm{CAlg}(\mathcal{S}p_{\mathrm{gl}}^{\wedge})$ is left adjoint to the forgetful functor $\mathrm{ev}_{\langle 1\rangle}$. By the vertical left adjointability in Theorem C.6 (v), the functor $\mathbb{E}_{\mathrm{gl}}$ sends $\mathcal{S}p_{\mathcal{F}}$ into $\mathrm{CAlg}(\mathcal{S}p_{\mathcal{F}}^{\wedge})$. The functor $\mathbb{P}_{\mathrm{gl}} = \lambda_{\mathrm{gl}} \circ \mathbb{E}_{\mathrm{gl}}$ sends $\mathcal{S}p_{\mathcal{F}}$ into $\mathrm{URing}_{\mathcal{F}}$ by part (i); so λ_{gl} sends the image of the restricted functor $\mathbb{E}_{\mathrm{gl}}\colon \mathcal{S}p_{\mathcal{F}} \longrightarrow \mathrm{CAlg}(\mathcal{S}p_{\mathrm{gl}}^{\wedge})_{\mathcal{F}}$ into $\mathrm{URing}_{\mathcal{F}}$. Since $\mathrm{ev}_{\langle 1\rangle}\colon\ \mathrm{CAlg}(\mathcal{S}p_{\mathcal{F}}^{\wedge}) \longrightarrow \mathcal{S}p_{\mathcal{F}}$ is monadic right adjoint, the category $\mathrm{CAlg}(\mathcal{S}p_{\mathcal{F}}^{\wedge})$ is generated under colimits by $\mathbb{E}_{\mathrm{gl}}(\mathcal{S}p_{\mathcal{F}})$. Since λ_{gl} preserves colimits and $\mathrm{URing}_{\mathcal{F}}$ is closed under colimits in $\mathrm{URing}_{\mathrm{gl}}$, the functor λ_{gl} sends all of $\mathrm{CAlg}(\mathcal{S}p_{\mathcal{F}}^{\wedge})$ into $\mathrm{URing}_{\mathcal{F}}$, proving (ii).

We prove the remaining claims (iii), (iv) and (v) together. By definition, the functor $u_{\mathrm{gl}}\colon\ \mathrm{URing}_{\mathrm{gl}} \longrightarrow \mathrm{CAlg}(\mathcal{S}p_{\mathrm{gl}}^{\wedge})$ takes the subcategory $\mathrm{URing}_{\mathcal{F}}$ to the subcategory $\mathrm{CAlg}(\mathcal{S}p_{\mathcal{F}}^{\wedge})$. By (ii), its left adjoint

λ_{gl} takes $\mathrm{CAlg}(\mathcal{S}p^{\wedge}_{\mathcal{F}})$ to $\mathrm{URing}_{\mathcal{F}}$. The restricted functors

$$\lambda_{\mathcal{F}} = \lambda_{\mathrm{gl}}|_{\mathrm{CAlg}(\mathcal{S}p^{\wedge}_{\mathcal{F}})} : \mathrm{CAlg}(\mathcal{S}p^{\wedge}_{\mathcal{F}}) \rightleftarrows \mathrm{URing}_{\mathcal{F}} : u|_{\mathrm{URing}_{\mathcal{F}}} = u_{\mathcal{F}}$$

then form another adjoint pair, with respect to the restricted adjunction unit and counit. The design property that $\lambda_{\mathcal{F}}$ is the restriction of λ_{gl} precisely means that the pullback square in (v) is vertically left adjointable. Since the restricted functor $u_{\mathcal{F}}\colon \mathrm{URing}_{\mathcal{F}} \longrightarrow \mathrm{CAlg}(\mathcal{S}p^{\wedge}_{\mathcal{F}})$ is a right adjoint, it is monadic by [20, Proposition 3.26].

The category $\mathrm{CAlg}(\mathcal{S}p_{\mathcal{F}})$ is coreflective in $\mathrm{CAlg}(\mathcal{S}p_{\mathrm{gl}})$. Since $u_{\mathcal{F}}$ and u_{gl} are monadic right adjoints, the ∞-category $\mathrm{URing}_{\mathcal{F}}$ is then coreflective inside $\mathrm{URing}_{\mathrm{gl}}$. In the equivalent language of algebras over monads (as opposed to monoidal right adjoints), this is shown in [30, Example 7.3.2.10]. To translate the result into our formulation using monadic right adjoints, we use the equivalence between the categories of monadic right adjoints and of monads facilitated by Lurie's Barr–Beck theorem [30, Theorem 4.7.3.5]. This equivalence is sketched without proof in [30, Remark 4.7.3.8], and fully proved in [20, Theorem 3.24].

Since $\mathrm{URing}_{\mathcal{F}}$ is closed under colimits inside $\mathrm{URing}_{\mathrm{gl}}$, and $\mathrm{CAlg}(\mathcal{S}p^{\wedge}_{\mathcal{F}})$ is closed under colimits inside $\mathrm{CAlg}(\mathcal{S}p^{\wedge}_{\mathrm{gl}})$, and since u_{gl} is also preserves colimits, also the restriction $u_{\mathcal{F}}\colon \mathrm{URing}_{\mathcal{F}} \longrightarrow \mathrm{CAlg}(\mathcal{S}p^{\wedge}_{\mathcal{F}})$ preserves colimits. Hence the underlying functor $u_{\mathcal{F}} \circ \lambda_{\mathcal{F}}\colon \mathrm{CAlg}(\mathcal{S}p^{\wedge}_{\mathcal{F}}) \longrightarrow \mathrm{CAlg}(\mathcal{S}p^{\wedge}_{\mathcal{F}})$ underlying the monad for $\mathrm{URing}_{\mathcal{F}}$ preserves all colimits. So $u_{\mathcal{F}} \circ \lambda_{\mathcal{F}}$ is in particular κ-accessible for every regular cardinal κ. Since $\mathrm{CAlg}(\mathcal{S}p^{\wedge}_{\mathcal{F}})$ is presentable and the monad is κ-accessible, the category of algebras are again presentable, see [20, Corollary 6.8]. Since $\mathrm{URing}_{\mathcal{F}}$ is equivalent to the category of algebras, it is presentable, too, proving (iv).

Since source and target of the colimit preserving functor $u_{\mathcal{F}}\colon \mathrm{URing}_{\mathcal{F}} \longrightarrow \mathrm{CAlg}(\mathcal{S}p^{\wedge}_{\mathcal{F}})$ are presentable, this functor admits a right adjoint by Lurie's adjoint functor theorem [28, Corollary 5.5.2.9]. Since we already know that $u_{\mathcal{F}}$ is monadic, the existence of a right adjoint makes it comonadic, too, proving (iii).

We already observed that the pullback square in (v) is vertically left adjointable. In this square, both horizontal functors also have right adjoints. In such situations, vertical left adjointability is equivalent to horizontal right adjointability, compare [12, Lemma A.10]. So the square is also horizontally right adjointable, proving (v). □

If the multiplicative class $\mathcal{F}$ is not closed under wreath products, Theorem C.10 fails. The minimal global family $\mathcal{F} = \{1\}$ of trivial groups already illustrates this: the free functor $\mathbb{P}_{\mathrm{gl}}\colon \mathcal{S}p_{\mathrm{gl}} \longrightarrow \mathrm{URing}_{\mathrm{gl}}$ takes the sphere spectrum (which is left $\{1\}$-induced) to $\mathbb{P}_{\mathrm{gl}}(\mathbb{S})$, whose underlying global spectrum $\bigvee_{n\geq 0} \Sigma^{\infty}_{+} B_{\mathrm{gl}}\Sigma_n$ is *not* left $\{1\}$-induced. So the functor $\mathbb{P}_{\mathrm{gl}}$ does *not* take $\mathcal{S}p_{\{1\}}$ to $\mathrm{URing}_{\{1\}}$; hence the functor $\lambda_{\mathrm{gl}}\colon \mathrm{CAlg}(\mathcal{S}p^{\wedge}_{\mathrm{gl}})\colon \mathcal{S}p_{\mathrm{gl}} \longrightarrow \mathrm{URing}_{\mathrm{gl}}$ does *not* takes the subcategory $\mathrm{CAlg}(\mathcal{S}p^{\wedge}_{\{1\}})$ to the subcategory $\mathrm{URing}_{\{1\}}$, either.

Remark C.11 (Ultra-commutative $\mathcal{F}$-approximations)**.** The previous theorem says, loosely speaking, that for multiplicative global classes $\mathcal{F}$ that are also closed under wreath products, also ultra-commutative ring spectra have well-behaved approximations by $\mathcal{F}$-left induced objects. This should be compared to Remark C.7, where such approximations were discussed in the category of commutative global ring spectra, and the global class was assumed multiplicative, but not necessarily closed under wreath products.

Indeed, the coreflectiveness of $\mathrm{URing}_{\mathcal{F}}$ in Theorem C.10 (iv) means that for every ultra-commutative ring spectrum R, there exists a morphism of ultra-commutative ring spectra $\delta\colon \rho^{uc}_{\mathcal{F}}(R) \longrightarrow R$ that enjoys the following equivalent universal properties:

(a) the pair $(\rho^{uc}_{\mathcal{F}}(R), \delta)$ represents the restriction of the functor $\mathrm{URing}_{\mathrm{gl}}(-, R)\colon (\mathrm{URing}_{\mathrm{gl}})^{\mathrm{op}} \longrightarrow \mathcal{S}$ to the subcategory $\mathrm{URing}_{\mathcal{F}}$. In other words, the underlying global spectrum of $\rho^{uc}_{\mathcal{F}}(R)$ is $\mathcal{F}$-left induced, and for every ultra-commutative ring spectrum S whose underlying global spectrum is $\mathcal{F}$-left induced, the map of mapping anima

$$\mathrm{URing}_{\mathrm{gl}}(S,\delta) \ : \ \mathrm{URing}_{\mathrm{gl}}(S,\rho^{uc}_{\mathcal{F}}(R)) \ \longrightarrow \ \mathrm{URing}_{\mathrm{gl}}(S,R)$$

is an equivalence.

(b) The morphism $\delta\colon \rho^{uc}_{\mathcal{F}}(R) \longrightarrow R$ is a terminal object in the slice category $(\mathrm{URing}_{\mathcal{F}})_{/R}$.

Both properties make precise that whenever the underlying global spectrum of S is $\mathcal{F}$-left induced, then every morphism $f\colon S \longrightarrow R$ of ultra-commutative ring spectra factors uniquely through $\delta\colon \rho^{uc}_{\mathcal{F}}(R) \longrightarrow R$.

Moreover, the horizontal right adjointability of the square in Theorem C.10 (v) means that the underlying morphism $u_{\mathrm{gl}}(\delta)\colon u_{\mathrm{gl}}(\rho^{uc}_{\mathcal{F}}(R)) \longrightarrow u_{\mathrm{gl}}(R)$ is a universal $\mathcal{F}$-approximation of the underlying commutative global ring spectrum of R as discussed in Remark C.7 above. So informally speaking,

$$u_{\mathrm{gl}}(\rho^{uc}_{\mathcal{F}}(R)) \ = \ \rho^{\wedge}_{\mathcal{F}}(u_{\mathrm{gl}}(R)) \ .$$

And when furthermore combined with the horizontal right adjointability of the square in Theorem C.6 (v), we obtain that the morphism of global spectra underlying $\delta\colon \rho^{uc}_{\mathcal{F}}(R) \longrightarrow R$ is a universal $\mathcal{F}$-approximation of the underlying global spectrum of R.

The essence of Theorem C.10 is thus that not only does there exist a 'best approximation of R by an $\mathcal{F}$-left induced object', there is furthermore no ambiguity about whether the term 'best approximation' is interpreted in the category $\mathrm{URing}_{\mathrm{gl}}$, in the less highly structure category, $\mathrm{CAlg}(\mathcal{S}p^{\wedge}_{\mathrm{gl}})$ or in the underlying category $\mathcal{S}p_{\mathrm{gl}}$.

Appendix D. Representation-graded power operations

In this appendix we review power operations and norm maps for ultra-commutative ring spectra in the representation-graded context. The construction is much like in the special case covered in [38, (5.1.2)], to which it specializes for $V = 0$. The construction is used to define the pre-Bott classes in (2.7).

Construction D.1 (Representation-graded power operations)**.** We let G be a compact Lie group. We recall that the wreath product $\Sigma_m \wr G$ with the symmetric group Σ_m is the semidirect product

$$\Sigma_m \wr G \ = \ \Sigma_m \ltimes G^m$$

formed with respect to the action of Σ_m by permuting the factors of G^m. So the multiplication in $\Sigma_m \wr G$ is given by

$$(\sigma;\, g_1,\dots,g_m)\cdot(\tau;\, k_1,\dots,k_m) \ = \ (\sigma\tau;\, g_{\tau(1)}k_1,\dots,g_{\tau(m)}k_m) \ .$$

For every G-space E, the wreath product $\Sigma_m \wr G$ acts on the space E^m by

$$(\sigma;\, g_1,\dots,g_m)\cdot(e_1,\dots,e_m) \ = \ (g_{\sigma^{-1}(1)}e_{\sigma^{-1}(1)},\dots,g_{\sigma^{-1}(m)}e_{\sigma^{-1}(m)}) \ .$$

For every G-representation V, this action even makes V^m a $(\Sigma_m \wr G)$-representation.

For a commutative orthogonal ring spectrum R, and for a G-representation U of G, we let

$$\mu_{U,\dots,U} \ : \ R(U)\times\dots\times R(U) \ \longrightarrow \ R(U\oplus\dots\oplus U)$$

denote the $(U,\dots,U)$-component of the iterated multiplication map of R. This map is $(\Sigma_m \wr G)$-equivariant because the multiplication of R is commutative. The m-th power operation

$$P^m \ : \ \pi^G_V(R) \ \longrightarrow \ \pi^{\Sigma_m \wr G}_{V^m}(R) \tag{D.2}$$

takes the class represented by a based G-map $f\colon S^{U\oplus V} \longrightarrow R(U)$ to the class of the composite $(\Sigma_m \wr G)$-map

$$S^{U^m\oplus V^m} \ \cong \ S^{(U\oplus V)^m} \ \xrightarrow{f^{\wedge m}} \ R(U)^{\wedge m} \ \xrightarrow{\mu_{U,\dots,U}} \ R(U^m) \ .$$

The unnamed homeomorphism shuffles factors. Clearly, this power construction takes G-homotopic maps to $(\Sigma_m \wr G)$-homotopic maps. And if we stabilize f by a G-representation W, then the composite gets stabilized by the $(\Sigma_m \wr G)$-representation W^m. The upshot is thus a well-defined map (D.2).

For $V = 0$, the map (D.2) specializes to the degree 0 power operation of [38, (5.1.2)]. These operations satisfy a whole list of relations that describe their interaction with each other, as well as with addition, multiplication, restriction homomorphism and transfers. For $V = 0$, the relations can be summarized by saying that the power operations make the ring valued global functor $\underline{\pi}_0(R)$ into a *global power functor* in the sense of [38, Definition 5.1.6], see [38, Theorem 5.1.11]. The power operations on representation-graded homotopy groups (D.2) satisfy suitable generalizations of the global power functor relations. We refrain from spelling out all these relations here, but mention one that we use in the proof of the multiplicativity of the pre-Bott classes in Theorem 2.10. The 'concatenation' group monomorphism $+\colon \Sigma_m \times \Sigma_n \longrightarrow \Sigma_{m+n}$ is defined by

$$(\sigma + \sigma')(i) \ = \ \begin{cases} \sigma(i) & \text{for } 1 \leq i \leq m, \text{ and} \\ \sigma'(i-m) + m & \text{for } m+1 \leq i \leq m+n. \end{cases}$$

An embedding of a product of wreath products is now defined by

$$\begin{aligned} \Phi_{m,n} \ : \ (\Sigma_m \wr G) \ \times \ (\Sigma_n \wr G) \qquad & \longrightarrow \quad \Sigma_{m+n} \wr G \\ ((\sigma;\, g_1, \dots, g_m),\, (\sigma';\, g_{m+1}, \dots, g_{m+n})) \ & \longmapsto \ (\sigma + \sigma';\, g_1, \dots, g_{m+n}) \ . \end{aligned}$$

With this notation, for all $x \in \pi_V^G(R)$, the power operations satisfy the relation

$$\Phi_{m,n}^*(P^{m+n}(x)) \ = \ P^m(x) \times P^n(x) \tag{D.3}$$

in $\pi_{V^m \oplus V^n}^{(\Sigma_m \wr G)\times(\Sigma_n \wr G)}(R)$, exploiting that $\Phi_{m,n}^*(V^{m+n}) \ = \ V^m \oplus V^n$ as representations of the group $(\Sigma_m \wr G) \times (\Sigma_n \wr G)$. This relation is proved by direct comparison of representatives.

Construction D.4 (Representation-graded norms)**.** For a finite index subgroup H of a compact Lie group G, and an H-representation W, we let

$$\mathrm{ind}_H^G(W) \ = \ \mathrm{map}^H(G, W) \ = \ \{ f \colon G \longrightarrow W \mid f(hg) = h \cdot f(g) \text{ for all } h \in H,\, g \in G \}$$

denote the induced G-representation, with G-action by $(\gamma \cdot f)(g) = f(g\gamma)$, and with inner product

$$\langle f, f' \rangle \ = \ \sum_{Hg \in H \backslash G} \langle f(g), f'(g) \rangle \ .$$

As we recall now, the power operations for ultra-commutative ring spectra give rise *norm maps*

$$N_H^G \ : \ \pi_W^H(R) \ \longrightarrow \ \pi_{\mathrm{ind}_H^G(W)}^G(R) \ . \tag{D.5}$$

The special case $W = 0$ is explained in [38, Remark 5.1.7], and we now generalize to representation-gradings.

For the construction we choose a transversal of H in G, i.e., an ordered m-tuple $\bar{g} = (g_1, \dots, g_m)$ of elements in disjoint right H-orbits such that

$$G \ = \ \bigcup\nolimits_{i=1}^m \, g_i H \ ,$$

where $m = [G : H]$ is the index. The wreath product $\Sigma_m \wr H$ acts freely and transitively from the right on the set of all such transversals, by the formula

$$(g_1, \dots, g_m) \cdot (\sigma;\, h_1, \dots, h_m) \ = \ (g_{\sigma(1)} h_1, \dots, g_{\sigma(m)} h_m) \ .$$

We obtain a continuous homomorphism

$$\Psi_{\bar{g}} \ : \ G \ \longrightarrow \ \Sigma_m \wr H \tag{D.6}$$

by requiring that
$$\gamma \cdot \bar g \ = \ \bar g \cdot \Psi_{\bar g}(\gamma) \ .$$
The vector space V^m becomes an orthogonal $(\Sigma_m \wr H)$-representation by
$$(\sigma;\, h_1, \dots, h_m) \cdot (v_1, \dots, v_m) \ = \ (h_{\sigma^{-1}(1)} v_{\sigma^{-1}(1)}, \dots, h_{\sigma^{-1}(m)} v_{\sigma^{-1}(m)}) \ ;$$
and the map
$$\phi_{\bar g} \ : \ \mathrm{ind}_H^G(W) \ = \ \mathrm{map}^H(G, W) \ \longrightarrow \ \Psi_{\bar g}^*(V^m) \ , \quad f \ \longmapsto \ (f(g_1^{-1}), \dots, f(g_m^{-1}))$$
is a G-equivariant isomorphism. We define the norm map (D.5) as the composite
$$\pi_W^H(R) \ \xrightarrow{\ P^m\ } \ \pi_{W^m}^{\Sigma_m \wr H} \ \xrightarrow{\ \Psi_{\bar g}^*\ } \ \pi^G_{\Psi^*_{\bar g}(W)}(R) \ \xrightarrow[\cong]{\ \phi^\flat_{\bar g}\ } \ \pi^G_{\mathrm{ind}_H^G W}(R) \ .$$
The last isomorphism $\phi^\flat_{\bar g}$ is precomposition with the equivariant homeomorphism $S^{\phi_{\bar g}} \colon S^{\mathrm{ind}_H^G(W)} \cong S^{\Psi^*_{\bar g}(W^m)}$.

Any other choice of transversal is of the form $\bar g \cdot \omega$ for a unique element $\omega \in \Sigma_m \wr H$, and $\Psi_{\bar g \omega} = c_\omega \circ \Phi_{\bar g}$, for $c_\omega \colon \Sigma_m \wr H \longrightarrow \Sigma_m \wr H$ the inner automorphism defined by $c_\omega(\gamma) = \omega^{-1}\gamma\omega$. Moreover, the composite isometry
$$\mathrm{ind}_H^G(W) \ \xrightarrow{\ \phi_{\bar g\omega}\ } \ \Psi^*_{\bar g\omega}(W^m) \ = \ \Psi^*_{\bar g}(c^*_\omega(W^m)) \ \xrightarrow{\ \Psi^*_{\bar g}(l_\omega)\ } \ \Psi^*_{\bar g}(W^m)$$
equals $\phi_{\bar g}$. Hence the following diagram commutes:
$$\begin{tikzcd}
 & & \pi^G_{\Psi^*_{\bar g}(W^m)}(R) \arrow[d, "(\Psi^*_{\bar g}(l_\omega))^\flat"'] \arrow[dr, "\phi^\flat_{\bar g}", "\cong"'] & \\
\pi^{\Sigma_m \wr H}_{W^m}(R) \arrow[urr, bend left, "\Psi^*_{\bar g}"] \arrow[r, "c^*_\omega = l^\flat_\omega"] \arrow[drr, bend right, "\Psi^*_{\bar g\omega}"'] & \pi^{\Sigma_m \wr H}_{c^*_\omega(W^m)}(R) \arrow[r, "\Psi^*_{\bar g}"'] & \pi^G_{\Psi^*_{\bar g}(c^*_\omega(W^m))}(R) \arrow[d, equal] & \pi^G_{\mathrm{ind}_H^G(W)}(R) \\
 & & \pi^G_{\Psi^*_{\bar g\omega}(W^m)}(R) \arrow[ur, "\phi^\flat_{\bar g\omega}"', "\cong"] &
\end{tikzcd}$$
We have used here that the restriction homomorphism $c^*_\omega \colon \pi^{\Sigma_m \wr H}_{W^m}(R) \longrightarrow \pi^{\Sigma_m \wr H}_{c^*_\omega(W^m)}(R)$ equals precomposition with the $(\Sigma_m \wr H)$-equivariant linear isometry $l_\omega \colon c^*_\omega(W^m) \longrightarrow W^m$ because R is a global spectrum, compare [38, Proposition 3.1.16]. Altogether this shows that different choices of transversals lead to same norm map, i.e., the norm map is independent of the choice of $\bar g$.

Proposition D.7. *Let R be an ultra-commutative ring spectrum, and let $v \in \pi_W^H(R)$ be an invertible representation-graded homotopy class.*

(i) *Then for every $m \geq 1$, the class $P^m(v)$ in $\pi^{\Sigma_m \wr H}_{W^m}(R)$ is invertible.*
(ii) *Let H be a finite index subgroup of a compact Lie group G. Then the class $N_H^G(v)$ in $\pi^G_{\mathrm{ind}_H^G(W)}(R)$ is invertible.*

Proof. (i) The morphism of H-spectra $v^\flat \colon R_H \wedge S^W \longrightarrow R_H$ was defined in (1.2). Proposition 1.5 provides an element $u \in \pi_0^H(R \wedge S^W)$ such that $v^\flat_*(u) = 1$ in $\pi_0^H(R)$. We suppose that u is represented by the continuous based H-map $f \colon S^U \longrightarrow R(U) \wedge S^W$. We define $P^m(u) \in \pi_0^{\Sigma_m \wr H}(R \wedge S^{W^m})$ as the class represented by the $(\Sigma_m \wr H)$-map
$$\begin{aligned} S^{U^m} \ = \ (S^U)^{\wedge m} \ &\xrightarrow{\ f^{\wedge m}\ } \ (R(U) \wedge S^W)^{\wedge m} \ \xrightarrow[\cong]{\ \text{shuffle}\ } R(U)^{\wedge m} \wedge S^{W^m} \\ &\xrightarrow{\ \mu_{U,\dots,U} \wedge S^{W^m}\ } R(U^m) \wedge S^{W^m} \ . \end{aligned}$$

Then this class satisfies the relation

$$(P^m(v))^\flat_*(P^m(u)) \;=\; 1$$

in the group $\pi_0^{\Sigma_m\wr H}(R)$. So the class $P^m(u)$ witnesses criterion (b) of Proposition 1.5 for the class $P^m(v)$, and thus $P^m(v)$ is invertible.

(ii) We let $\bar g = (g_1,\dots,g_m)$ be any transversal of H in G. The class $P^m(v)$ is invertible by (i), so $\Psi^*_{\bar g}(P^m(v))$ is invertible by Proposition 1.12. The norm $N_H^G(v)$ differs from $\Psi^*_{\bar g}(P^m(v))$ by an isometry of indexing representations, so $N_H^G(v)$ is invertible, too. □

Power operations are *not* generally additive. However, they become additive upon passage to geometric fixed points, as we now recall.

Proposition D.8. *Let R be an ultra-commutative ring spectrum. Let H be a finite index subgroup of a compact Lie group G. Then for all $x,y\in\pi_0^H(R)$, the relation*

$$\Phi^G(N_H^G(x+y)) \;=\; \Phi^G(N_H^G(x)) + \Phi^G(N_H^G(y))$$

holds in the group $\Phi_0^G(R)$.

Proof. We choose a transversal $\bar g=(g_1,\dots,g_m)$ of H in G. Then the composite

$$G \xrightarrow{\ \Psi_{\bar g}\ } \Sigma_m\wr H \xrightarrow{\ \mathrm{proj}\ } \Sigma_m$$

classifies the action of G on G/H by left translation. Since this action is transitive, the image of the composite is a transitive subgroup of Σ_m, and hence *not* subconjugate to $\Sigma_k\times\Sigma_{m-k}$ for any $1\le k\le m-1$. Consequently, the image $\bar G=\Psi_{\bar g}(G)$ of $\Psi_{\bar g}\colon G\longrightarrow \Sigma_m\wr H$ is not subconjugate to $(\Sigma_k\wr H)\times(\Sigma_{m-k}\wr H)$ for any $1\le k\le m-1$. Thus in the double coset formula for $\mathrm{res}^{\Sigma_m\wr H}_{\bar G}\circ \mathrm{tr}^{\Sigma_m\wr H}_{(\Sigma_k\wr H)\times(\Sigma_{m-k}\wr H)}$, all summands are transfers from proper subgroups of $\bar G$. Since transfers commute with inflation, the composite

$$\Psi^*_{\bar g}\circ \mathrm{tr}^{\Sigma_m\wr H}_{(\Sigma_k\wr H)\times(\Sigma_{m-k}\wr H)}$$

lands in the subgroup generated by transfers from proper subgroups of G, for $1\le k\le m-1$. The geometric fixed point homomorphism $\Phi^G\colon \pi_0^G(R)\longrightarrow\Phi_0^G(R)$ vanishes on transfers from proper subgroups of G, see [38, Proposition 3.3.11 (ii)]. Hence we conclude that

$$\begin{aligned}
\Phi^G(N_H^G(x+y)) \;&=\; \Phi^G(\Psi^*_{\bar g}(P^m(x+y)))\\
&=\; \sum_{k=0}^m \Phi^G(\Psi^*_{\bar g}(\mathrm{tr}^{\Sigma_m\wr H}_{(\Sigma_k\wr H)\times(\Sigma_{m-k}\wr H)}(P^k(x)\times P^{m-k}(y))))\\
&=\; \Phi^G(\Psi^*_{\bar g}(P^m(x))) + \Phi^G(\Psi^*_{\bar g}(P^m(y))) \;=\; \Phi^G(N_H^G(x)) + \Phi^G(N_H^G(y))\ .
\end{aligned}$$

The second equation is the formula for a power operation on a sum in a global power functor, proved for $\underline{\pi}_0(R)$ in [38, Theorem 5.1.11]. □

Appendix E. $RO(G)$-gradings

The topics of '$RO(G)$-gradings' in equivariant stable homotopy theory has a colorful track record, and provides some pitfalls to be avoided. Readers unfamiliar with the subtleties and pitfalls of $RO(G)$-gradings are encouraged to consult Adams' entertaining introduction to [1, §6]. Since we want to use $RO(G)$-graded multiplicative structure, we give a brief review of how we handle these issues.

Construction E.1 ($RO(G)$-grading)**.** Lewis and Mandell [26, Theorem A.1] and Dugger [14] show that one can make 'consistent' choices of invertible G-spectra for virtual representations, in the following sense. We consider the representation ring $RO(G)$ as a discrete symmetric monoidal category, i.e., with only identity morphisms, and with monoidal product given by addition in $RO(G)$. This symmetric monoidal structure is extremely strict, in that all coherence isomorphisms, for unitality, associativity, and even commutativity, are identities. The construction of [14, 26] provides a strong monoidal functor

$$\mathbb{S}_G^\bullet \ : \ RO(G) \longrightarrow h(\mathcal{S}p_G^\wedge) \tag{E.2}$$

with values in invertible objects, and with a particular kind of symmetry behavior. We emphasize that the functor is not *symmetric* monoidal, and the construction is not 'canonical', and that it makes essential use of the fact that the abelian group $RO(G)$ is free. In fact, these phenomena already appear when G is trivial group, see Example E.10 below.

We review the construction of the strong monoidal functor (E.2) in some level of detail, adapted to our context. It begins by choosing representatives for the isomorphism classes of irreducible orthogonal G-representations and enumerating them $\lambda_0, \lambda_1, \lambda_2, \dots$. If the group G is finite, then this is a finite list; otherwise, it is countably infinite. One may want to insist that λ_0 is the vector space $\mathbb{R}$ with trivial action, so that the part of the grading accounted for by the span of λ_0 in $RU(G)$ becomes the usual $\mathbb{Z}$-grading. Arguably, the most essential choice in the construction is that of the monoidal inverses, by which we mean invertible G-spectra $\mathbb{S}_G^{-\lambda_i}$ together with an isomorphism

$$\epsilon_i \ : \ \mathbb{S}_G^{-\lambda_i} \wedge \mathbb{S}_G^{\lambda_i} \ = \ \mathbb{S}_G^{-\lambda_i} \wedge S^{\lambda_i} \ \xrightarrow{\cong} \ \mathbb{S}_G \tag{E.3}$$

in the homotopy category of G-spectra, where $\mathbb{S}_G^{\lambda_i} = \gamma_G(\Sigma^\infty S^{\lambda_i})$ is the suspension spectrum of the sphere S^{λ_i}. Of course, the objects $\mathbb{S}_G^{-\lambda_i}$ are unique up to isomorphism in $h(\mathcal{S}p_G)$, but even if the choice of inverse object is fixed, the equivalence (E.3) that witnesses the inverse property can be changed by any unit in $\pi_0^G(\mathbb{S})$ to form a new witness. There is no intrinsic way to make 'natural' choices of $(\mathbb{S}_G^{-\lambda_i}, \epsilon_i)$; and if the objects $\mathbb{S}_G^{-\lambda_i}$ are fixed, the set of choices for the equivalences ϵ_i is a torsor over the abelian group $\mathrm{Hom}(RO(G), \pi_0^G(\mathbb{S})^\times)$. We illustrate in Example E.10 how the choices effect the resulting $RO(G)$-graded rings.

With all these choices made, Lewis–Mandell and Dugger specify a procedure to define isomorphisms

$$\omega_{\alpha,\beta} \ : \ \mathbb{S}_G^{\alpha+\beta} \ \xrightarrow{\cong} \ \mathbb{S}_G^\alpha \wedge \mathbb{S}_G^\beta \tag{E.4}$$

for all $\alpha, \beta \in RO(G)$, by using only associativity, unitality and symmetry isomorphism, and the chosen equivalences (E.3) to 'cancel' irreducible summands that appear in α and β with opposite signs. The main result is that the resulting data satisfies unitality and associativity constraints, i.e., the isomorphisms $\omega_{\alpha,\beta}$ form a strong monoidal structure on the unique functor $\mathbb{S}_G^\bullet \colon RU(G) \longrightarrow h(\mathcal{S}p_G^\wedge)$ that sends α to $\mathbb{S}_G^\alpha$; see [26, Proposition A.5] and [14, Proposition 6.1] for details. In particular, the associativity constraint says that the diagram

$$\begin{tikzcd}
(\mathbb{S}_G^\alpha \wedge \mathbb{S}_G^\beta) \wedge \mathbb{S}_G^\gamma \arrow[r, "\text{associativity}"] \arrow[d, "\omega_{\alpha,\beta} \wedge \mathbb{S}_G^\gamma"'] & \mathbb{S}_G^\alpha \wedge (\mathbb{S}_G^\beta \wedge \mathbb{S}_G^\gamma) \arrow[r, "\mathbb{S}_G^\alpha \wedge \omega_{\beta,\gamma}"] & \mathbb{S}_G^\alpha \wedge \mathbb{S}_G^{\beta+\gamma} \arrow[d, "\omega_{\alpha,\beta+\gamma}"] \\
\mathbb{S}_G^{\alpha+\beta} \wedge \mathbb{S}_G^\gamma \arrow[rr, "\omega_{\alpha+\beta,\gamma}"'] & & \mathbb{S}_G^{\alpha+\beta+\gamma}
\end{tikzcd}$$

commutes for all $\alpha, \beta, \gamma \in RO(G)$.

The monoidal functor $\mathbb{S}_G^\bullet$ is *not* symmetric; however, since $\mathbb{S}_G^{\alpha+\beta} = \mathbb{S}_G^{\beta+\alpha}$ is invertible, there is a unique invertible element $t_{\alpha,\beta} \in \pi_0^G(\mathbb{S})$ that measures the deviation from symmetry, in the sense that the following diagram commutes:

$$\begin{tikzcd}
\mathbb{S}_G^\alpha \wedge \mathbb{S}_G^\beta \arrow[r, "\text{symmetry}"] \arrow[d, "\omega_{\alpha,\beta}"'] & \mathbb{S}_G^\alpha \wedge \mathbb{S}_G^\beta \arrow[d, "\omega_{\beta,\alpha}"] \\
\mathbb{S}_G^{\alpha+\beta} \arrow[r, "t_{\alpha,\beta}\cdot -"'] & \mathbb{S}_G^{\beta+\alpha}
\end{tikzcd}$$

Then clearly $t_{\alpha,\beta} = t_{\beta,\alpha}^{-1}$, and $t_{\alpha,0} = 1$ by the unitality condition. The hexagon coherence conditions between the associativity and symmetry isomorphisms in $h(\mathcal{S}p_G^\wedge)$, and the fact that the isomorphisms $\omega_{\alpha,\beta}$ are associative yields the relations

$$t_{\alpha,\gamma} \cdot t_{\beta,\gamma} \;=\; t_{\alpha+\beta,\gamma} \quad \text{and} \quad t_{\alpha,\beta} \cdot t_{\alpha,\gamma} \;=\; t_{\alpha,\beta+\gamma} \tag{E.5}$$

for all α, β, γ in $RO(G)$. Moreover, $\varepsilon_\alpha = t_{\alpha,\alpha}$ is the element in $\pi_0^G(\mathbb{S})$ such that

$$\varepsilon_\alpha \cdot \mathrm{Id}_{\mathbb{S}_G^\alpha \wedge \mathbb{S}_G^\alpha} \;=\; \tau_{\mathbb{S}_G^\alpha, \mathbb{S}_G^\alpha} \;:\; \mathbb{S}_G^\alpha \wedge \mathbb{S}_G^\alpha \;\longrightarrow\; \mathbb{S}_G^\alpha \wedge \mathbb{S}_G^\alpha \ .$$

This element satisfies $\varepsilon_\alpha^2 = 1$, and it equals the trace of the identity, and hence the Euler characteristic of the invertible object $\mathbb{S}_G^\alpha$.

The relations (E.5) imply that the units $t_{\alpha,\beta}$ are determined by the special cases where $\alpha = [\lambda]$ and $\beta = [\mu]$ are the classes of two irreducible representations λ and μ. In that case,

$$t_{[\lambda],[\mu]} \;=\; \begin{cases} 1 & \text{if } \lambda \ncong \mu, \text{ and} \\ \varepsilon_{[\lambda]} & \text{if } \lambda \cong \mu. \end{cases}$$

So in fact

$$t_{\alpha,\beta} \;=\; \prod_\lambda (\varepsilon_{[\lambda]})^{\mu(\lambda,\alpha)\cdot\mu(\lambda,\beta)} \ ,$$

where the product is over a set of representatives of the isomorphism classes of irreducible representations, and $\mu(\lambda,\alpha) \in \mathbb{Z}$ is the multiplicity of λ in α, compare [14, Proposition 1.2].

The lack of symmetry is already visible for $G = \{1\}$, where $RO(\{1\}) = \mathbb{Z}$ and $\tau_{1,1} = -1$, simply because the twist isomorphism of $S^1 \wedge S^1$ has degree -1. Another instructive example is $G = C_2$ and $\sigma \in RO(C_2)$ the class of the sign representation on $\mathbb{R}$. In this case

$$\varepsilon_\sigma \;=\; 1 - \mathrm{tr}_{\{1\}}^{C_2}(1) \;\in \pi_0^{C_2}(\mathbb{S}_{C_2}) \ .$$

Construction E.6 ($RO(G)$-graded homotopy groups)**.** The strong symmetric monoidal functor $\mathbb{S}_G^\bullet$ from (E.2) allows us to introduce $RO(G)$-graded G-equivariant homotopy groups, as follows. For a G-spectrum X and a virtual representation $\alpha \in RO(G)$, we set

$$\pi_\alpha^G(X) \;=\; h(\mathcal{S}p_G)(\mathbb{S}_G^\alpha, X) \ ,$$

the abelian group of morphisms, in the homotopy category of G-spectra, from $\mathbb{S}_G^\alpha$. Since the object $\mathbb{S}_G^\alpha$ is invertible, it is in particular compact in $h(\mathcal{S}p_G)$, so the functor π_α^G sends arbitrary coproducts to wedges. We shall write

$$\pi_\star^G(X) \;=\; \{\pi_\alpha^G(X)\}_{\alpha \in RO(G)}$$

for the resulting $RO(G)$-graded abelian group.

If Y is another G-spectrum and $\beta \in RO(G)$, we obtain a biadditive pairing

$$\begin{aligned} \wedge \;:\; \pi_\alpha^G(X) \times \pi_\beta^G(Y) \quad &= \quad h(\mathcal{S}p_G)(\mathbb{S}_G^\alpha, X) \times h(\mathcal{S}p_G)(\mathbb{S}_G^\beta, Y) \\ &\xrightarrow{\ \wedge\ } h(\mathcal{S}p_G)(\mathbb{S}_G^\alpha \wedge \mathbb{S}_G^\beta, X \wedge Y) \\ &\xrightarrow[\cong]{-\circ\omega_{\alpha,\beta}} h(\mathcal{S}p_G)(\mathbb{S}_G^{\alpha+\beta}, X \wedge Y) \;=\; \pi_{\alpha+\beta}^G(X \wedge Y) \ . \end{aligned} \tag{E.7}$$

The isomorphism $\omega_{\alpha,\beta}\colon \mathbb{S}_G^{\alpha+\beta} \cong \mathbb{S}_G^\alpha \wedge \mathbb{S}_G^\beta$ is the monoidal structure morphism (E.4).

It is a routine check, using that the functor $\mathbb{S}_G^\bullet$ is monoidal, that these pairings are associative and unital. Since the functor $\mathbb{S}_G^\bullet$ is not symmetric monoidal, the pairing is not strictly commutative, which is already the case for the usual $\mathbb{Z}$-grading in stable homotopy theory. The deviation from strict commutativity is measured by the units $t_{\alpha,\beta}$, in the sense that

$$y \wedge x \ = \ t_{\alpha,\beta} \cdot (\tau_{X,Y})_*(x \wedge y) \quad \text{in } \pi^G_{\beta+\alpha}(Y \wedge X),$$

for all $x \in \pi^G_\alpha(X)$ and $y \in \pi^G_\beta(Y)$. Here $\tau_{X,Y}\colon X \wedge Y \longrightarrow Y \wedge X$ denotes the symmetry equivalence for the smash product of G-spectra.

Now we let R be a homotopy G-ring spectrum with multiplication morphism $\mu\colon R \wedge R \longrightarrow R$ in $h(\mathcal{S}p_G^\wedge)$. Then the composite maps

$$\cdot \ : \ \pi^G_\alpha(R) \times \pi^G_\beta(R) \ \xrightarrow{\ \wedge\ } \ \pi^G_{\alpha+\beta}(R \wedge R) \ \xrightarrow{\ \mu_*\ } \ \pi^G_{\alpha+\beta}(R)$$

enhance $\pi^G_\star(R)$ to an $RO(G)$-graded ring. And for every left homotopy R-module M in $h(\mathcal{S}p_G^\wedge)$, the composite maps

$$\pi^G_\alpha(R) \times \pi^G_\beta(M) \ \xrightarrow{\ \wedge\ } \ \pi^G_{\alpha+\beta}(R \wedge M) \ \xrightarrow{\ \mathrm{act}_*\ } \ \pi^G_{\alpha+\beta}(R)$$

enhance $\pi^G_\star(M)$ to an $RO(G)$-graded left $\pi^G_\star(R)$-module. If the multiplication of R is homotopy-commutative, then $\pi^G_\star(R)$ is a 'graded-commutative' $RO(G)$-graded ring, in the sense that

$$y \cdot x \ = \ t_{\alpha,\beta} \cdot (x \cdot y) \quad \text{in } \pi^G_{\beta+\alpha}(R),$$

for all $x \in \pi^G_\alpha(R)$ and $y \in \pi^G_\beta(R)$

Now we let $\alpha \in RO(G)$ be a virtual G-representation. For an equivariant homotopy class $x \in \pi^G_\alpha(R)$, we define a morphism $x^\flat\colon R \wedge \mathbb{S}_G^\alpha \longrightarrow R$ as the composite

$$R \wedge \mathbb{S}_G^\alpha \ \xrightarrow{\ R\wedge x\ } \ R \wedge R \ \xrightarrow{\ \mu\ } \ R \ .$$

This morphism satisfies

$$x^\flat_*(1 \wedge \mathbb{S}_G^\alpha) \ = \ x \ ,$$

where $1 \wedge \mathbb{S}_G^\alpha\colon \mathbb{S}_G^\alpha \longrightarrow R \wedge \mathbb{S}_G^\alpha$ is an element of $\pi^G_\alpha(R \wedge \mathbb{S}_G^\alpha)$.

The invertible G-spectrum $\mathbb{S}_G^\alpha$ represents the functor $\pi^G_\alpha\colon h(\mathcal{S}p_G) \longrightarrow \mathcal{A}b$, by the very definition of π^G_α. So if R underlies a G-ring spectrum R, i.e., an object in $\mathrm{Alg}(\mathcal{S}p_G^\wedge)$, then $R \wedge \mathbb{S}_G^\alpha$ underlies an R-module spectrum that represents the functor $\pi^G_\alpha\colon h(\mathrm{LMod}_R(\mathcal{S}p_G^\wedge)) \longrightarrow \mathcal{A}b$, with respect to the class $1 \wedge \mathbb{S}_G^\alpha$, and $x^\flat$ underlies a morphism of left R-modules.

Proposition E.8. *Let G be a compact Lie group, and let R be a homotopy G-ring spectrum. For an $RO(G)$-graded homotopy class $x \in \pi^G_\alpha(R)$, the following two conditions are equivalent.*

(a) *The class x is a graded unit in the $RO(G)$-graded homotopy ring $\pi^G_\star(R)$.*

(b) *The morphism of G-spectra $x^\flat\colon R \wedge \mathbb{S}_G^\alpha \longrightarrow R$ is an equivalence.*

Proof. (a)$\Longrightarrow$(b) We let $y \in \pi^G_{-\alpha}(R)$ be the inverse to x, i.e., such that $x \cdot y = 1 = y \cdot x$. Expanding the definitions of $x^\flat$ and $y^\flat$, and exploiting the associativity and unitality of the multiplication of R shows that the composite

$$R \cong R \wedge \mathbb{S}_G^\alpha \wedge \mathbb{S}_G^{-\alpha} \ \xrightarrow{\ x^\flat \wedge \mathbb{S}_G^{-\alpha}\ } \ R \wedge \mathbb{S}_G^{-\alpha} \ \xrightarrow{\ y^\flat\ } \ R$$

In particular, the morphism $y^\flat$ has a right inverse. And because $\mathbb{S}_G^{-\alpha}$ is invertible, this also proves that $x^\flat$ has a left inverse. The previous argument applied to the pair (y,x) instead of (x,y) shows that $x^\flat$ has a right inverse. Since $x^\flat$ has both a left and right inverse, it is an isomorphism in $h(\mathrm{LMod}_R(\mathcal{S}p_G^\wedge))$, and thus in particular an equivalence of underlying G-spectra.

(b)$\Longrightarrow$(a) Since the morphism $x^\flat$ is an equivalence, the composite

$$\begin{aligned}\pi^G_{-\alpha}(R) \ = \ h(\mathcal{S}p_G)(\mathbb{S}^{-\alpha}_G, R) \ &\xrightarrow[\cong]{-\wedge\mathbb{S}^\alpha_G} h(\mathcal{S}p_G)(\mathbb{S}^{-\alpha}_G\wedge\mathbb{S}^\alpha_G, R\wedge\mathbb{S}^\alpha_G) \\ &\xrightarrow[\cong]{-\circ\omega_{-\alpha,\alpha}} h(\mathcal{S}p_G)(\mathbb{S}^0_G, R\wedge\mathbb{S}^\alpha_G) \ = \ \pi^G_0(R\wedge\mathbb{S}^\alpha_G) \ \xrightarrow[\cong]{x^\flat_*} \ \pi^G_0(R)\end{aligned}$$

is bijective. This composite is right multiplication by the class x, by definition of the $RO(G)$-graded multiplication. Surjectivity of right multiplication yields a class $y\in\pi^G_{-\alpha}(R)$ such that $y\cdot x = 1$. Then

$$(x\cdot y)\cdot x \ = \ x\cdot(y\cdot x) \ = \ x\cdot 1 \ = \ 1\cdot x \ .$$

Injectivity of right multiplication thus yields $x\cdot y = 1$. So x is a graded unit. □

Remark E.9 (Representation-graded versus $RO(G)$-graded invertibility)**.** Invertibility in the sense of Definition 1.3 is essentially the same as invertibility in the $RO(G)$-graded homotopy ring $\pi^G_\star(R)$. The caveat is that while the representation-graded homotopy group $\pi^G_V(R)$ is isomorphic to the $RO(G)$-graded homotopy group $\pi^G_{[V]}(R)$, there is no preferred or 'canonical' way to identify them.

However, since the G-spectra $\mathbb{S}^{[V]}_G$ and $\Sigma^\infty S^V$ are equivalent, the group $\pi^G_{[V]}(\Sigma^\infty S^V)$ is a free module of rank 1 over the Burnside ring $\pi^G_0(\mathbb{S})$, and the isomorphisms $\mathbb{S}^{[V]}_G\cong\Sigma^\infty S^V$ in $h(\mathcal{S}p_G)$ are precisely the $\pi^G_0(\mathbb{S})$-module generators of $\pi^G_{[V]}(\Sigma^\infty S^V)$. Any equivalence of G-spectra $\psi\colon\mathbb{S}^{[V]}_G\sim\Sigma^\infty S^V$ induces an isomorphism

$$\pi^G_V(R) \ \cong \ h(\mathcal{S}p_G)(\Sigma^\infty S^V, R) \ \xrightarrow[\cong]{-\circ\psi} \ h(\mathcal{S}p_G)(\mathbb{S}^{[V]}, R) \ = \ \pi^G_{[V]}(R)$$

that sends a representation-graded homotopy class $v\in\pi^G_V(R)$ to an $RO(G)$-graded homotopy class $\psi^*(v)\in\pi^G_{[V]}(R)$. Any two choices of equivalence differ by a unit in the Burnside ring $\pi^G_0(\mathbb{S}_G)$, which also changes the class $\psi^*(v)$ by a unit. In particular, whether or not the class $\psi^*(v)$ is an $RO(G)$-graded unit is independent of the choice of identification. For $RO(G)$-graded homotopy classes $x\in\pi^G_\alpha(R)$, being a graded unit is equivalent to the morphism $x^\flat\colon R\wedge\mathbb{S}^\alpha_G\longrightarrow R$ being an equivalence, see Proposition E.8. So we conclude that $v\in\pi^G_V(R)$ is invertible in the sense of Definition 1.3 if and only if for some (hence any) equivalence of G-spectra $\psi\colon\mathbb{S}^{[V]}_G\sim\Sigma^\infty S^V$, the class $\psi^*(v)$ is a graded unit in the $RO(G)$-graded ring $\pi^G_\star(R)$.

Example E.10. As mentioned several times above, the construction of the functor $\mathbb{S}^\bullet_G\colon RO(G)\longrightarrow h(\mathcal{S}p^\wedge_G)$ involves non-canonical choices that affect the resulting $RO(G)$-grading. This phenomenon already exists for the trivial group $G=\{1\}$, and we illustrate it in this case. For $n\geq 0$ we set $\mathbb{S}^n=\Sigma^\infty S^n$, the suspension spectrum of the n-sphere. Suppose that Alice has chosen an invertible spectrum $\mathbb{S}^{-1}$ and an equivalence $\epsilon^A\colon\mathbb{S}^{-1}\wedge\mathbb{S}^1\sim\mathbb{S}^0$. And suppose that Bob chooses the same -1-sphere $\mathbb{S}^{-1}$, but the *negative* equivalence

$$\epsilon^B \ = \ -\epsilon^A \ : \ \mathbb{S}^{-1}\wedge\mathbb{S}^1\sim\mathbb{S}^0 \ .$$

In the construction of the equivalence

$$\omega_{k,l} \ : \ \mathbb{S}^{k+l} \ \xrightarrow{\cong} \ \mathbb{S}^k\wedge\mathbb{S}^l \ ,$$

for $k,l\in\mathbb{Z}$, the previous choice of equivalence is used to cancel 'positive' contributions from one of the two integers against 'negative' contributions from the other one. So the chosen equivalence from $\mathbb{S}^{-1}\wedge\mathbb{S}^1$ to $\mathbb{S}^0$ is not invoked if k and l have the same sign, and it is invoked $|k+l|$ times if k and l have opposite signs. Said differently, the equivalence is invoked

$$c(k,l) \ = \ \frac{|k|+|l|-|k+l|}{2}$$

many times. So Alice's and Bob's equivalences are related by

$$\omega^B_{k,l} \;=\; (-1)^{c(k,l)} \cdot \omega^A_{k,l} \ .$$

Consequently, the homotopy group functors

$$\pi_\star \ : \ h(\mathcal{S}p^\wedge) \ \longrightarrow \ \mathcal{A}b$$

obtained by Alice and Bob are equal as graded abelian groups, but come with *different* multiplications, related by

$$x \cdot_B y \;=\; (-1)^{c(k,l)} \cdot x \cdot_A y \ .$$

Both Alice's and Bob's multiplications satisfy the same commutativity rule, namely the usual one, with sign $(-1)^{kl}$ occurring whenever two classes of degrees k and l switch places. And while Alice's and Bob's multiplications are different, they are nevertheless naturally isomorphic.

Construction E.11. We let G be a compact Lie group. As we shall now explain, for *unitary* G-representations V, one can make choices of isomorphisms

$$\kappa_V \ : \ \mathbb{S}_G^{[V]} \ \cong \ \Sigma^\infty S^V$$

in the homotopy category of G-spectra that are multiplicative and consistent under complex equivariant isometries. The key fact that makes this possible is that automorphism groups of unitary representations are connected. We shall use these isomorphisms in (5.5) to turn the pre-Bott classes from representation-graded homotopy classes into $RO(G)$-graded homotopy classes.

To this end, we choose representatives $\{\mu_i\}_{i\in I}$ for the isomorphism classes of irreducible unitary G-representations. For every chosen representative, we choose an equivalence of G-spectra $\kappa_i\colon \mathbb{S}_G^{[\mu_i]} \cong \Sigma^\infty S^{\mu_i}$, where $[\mu_i]$ is the class in $RO(G)$ of the underlying orthogonal G-representation of μ_i.

Now we let V be any unitary G-representation. We choose a $\mathbb{C}$-linear G-equivariant isometry

$$\psi \ : \ \mu_{i_1} \oplus \dots \oplus \mu_{i_k} \ \cong \ V$$

for suitable $i_1, \dots, i_k \in I$. We define an isomorphism

$$\kappa_V \ : \ \mathbb{S}_G^{[V]} \ \cong \ \Sigma^\infty S^V$$

in the homotopy category of G-spectra as the composite

$$\begin{aligned}
\mathbb{S}_G^{[V]} \ = \ \mathbb{S}_G^{[\mu_{i_1}]+\dots+[\mu_k]} \ &\xrightarrow[\cong]{\omega_{[\mu_{i_1}],\dots,[\mu_{i_k}]}} \mathbb{S}_G^{[\mu_{i_1}]} \wedge \ldots \wedge \mathbb{S}_G^{[\mu_k]} \\
&\xrightarrow[\cong]{\kappa_{i_1}\wedge\ldots\wedge\kappa_{i_k}} (\Sigma^\infty S^{\mu_{i_1}}) \wedge \ldots \wedge (\Sigma^\infty S^{\mu_{i_k}}) \\
&\cong \quad \Sigma^\infty S^{\mu_{i_1}\oplus\dots\oplus\mu_{i_k}} \xrightarrow[\cong]{\Sigma^\infty S^\psi} \Sigma^\infty S^V \ .
\end{aligned}$$

The first isomorphism is the multi-index extension of the associativity isomorphism (E.4), where we have suppressed associativity coherence isomorphisms. Since the compact Lie group of $\mathbb{C}$-linear self-isometries of any unitary representation is connected, any two choices of ψ are homotopic through $\mathbb{C}$-linear equivariant isometries, and hence the isomorphism $\kappa_V\colon \mathbb{S}_G^{[V]} \cong \Sigma^\infty S^V$ in $h(\mathcal{S}p_G)$ does not depend on the choice of ψ. Similarly, because the relevant automorphism groups are connected:

- for every $\mathbb{C}$-linear isometry $\varphi\colon V \cong W$, the composite

$$\mathbb{S}_G^{[W]} \ = \ \mathbb{S}_G^{[V]} \ \xrightarrow{\kappa_V} \ \Sigma^\infty S^V \ \xrightarrow{\Sigma^\infty S^\varphi} \ \Sigma^\infty S^W$$

  equals κ_W, and

- the composite
$$\mathbb{S}_G^{[V\oplus W]} \xrightarrow[\cong]{\omega_{[V],[W]}} \mathbb{S}_G^{[V]} \wedge \mathbb{S}_G^{[W]} \xrightarrow[\cong]{\kappa_V \wedge \kappa_W} (\Sigma^\infty S^V) \wedge (\Sigma^\infty S^W) \cong \Sigma^\infty S^{V\oplus W}$$
equals $\kappa_{V\oplus W}$.

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