

THE UNIVERSAL PROPERTY OF BORDISM OF COMMUTING INVOLUTIONS

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ABSTRACT. We propose a formalism to capture the structure of equivariant bordism of smooth manifolds with commuting involutions. We introduce the concept of an *oriented* el_2^{RO} -algebra, an algebraic structure featuring representation graded rings for all elementary abelian 2-groups, connected by restriction homomorphisms, a pre-Euler class, and an inverse Thom class; this data is subject to one exactness property. Besides equivariant bordism, oriented global ring spectra also give rise to oriented el_2^{RO} -algebras, so examples abound. Inverting the inverse Thom classes yields a global 2-torsion group law. In this sense, our oriented el_2^{RO} -algebras are delocalized generalizations of global 2-torsion group laws.

Our main result shows that equivariant bordism for elementary abelian 2-groups is an initial oriented el_2^{RO} -algebra. Several other interesting equivariant homology theories can also be characterized, on elementary abelian 2-groups, by similar universal properties. We prove that stable equivariant bordism is an initial el_2^{RO} -algebra with an invertible orientation; that Bredon homology with constant mod 2 coefficients is an initial el_2^{RO} -algebra with an additive orientation; and that Borel equivariant homology with mod 2 coefficients is an initial el_2^{RO} -algebra with an orientation that is both additive and invertible.

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INTRODUCTION

The purpose of this paper is to unveil the universal property of equivariant bordism for elementary abelian 2-groups. More precisely, we will show that equivariant bordism of manifolds with commuting involutions is an initial example of a specific kind of algebraic structure that we call an *oriented* el_2^{RO} -algebra. The superscript ‘*RO*’ refers to the feature that gradings by

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real representations are hard wired into our theory. Our formalism also allows to characterize several other interesting equivariant homology theories, on elementary abelian 2-groups, by similar universal properties in the category of oriented el_2^{RO} -algebras; examples include stable equivariant bordism, Bredon homology with constant mod 2 coefficients, and Borel cohomology with mod 2 coefficients. In short, stable equivariant bordism is initial among oriented el_2^{RO} -algebras equipped with an invertible orientation; Bredon homology is initial among oriented el_2^{RO} -algebras equipped with an additive orientation; and Borel cohomology is initial among oriented el_2^{RO} -algebras equipped with an orientation that simultaneously invertible and additive.

The key new insight that we wish to promote in this paper is that the algebraic structure of an orientable el_2^{RO} -algebra arises very naturally in the study of equivariant homology theories for elementary abelian 2-groups, and that it greatly clarifies and conceptualizes the ‘true’ nature of several prominent equivariant theories. Indeed, for all the theories mentioned in the previous paragraph, there is a vast literature of explicit calculations, in particular for the group with two elements. Examples include [2, 4, 5, 6, 7, 10, 12, 15, 28, 29, 30] for equivariant bordism, [9, 15, 28] for stable equivariant bordism, and [3, 11, 14, 18, 19, 20, 21, 22, 26, 27] for Bredon homology and cohomology. Many of the answers are in terms of lists of generators and relations. While often quite explicit, the present authors do not find many of these calculations conceptually enlightening, and contemplating the answers left us wondering what it actually means to ‘calculate’ or understand such kinds of mathematical structures. Furthermore, none of the previous results we are aware of relate to Quillen’s algebraic interpretation [24] of the bordism ring \mathcal{N}_* as the 2-torsion Lazard ring, carrying the universal formal group law with trivial 2-series.

Our approach shifts the emphasis away from explicit calculations of equivariant bordism or cohomology rings for individual groups of equivariance; we focus on the question: ‘What do the equivariant rings represent?’ A lesson we learned is that for getting interesting answers, one might want to:

- consider all elementary abelian 2-groups simultaneously, keeping track of the contravariant functoriality in group homomorphisms, and
- remember and exploit the RO -grading by real representations.

We implement these two aspects in the structure an *oriented el_2^{RO} -algebra*. Central to this notion are two distinguished classes of elements, the ‘pre-Euler classes’ a_λ and the ‘inverse Thom classes’ t_λ , indexed by all nontrivial characters λ of elementary abelian 2-groups. In the case of equivariant bordism, a_λ equals the bordism class of the inclusion $\{0\} \rightarrow S^\lambda$ of the origin into the representation sphere associated to λ , while t_λ is given by the bordism class of the identity map $S^\lambda \rightarrow S^\lambda$. In the case of Bredon homology, the classes a_λ and t_λ generate the entire representation-graded ring, subject to certain explicit polynomial relations [18]. The structure is subject to one key regularity property: multiplication by the pre-Euler classes of nontrivial characters must be injective, with cokernel given by the value at the kernel of the given character, while the classes t_λ must restrict to the unit at the trivial group. The precise definition of an oriented el_2^{RO} -algebra will be given in Definition 1.4. Notably, the category of oriented el_2^{RO} -algebras contains the category of 2-torsion formal group laws as a reflective subcategory via the ‘complete t -invertible’ objects, see Section 8. Hence, in very loose terms, an oriented el_2^{RO} -algebra can be viewed as a delocalized and decompleted version of a 2-torsion formal group law.

After this attempt to provide motivation for our formalism, we now list our main results. The first of these, to be proved as Theorem 7.14 below, makes the title of this paper precise:

Theorem A. *Equivariant bordism is an initial oriented el_2^{RO} -algebra.*

We emphasize that in contrast to all previous results on equivariant bordism that we are aware of, Theorem A characterizes equivariant bordism in absolute terms rather than relative to the non-equivariant bordism ring \mathcal{N}_* . In particular, no a priori connection to formal group laws, nor any \mathcal{N}_* -algebra structure, are directly built into the definition of an oriented el_2^{RO} -algebra. Still, the regularity properties of the pre-Euler classes together with the functoriality in elementary abelian 2-groups indirectly induce non-trivial structure also over the trivial group, namely that of a 2-torsion formal group law. As such, Theorem A can be viewed as a refinement of Quillen's theorem [24]; however, our Theorem A does not provide an independent proof of Quillen's result, as we rely on it as central input.

We now turn to our next result. Stable equivariant bordism is a certain localization of equivariant bordism first considered by Bröcker and Hook [9, §2]. We call an orientation of an el_2^{RO} -algebra *invertible* if the inverse Thom classes for all characters of elementary abelian 2-groups act invertibly, see Definition 8.1. Stable equivariant bordism is, essentially by design, the oriented el_2^{RO} -algebra obtained from equivariant bordism by inverting all those inverse Thom classes. The following result is thus an immediate corollary of Theorem A:

Theorem B. *Stable equivariant bordism is an initial invertibly oriented el_2^{RO} -algebra.*

Theorem B will be proved as Corollary 8.6 below. As we explain at the end of Section 8, combining Theorem B with the main result of Bröcker and Hook [9] yields an independent proof of a recent result of the first author, namely that the global 2-torsion group law carried by the global Thom spectrum \mathbf{MO} is initial, see [17, Theorem D]. In fact, this earlier result of the first author was a key motivation for the present project, and some of our arguments are inspired by arguments from [17].

The next interesting example is Bredon homology with coefficients in the constant Mackey functor with value \mathbb{F}_2 ; by definition, the \mathbb{Z} -graded part of this theory is a copy of \mathbb{F}_2 concentrated in dimension 0. These Bredon homology groups have attracted a certain amount of attention, and we listed specific references with many explicit calculations above. In this paper we show an algebraic universal property for the coefficients of Bredon homology. The oriented el_2^{RO} -algebra of Bredon homology has a unique orientation, and this orientation is *additive* in the sense of Definition 3.3. We use the adjective 'additive' because the formal group law arising from an additive orientation is the additive one, see Example 4.8.

Theorem C. *Bredon homology with constant \mathbb{F}_2 -coefficients is an initial additively oriented el_2^{RO} -algebra.*

Theorem C is proved as Theorem 3.5 below. In our paper [18], we provide an explicit presentation of the representation-graded Bredon homology rings with constant \mathbb{F}_2 -coefficients.

The last example whose universal property we discuss is the mod 2 global Borel spectrum $b(H\mathbb{F}_2)$, representing Borel equivariant cohomology with \mathbb{F}_2 -coefficients. The associated el_2^{RO} -algebra has a unique orientation, and this unique orientation is both additive and invertible. We prove in Theorem 8.8 that mod 2 global Borel cohomology is characterized by such an orientation:

Theorem D. *Borel cohomology with \mathbb{F}_2 -coefficients is initial among oriented el_2^{RO} -algebras whose orientation is both additive and invertible.*

In Section 9 we explain how our theory can be used to reprove various previous results on equivariant bordism, in particular the Conner–Floyd exact sequence [12], Alexander's description [2] of an explicit basis over \mathcal{N}_* for the bordism ring of involutions, and Firsching's pullback

square [15]. In all cases, the results in fact hold for a large class of oriented el_2^{RO} -algebras, and not just for the initial one given by equivariant bordism.

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1. ORIENTED el_2^{RO} -ALGEBRAS

In this section we introduce our main characters, the *oriented el_2^{RO} -algebras*. One of the key points of this paper is the insight that to unveil the universal property of equivariant bordism for elementary abelian 2-groups, one needs to take the grading by representations into account. More precisely, the proper book keeping needs the ‘effective’ part of the RO -grading, i.e., the part that keeps track of the integer graded bordism groups of representation spheres.

We start by fixing notation and terminology to deal with the RO -grading. For every finite group G , we define a submonoid I_G of the real representation ring $RO(G)$ by

$$I_G = \{m \in RO(G) : m = k - V \text{ for some } k \in \mathbb{Z} \text{ and some } G\text{-representation } V\}.$$

So the elements of I_G are formal differences of integers and actual representations. These submonoids are closed under restriction along homomorphisms $\alpha : K \rightarrow G$ between finite groups, i.e., $\alpha^* : RO(G) \rightarrow RO(K)$ sends I_G to I_K . We emphasize that unless the group G is trivial, the abelian monoid I_G is *not* a group.

Construction 1.1 (The category el_2^{RO}). We introduce a category el_2^{RO} of effective representations of elementary abelian 2-groups. Objects of el_2^{RO} are pairs (A, m) , where A is an elementary abelian 2-group and $m \in I_A$. Morphisms from (B, n) to (A, m) in the category el_2^{RO} are group homomorphisms $\alpha : B \rightarrow A$ such that $\alpha^*(m) = n$.

We give the category el_2^{RO} a symmetric monoidal structure as follows. On objects, we set

$$(A, m) \times (B, n) = (A \times B, m \oplus n),$$

where

$$m \oplus n = p_1^*(m) + p_2^*(n)$$

is the external sum, with $p_1 : A \times B \rightarrow A$ and $p_2 : A \times B \rightarrow B$ the projections to the two factors. On morphisms, the symmetric monoidal structure is given by product of homomorphisms. The symmetry and associativity isomorphisms are the symmetry and associativity isomorphism for the product of groups. The unit object is $(1, 0)$, the pair consisting of the trivial group 1 and its zero representation. A symmetric monoidal structure on a category is at the same time a symmetric monoidal structure on its opposite.

Definition 1.2. An el_2^{RO} -algebra is a lax symmetric monoidal functor

$$X : (\text{el}_2^{RO})^{\text{op}} \rightarrow \mathbb{F}_2\text{-mod}$$

to the category of \mathbb{F}_2 -vector spaces under tensor product. A *morphism* of el_2^{RO} -algebras is a monoidal transformation.

Since symmetry is part of the definition of an el_2^{RO} -algebra, a more precise name for these objects would have been *commutative el_2^{RO} -algebras*. Since we will not consider non-commutative el_2^{RO} -algebras (i.e., non-symmetric lax monoidal functors), we have opted for the simpler name.

Expanding the definition reveals that an el_2^{RO} -algebra X consists of

- \mathbb{F}_2 -vector spaces $X(A, m)$ for every elementary abelian 2-group A and every $m \in I_A$,
- a restriction homomorphism

$$\alpha^* : X(A, m) \longrightarrow X(B, \alpha^*(m)) ,$$

for every group homomorphism $\alpha : B \longrightarrow A$ and every $m \in I_A$,

- associative and commutative pairings

$$\times : X(A, m) \otimes X(B, n) \longrightarrow X(A \times B, m \oplus n)$$

with a multiplicative unit $1 \in X(1, 0)$.

Moreover, the restriction homomorphisms must be contravariantly functorial and strictly compatible with the multiplicative structure. A morphism $f : X \longrightarrow Y$ of el_2^{RO} -algebras consists of additive maps $f(A, m) : X(A, m) \longrightarrow Y(A, m)$ such that $f(1, 0)(1) = 1$ and

$$f(A \times B, m \oplus n)(x \times y) = f(A, m)(x) \times f(B, n)(y)$$

for all $x \in X(A, m)$ and $y \in X(B, n)$.

The multiplicative structure could alternatively be given in ‘internal form’

$$\cdot : X(A, m) \otimes X(A, n) \longrightarrow X(A, m + n) ,$$

as the composite

$$X(A, m) \otimes X(A, n) \xrightarrow{\times} X(A \times A, m \oplus n) \xrightarrow{\Delta^*} X(A, m + n) ,$$

where $\Delta : A \longrightarrow A \times A$ is the diagonal. For fixed A and varying m , this makes the collection of groups $X(A, m)$ into a commutative I_A -graded \mathbb{F}_2 -algebra. We will often employ the notation $X(A, \star)$ for this I_A -graded \mathbb{F}_2 -algebra. The abelian groups $X(A, k)$ for $k \in \mathbb{Z}$ form a commutative \mathbb{Z} -graded \mathbb{F}_2 -algebra, the *integer graded subring* of this I_A -graded \mathbb{F}_2 -algebra $X(A, \star)$, which we denote by $X(A)_*$. The external multiplication can be recovered from the internal multiplication as the composite

$$X(A, m) \otimes X(B, n) \xrightarrow{p_1^* \otimes p_2^*} X(A \times B, p_1^*(m)) \otimes X(A \times B, p_2^*(n)) \xrightarrow{\cdot} X(A \times B, m \oplus n) .$$

Remark 1.3. The cautious reader might wonder why we consider functors on *isomorphism classes* of representations, and why we abuse notation and use the same symbol for a representation and for its class in $RO(G)$. In other words: why does our theory not take automorphisms of representations into account? The reason is that the automorphisms of real representations, and even all equivariant homotopy self-equivalences of linear spheres, are invisible to the theories we consider. We explain this in detail in Theorem 2.3, but here is already a brief sketch. All theories relevant to us arise from global ring spectra E in which $\text{tr}_1^C(1) = 0$, i.e., the transfer from the trivial group to the two element group C vanishes in $\pi_0^C(E)$. This property has strong consequences for the coefficients of the theory: for every compact Lie group G , the unit ring map $\pi_0^G(\mathbb{S}) \longrightarrow \pi_0^G(E)$ from the G -equivariant 0-stem sends all units to 1. Consequently, all G -equivariant homotopy self-equivalences of spheres of real G -representations induce the identity in the G -homology theory represented by E .

In the following we write $C = \{\pm 1\}$ for the group with two elements. A *character* of a group G is a homomorphism $\lambda : G \longrightarrow C$. We shall consistently confuse such characters with the associated 1-dimensional G -representation on \mathbb{R} with G -action $g \cdot x = \lambda(g) \cdot x$, and with the class of this representation in the representation ring $RO(G)$. We write σ for the sign representation of C on \mathbb{R} , corresponding to the identity character of C .

Definition 1.4 (Orientable el_2^{RO} -algebras). An *orientable* el_2^{RO} -algebra is a pair (X, a) consisting of an el_2^{RO} -algebra X and a class $a \in X(C, -\sigma)$ with the following property: for every elementary abelian 2-group A , every $m \in I_A$ and every nontrivial A -character λ , the sequence

$$0 \longrightarrow X(A, m) \xrightarrow{a_\lambda \cdot -} X(A, m - \lambda) \xrightarrow{\text{res}_K^A} X(K, \text{res}_K^A(m) - 1) \longrightarrow 0$$

is exact, where K is the kernel of λ , and $a_\lambda = \lambda^*(a)$. We refer to the class a as the *pre-Euler class* of the orientable el_2^{RO} -algebra. A *morphism* of orientable el_2^{RO} -algebras is a morphism of el_2^{RO} -algebras that takes the pre-Euler class to the pre-Euler class.

For an orientable el_2^{RO} -algebra (X, a) we already employed the notation $a_\lambda = \lambda^*(a)$ for A -characters λ . Then $a_\lambda = 0$ when λ is the trivial character; and multiplication by a_λ is injective for all nontrivial characters λ . If V is an A -representation, we define

$$a_V = \prod_{\lambda \in \text{Hom}(A, C)} a_\lambda^{m_\lambda} \in X(A, -V),$$

where m_λ is the multiplicity of the character λ in V . Then $a_V = 0$ whenever V has nontrivial A -fixed points; and multiplication by a_V is injective whenever $V^A = 0$. These a -classes are multiplicative and natural for restriction homomorphisms, i.e.,

$$a_V \cdot a_W = a_{V \oplus W} \quad \text{and} \quad \beta^*(a_V) = a_{\beta^*(V)}$$

for all homomorphisms $\beta : B \longrightarrow A$ between elementary abelian 2-groups.

In an orientable el_2^{RO} -algebra, divisibility by the a -classes of general representations can be tested separately on the isotypical summands. The next proposition makes this precise.

Proposition 1.5. *Let (X, a) be an orientable el_2^{RO} -algebra, and let A be an elementary abelian 2-group.*

- (i) *Let λ be a nontrivial A -character with kernel K . Let x and y be homogeneous elements of $X(A, \star)$. Suppose that $\text{res}_K^A(y)$ is a non zero-divisor in $X(K, \star)$. Then x is divisible by a_λ^n if and only if xy is divisible by a_λ^n .*
- (ii) *Let V be an A -representation with trivial A -fixed points. Let z be an homogeneous element of $X(A, \star)$ that is divisible by $a_\lambda^{m_\lambda}$ for every nontrivial character λ of A , where m_λ is the multiplicity of λ in V . Then z is divisible by a_V .*

Proof. (i) If x is divisible by a_λ^n , then clearly so is xy . We prove the other direction by induction on n . The statement is vacuous for $n = 0$. Now let $n \geq 1$. By the induction hypothesis we know that x is divisible by a_λ^{n-1} . Hence there exists an element x' such that $x = x' a_\lambda^{n-1}$. Since $x' y a_\lambda^{n-1} = xy$, and $x' y$ is the unique element with this property, we know that $x' y$ must be divisible by a_λ . Hence the restriction $\text{res}_K^A(x' y) = \text{res}_K^A(x') \text{res}_K^A(y)$ is trivial. Since $\text{res}_K^A(y)$ is a non zero-divisor, this implies that $\text{res}_K^A(x') = 0$. Hence x' is divisible by a_λ , and therefore x is divisible by a_λ^n . This finishes the proof.

(ii) We argue by induction on the number of nontrivial isotypical summands of V . There is nothing to show if $V = 0$. Otherwise we let χ be a nontrivial A -character that occurs in V . Then $V = \chi^{m_\chi} \oplus U$ for an A -representation U with fewer nontrivial isotypical summands. We let $z \in X(A, m)$ be divisible by $a_\lambda^{m_\lambda}$ for all nontrivial A -characters λ . Then we know by induction that z is divisible by a_U , say $z = x \cdot a_U$ for some $x \in X(A, m + U)$. We let K denote the kernel of χ . Since χ does not occur in U , the representation U has trivial K -fixed points. So $\text{res}_K^A(a_U) = a_{U|_K}$ is a non zero-divisor in $X(K, \star)$. Since $x \cdot a_U = z$ is divisible by $a_\chi^{m_\chi}$, part (i) shows that x is divisible by $a_\chi^{m_\chi}$. So $z = x \cdot a_U$ is divisible by $a_\chi^{m_\chi} \cdot a_U = a_V$. \square

Definition 1.6 (Inverse Thom class). An *inverse Thom class* of an orientable el_2^{RO} -algebra (X, a) is a class $t \in X(C, 1 - \sigma)$ such that $\text{res}_1^C(t) = 1$, the multiplicative unit in $X(1, 0)$. An *oriented el_2^{RO} -algebra* is a triple (X, a, t) consisting of an orientable el_2^{RO} -algebra and an inverse Thom class.

Remark 1.7 (The choice of inverse Thom class). If (X, a) is an orientable el_2^{RO} -algebra, then in particular the restriction homomorphism

$$\text{res}_1^C : X(C, 1 - \sigma) \longrightarrow X(1, 0)$$

is surjective. So an inverse Thom class exists. Moreover, any two inverse Thom classes differ by an element of the form $a \cdot y$, for a unique $y \in X(C, 1)$; so the set of inverse Thom classes is a torsor over the additive group $X(C, 1)$. In some important cases, the group $X(C, 1)$ is trivial, so that there is a unique inverse Thom class. This holds for example for equivariant bordism, and for Bredon homology.

The choice of inverse Thom class is essential, and the automorphism group of an orientable el_2^{RO} -algebra need not act transitively on the set of inverse Thom classes. An explicit example is the orientable el_2^{RO} -algebra $(H \wedge H)^\sharp$ arising from the ‘global dual Steenrod algebra’, where $H = H\mathbb{F}_2$ is the global Eilenberg-MacLane spectrum of the constant Mackey functor. As we plan to discuss elsewhere, $(H \wedge H)^\sharp$ has exactly four inverse Thom classes, falling into two distinct orbits under the automorphism group of $(H \wedge H)^\sharp$.

Similarly as for the pre-Euler class, we will use the notation

$$t_\lambda = \lambda^*(t) \in X(A, 1 - \lambda)$$

for an inverse Thom class t and an A -character λ . The inverse Thom class of the trivial character is the multiplicative unit 1. And we extend the inverse Thom classes to general A -representations by defining

$$(1.8) \quad t_V = \prod_{\lambda \in \text{Hom}(A, C)} t_\lambda^{m_\lambda},$$

where m_λ is the multiplicity of the character λ in V . These inverse Thom classes are also multiplicative and natural for restriction, i.e.,

$$t_V \cdot t_W = t_{V \oplus W} \quad \text{and} \quad \beta^*(t_V) = t_{\beta^*(V)}$$

for all homomorphisms $\beta : B \longrightarrow A$ between elementary abelian 2-groups.

Proposition 1.9. *Let (X, a, t) be an oriented el_2^{RO} -algebra.*

(i) *For every subgroup B of A , and for all $m \in I_A$, the restriction homomorphism*

$$\text{res}_B^A : X(A, m) \longrightarrow X(B, \text{res}_K^A(m))$$

is surjective.

(ii) *The I_A -graded algebra $X(A, \star)$ is generated by its \mathbb{Z} -graded subalgebra $X(A)_*$ and the classes a_λ and t_λ for all nontrivial A -characters λ .*

Proof. We prove both statements together by induction over the rank of A . The induction starts when A is the trivial group, in which case there is nothing to show. Now we let A be a nontrivial elementary abelian 2-group, and we assume that (i) and (ii) hold for all proper subgroups of A .

To prove (i) we may assume that B is a proper subgroup of A . We write $m = k - W$ for an A -representation W with $W^A = 0$. By (ii) for B , it suffices to show that all classes of the form $x \cdot a_U \cdot t_V$ are in the image of the restriction homomorphism, whenever U and V are B -representations such that $U \oplus V = W|_B$, and $x \in X(B)_k$. Because W is a sum of

1-dimensional A -representations, we may choose an A -equivariant decomposition $W = \bar{U} \oplus \bar{V}$ such that $\bar{U}|_B \cong U$ and $\bar{V}|_B \cong V$. We also choose a homomorphism $r : A \rightarrow B$ that is a retraction to the inclusion. Then

$$\text{res}_B^A(r^*(x) \cdot a_{\bar{U}} \cdot t_{\bar{V}}) = x \cdot a_U \cdot t_V ,$$

and we have shown (i) for A .

For (ii) we also fix a representation grading $m = k - W$, with $W^A = 0$. We prove the claim for all classes in $X(A, k - W)$ by induction over the dimension of W . There is nothing to show if $W = 0$. Otherwise we write $W = V \oplus \lambda$ for an A -representation V and a nontrivial A -character λ , with kernel K . By part (i), the restriction map $\text{res}_K^A : X(A, k - 1 - V) \rightarrow X(K, k - 1 - V|_K)$ is surjective. So given $x \in X(A, k - W)$, there is a class $z \in X(A, k - 1 - V)$ such that $\text{res}_K^A(z) = \text{res}_K^A(x)$. Then

$$\text{res}_K^A(x + t_\lambda \cdot z) = \text{res}_K^A(x) + \text{res}_K^A(z) = 0 .$$

The fundamental exactness property of an oriented el_2^{RO} -algebra provides a class $y \in X(A, k - V)$ such that $a_\lambda \cdot y = x + t_\lambda \cdot z$. By the inductive hypothesis, the classes y and z are $X(A)_*$ -linear combinations of products of a -classes and t -classes. Hence the same is true for $x = y \cdot a_\lambda + z \cdot t_\lambda$, and we have shown (ii). \square

We close this section with an easy but useful fact: a morphism of orientable el_2^{RO} -algebras is already an isomorphism if its restriction to all the integer graded subrings is.

Proposition 1.10. *Let $f : X \rightarrow Y$ be a morphism of orientable el_2^{RO} -algebras. If all the homomorphisms of \mathbb{Z} -graded rings $f(A)_* : X(A)_* \rightarrow Y(A)_*$ are bijective, then f is an isomorphism of el_2^{RO} -algebras.*

Proof. We show that for all elementary abelian 2-groups A , all $k \in \mathbb{Z}$ and all A -representations W with $W^A = 0$, the map $f(A, k - W) : X(A, k - W) \rightarrow Y(A, k - W)$ is an isomorphism. We argue by induction on the dimension of W . For $W = 0$ the claim is true by hypothesis. If W is nonzero, we choose a nontrivial A -character λ such that $W = V \oplus \lambda$. Then the outer two maps in the following commutative diagram are isomorphisms by induction, because V has smaller dimension than W :

$$\begin{array}{ccccccc} 0 & \longrightarrow & X(A, k - V) & \xrightarrow{a_\lambda \cdot -} & X(A, k - W) & \xrightarrow{\text{res}_K^A} & X(K, k - 1 - V|_K) \longrightarrow 0 \\ & & \downarrow f(A, k - V) & & \downarrow f(A, k - W) & & \downarrow f(K, k - 1 - V|_K) \\ 0 & \longrightarrow & Y(A, k - V) & \xrightarrow{a_\lambda \cdot -} & Y(A, k - W) & \xrightarrow{\text{res}_K^A} & Y(K, k - 1 - V|_K) \longrightarrow 0 \end{array}$$

Since both rows are exact, the middle map is an isomorphism. \square

2. ORIENTED GLOBAL RING SPECTRA

In this section we explain how orientable global ring spectra give rise to orientable el_2^{RO} -algebras, see Construction 2.4 and Theorem 2.5, leading to a wealth of interesting examples. For the purpose of this paper, a *global ring spectrum* is a commutative monoid object in the global stable homotopy category of [25, Section 4.4], under the globally derived smash product [25, (4.2.25)]. So informally speaking, a global ring spectrum is a consistent collection of G -equivariant homotopy ring spectra, for every compact Lie group G .

In this paper, we mostly care about the underlying G -spectra of a global spectrum when G is an elementary abelian 2-group. So we could have worked in the el_2 -global stable homotopy category, the Bousfield localization of the global stable homotopy category that is designed so

that it only ‘sees’ the values of a global spectrum at the global family el_2 of elementary abelian 2-groups. However, since every el_2 -global ring spectrum is underlying a global ring spectrum (in many different ways), working with global ring spectra is no loss of generality. Moreover, some interesting results, specifically Proposition 2.1 and Theorem 2.3 below, refer to general compact Lie groups.

We recall that a *global functor* in the sense of [25, Definition 4.2.2] is an additive functor from the global Burnside category of [25, Construction 4.2.1] to the category of abelian groups. In more explicit terms, a global functor specifies values on all compact Lie groups, restriction homomorphisms along continuous group homomorphisms, and transfers along inclusions of closed subgroups; these data have to satisfy a short list of explicit relations that can be found after Theorem 4.2.6 of [25]. The primary invariant of a global spectrum E is the homotopy group global functor $\underline{\pi}_0(E) = \{\pi_0^G(E)\}_G$ that records the 0-th equivariant homotopy groups for all compact Lie groups, along with the restriction and transfer maps that relate them, see [25, Example 4.2.3]. One of the transfer homomorphisms plays a particularly important role for this paper, namely the transfer

$$\text{tr}_1^C : \pi_0(E) \longrightarrow \pi_0^C(E)$$

from the homotopy group of the underlying non-equivariant spectrum to the C -equivariant homotopy group.

In the following proposition, \mathbb{S} denotes the global sphere spectrum, so that $\underline{\pi}_0(\mathbb{S})$ is the Burnside ring global functor, compare [25, Example 4.2.7]. The value of $\underline{\pi}_0(\mathbb{S})$ at a compact Lie group G is the G -equivariant stable stem $\pi_0^G(\mathbb{S}) = \text{colim}[S^V, S^V]_*^G$, where the colimit is over finite-dimensional G -subrepresentations of a complete G -universe. When G is finite, $\pi_0^G(\mathbb{S})$ is isomorphic to the Grothendieck ring of isomorphism classes of finite G -sets.

For every closed subgroup H of G , the ring homomorphism

$$\Phi^H : \pi_0^G(\mathbb{S}) \longrightarrow \mathbb{Z}$$

sends the class of a based continuous G -map $f : S^V \longrightarrow S^V$ to the degree of the restriction $f^H : S^{V^H} \longrightarrow S^{V^H}$ to H -fixed points. These homomorphisms are jointly injective, i.e., classes in $\pi_0^G(\mathbb{S})$ are detected by the collection of fixed point degrees, see for example [25, Theorem 3.3.15].

Proposition 2.1. *Let $\langle \text{tr}_1^C \rangle$ denote the global subfunctor of $\underline{\pi}_0(\mathbb{S})$ generated by the transfer $\text{tr}_1^C(1)$ in $\pi_0^C(\mathbb{S})$. Let G be a compact Lie group.*

- (i) *A class $x \in \pi_0^G(\mathbb{S})$ belongs to $\langle \text{tr}_1^C \rangle(G)$ if and only if for every closed subgroup H of G with finite Weyl group, the integer $\Phi^H(x)$ is even.*
- (ii) *For every unit $u \in \pi_0^G(\mathbb{S})$, the class $u - 1$ belongs to $\langle \text{tr}_1^C \rangle(G)$.*

Proof. (i) As we show in the proof of [25, Proposition 6.1.45], the value $\langle \text{tr}_1^C \rangle(G)$ is the subgroup of $\pi_0^G(\mathbb{S})$ generated by $2 \cdot \pi_0^G(\mathbb{S})$ and by the transfers $t_H^G = \text{tr}_H^G(1_H)$ for all closed subgroups H of G whose Weyl group is finite and of even order. The degree homomorphism $\Phi^K : \pi_0^G(\mathbb{S}) \longrightarrow \mathbb{Z}$ is additive, so it takes even values on all classes in $2 \cdot \pi_0^G(\mathbb{S})$. If the Weyl group of H in G is finite of even order, then there is a closed subgroup L of G that contains H as a normal subgroup of index 2. Then

$$\Phi^K(t_H^G) = \Phi^K(\text{res}_K^G(\text{tr}_L^G(\text{tr}_H^L(1_H)))) .$$

The Weyl group $W_G H$ acts freely from the right on the coset space G/H , and hence it acts freely and smoothly on the smooth closed manifold $(G/H)^K$. By [25, Proposition 3.4.2 (ii)], the integer $\Phi^K(t_H^G)$ is divisible by the order of $W_G H$, and hence even. We have thus shown that every class in $\langle \text{tr}_1^C \rangle(G)$ has even fixed point degrees.

For the converse we consider a class with even fixed point degrees. By the previous paragraph, any such class is congruent modulo $\langle \mathrm{tr}_1^C \rangle(G)$ to a class of the form

$$t_{K_1}^G + \cdots + t_{K_n}^G$$

for pairwise non-conjugate closed subgroups K_1, \dots, K_n of G whose Weyl groups are finite and of odd order. If the above sum had at least one summand, we assume without loss of generality that K_1 is maximal with respect to subconjugacy among the groups that occur, i.e., K_1 is not subconjugate to K_i for $i = 2, \dots, n$. Then for $i = 2, \dots, n$, the fixed point set $(G/K_i)^{K_1}$ is empty, and thus $\Phi^{K_1}(t_{K_i}^G) = 0$. Hence

$$\Phi^{K_1}(t_{K_1}^G + \cdots + t_{K_n}^G) = \Phi^{K_1}(t_{K_1}^G) = |W_G K_1|.$$

This contradicts the assumption that all fixed point degrees are even. So the above sum must be empty, i.e., the given class belongs to $\langle \mathrm{tr}_1^C \rangle(G)$.

(ii) The degree homomorphism $\Phi^H : \pi_0^G(\mathbb{S}) \rightarrow \mathbb{Z}$ is multiplicative, so it sends all units of $\pi_0^G(\mathbb{S})$ to $\{\pm 1\}$. Hence $\Phi^H(u-1)$ is even, and part (i) proves the claim. \square

Now we introduce orientable global ring spectra, our main source of examples for orientable el_2^{RO} -algebras.

Definition 2.2. A global ring spectrum E is *orientable* if the class $\mathrm{tr}_1^C(1)$ in the group $E_0^C = \pi_0^C(E)$ is trivial.

Some important examples of orientable global ring spectra are the mod 2 global Eilenberg-MacLane spectrum $H\mathbb{F}_2$, see Example 3.2, and the global Thom spectrum \mathbf{mO} defined in [25, Example 6.1.24]. These are specific global equivariant forms of the mod 2 Eilenberg-MacLane spectrum and the classical real bordism spectrum. The global ring spectrum $H\mathbb{F}_2$ represents Bredon homology with constant coefficients, and the global ring spectrum \mathbf{mO} realizes the orientable el_2^{RO} -algebra \mathcal{N} of equivariant bordism. We return to these examples in much detail in Sections 3 and 6, respectively. Further examples that we discuss below are the global Thom ring spectrum \mathbf{MO} defined in [25, Example 6.1.7] that represents stable equivariant bordism, see Corollary 8.6 and the remarks immediately thereafter; and the global Borel ring spectrum associated to the mod 2 Eilenberg-MacLane spectrum, see Theorem 8.8. For these four examples, we establish universal characterizations of the associated oriented el_2^{RO} -algebra, see Theorem 3.5, Theorem 7.14, Corollary 8.6 and Theorem 8.8, respectively. The global smash product of an orientable global ring spectrum with any other global ring spectrum is again orientable, so this yields many further examples.

The next theorem shows that orientability of a global ring spectrum has rather strong consequences for the homotopy group global functor.

Theorem 2.3. *Let E be an orientable global ring spectrum, and let G be a compact Lie group.*

- (i) *All G -equivariant homotopy groups of E are \mathbb{F}_2 -vector spaces.*
- (ii) *The ring homomorphism $\eta_* : \pi_0^G(\mathbb{S}) \rightarrow \pi_0^G(E) = E_0^G$ induced by the unit morphism $\eta : \mathbb{S} \rightarrow E$ sends all units of $\pi_0^G(\mathbb{S})$ to 1.*
- (iii) *For every G -representation V and every G -equivariant based homotopy self-equivalence $\psi : S^V \rightarrow S^V$, the automorphism $E_*^G(S^\psi)$ of $E_*^G(S^V)$ is the identity.*

Proof. (i) The vanishing of the class $\mathrm{tr}_1^C(1)$ yields

$$2 = \mathrm{res}_1^C(\mathrm{tr}_1^C(1)) = 0$$

in $\pi_0(E)$. Since all equivariant homotopy groups of E are modules over $\pi_0(E)$ via inflation, they are all \mathbb{F}_2 -vector spaces.

(ii) As G varies over all compact Lie groups, the homomorphisms $\eta_* : \pi_0^G(\mathbb{S}) \longrightarrow \pi_0^G(E)$ form a morphism of global functors $\eta_* : \underline{\pi}_0(\mathbb{S}) \longrightarrow \underline{\pi}_0(E)$. Because $\mathrm{tr}_1^C(1) = 0$ in $\pi_0^C(E)$, the global subfunctor $\langle \mathrm{tr}_1^C \rangle$ of $\underline{\pi}_0(\mathbb{S})$ maps to 0 under the morphism η_* . Hence by Proposition 2.1 (ii), the homomorphism η_* sends all units in $\pi_0^G(\mathbb{S})$ to 1.

(iii) The automorphism $E_*^G(S^\psi)$ of $E_*^G(S^V)$ coincides with multiplication by the image of $[\psi] \in \pi_0^G(\mathbb{S})$ under the ring homomorphism $\eta_* : \pi_0^G(\mathbb{S}) \longrightarrow \pi_0^G(E)$. Since ψ is a homotopy self-equivalence, the class $[\psi]$ is a unit in $\pi_0^G(\mathbb{S})$, so part (ii) proves the claim. \square

Construction 2.4 (The el_2^{RO} -algebra of an orientable global ring spectrum). We let E be an orientable global ring spectrum. We will now define an associated el_2^{RO} -algebra E^\sharp . We write any given $m \in I_A$ as $m = k - V$ for an integer k and an A -representation V with trivial fixed points. Then k is unique, and V is unique up to isomorphism. We set

$$E^\sharp(A, m) = E^\sharp(A, k - V) = E_k^A(S^V) .$$

A key point is that this assignment is independent up to preferred isomorphism of the choice of V . Indeed, an isomorphism of A -representations $\psi : V \longrightarrow W$ induces an equivariant homeomorphism $S^\psi : S^V \longrightarrow S^W$, and hence an isomorphism of equivariant homotopy groups

$$E_k^A(S^\psi) : E_k^A(S^V) \longrightarrow E_k^A(S^W) .$$

Theorem 2.3 (iii) guarantees that this isomorphism does not depend on the choice of ψ .

The restriction map along a morphism $\alpha : (B, \alpha^*(m)) \longrightarrow (A, m)$ in el_2^{RO} is defined as follows. We choose an A -representation V and a B -representation W without fixed points so that $m = k - V$ and $\alpha^*(m) = l - W$ for integers $k \geq l$. There is then an isomorphism of A -representations

$$\psi : \alpha^*(V) \xrightarrow{\cong} W \oplus \mathbb{R}^{k-l} .$$

We define

$$\alpha^* : E^\sharp(A, m) \longrightarrow E^\sharp(B, \alpha^*(m))$$

as the composite

$$E_k^A(S^V) \xrightarrow{\alpha^*} E_k^B(S^{\alpha^*(V)}) \xrightarrow{E_k^B(S^\psi)} E_k^B(S^{W \oplus \mathbb{R}^{k-l}}) \xleftarrow[\cong]{-\wedge S^{k-l}} E_l^B(S^W) .$$

The first map is the restriction homomorphism of equivariant homotopy groups [25, Construction 3.1.15], and the final isomorphism pointing backwards is the suspension isomorphism. Theorem 2.3 (iii) guarantees that α^* does not depend on the choice of ψ . The multiplication pairings are defined as the composite

$$\begin{aligned} E^\sharp(A, m) \times E^\sharp(B, n) &= E_k^A(S^V) \times E_l^B(S^W) \\ &\xrightarrow{\wedge} \pi_{k+l}^{A \times B}(E \wedge S^V \wedge E \wedge S^W) \\ &\xrightarrow{\mu_*} \pi_{k+l}^{A \times B}(E \wedge S^{V \oplus W}) = E^\sharp(A \times B, m + n) . \end{aligned}$$

Here we have exploited that $m + n = (k + l) - (V \oplus W)$ in $I_{A \times B}$.

The fixed point inclusion $S^0 \longrightarrow S^\sigma$ defines an element $a \in \pi_0^C(\mathbb{S} \wedge S^\sigma)$ that is sometimes called the *(pre-)Euler class* of the sign representation; we abuse notation and use the same symbol for the image

$$a \in E_0^C(S^\sigma) = E^\sharp(C, -\sigma)$$

under the unit morphism $\mathbb{S} \longrightarrow E$.

Theorem 2.5. *For every orientable global ring spectrum E , the pair (E^\sharp, a) is an orientable el_2^{RO} -algebra.*

Proof. We exploit that the sequence of equivariant homotopy groups

$$E_1^C(S^\sigma) \xrightarrow{\text{res}_e^C} E_1(S^1) \xrightarrow{\text{tr}_1^C} E_1^C(S^1)$$

is exact. Because E is orientable, we have $\text{tr}_1^C(1 \wedge S^1) = \text{tr}_1^C(1) \wedge S^1 = 0$, and we can choose a class $t \in E_1^C(S^\sigma)$ that restricts to $1 \wedge S^1$. We use t to define classes $t_V \in E_d^A(S^V)$ as in (1.8), for all representations V of elementary abelian 2-groups A , where $d = \dim(V)$. We claim that the following three properties hold for all A -representations W :

- (i) For every subgroup B of A , the restriction homomorphism

$$\text{res}_B^A : E_k^A(S^W) \longrightarrow E_k^B(S^{W|_B})$$

is surjective.

- (ii) Suppose that $W = V \oplus \lambda$ for a nontrivial A -character λ with kernel K . Then the following sequence is exact:

$$0 \longrightarrow E_k^A(S^V) \xrightarrow{a_\lambda} E_k^A(S^{V \oplus \lambda}) \xrightarrow{\text{res}_K^A} E_{k-1}^K(S^{V|_K}) \longrightarrow 0$$

- (iii) The graded \mathbb{F}_2 -vector space $E_*^A(S^W)$ is generated as a graded E_*^A -module by the classes $a_U \cdot t_V$ for all A -representations U and V such that $U \oplus V = W$.

Part (ii) is the exactness property that shows that (E^\sharp, a) is an orientable el_2^{RO} -algebra.

The argument is similar as in the proof of Proposition 1.9: we prove all three statements together by induction over the rank of A . The induction starts when A is the trivial group, in which case there is nothing to show. Now we let A be a nontrivial elementary abelian 2-group, and we assume that parts (i)–(iii) hold for all proper subgroups of A .

We start by proving (i), where we may assume that B is a proper subgroup of A . By part (iii) for B , it suffices to show that all classes of the form $x \cdot a_U \cdot t_V$ are in the image of the restriction homomorphism, whenever U and V are B -representations such that $U \oplus V = W|_B$, and x is a homogeneous element of E_*^B . Because W is a sum of 1-dimensional A -representations, we may choose an A -equivariant decomposition $W = \bar{U} \oplus \bar{V}$ such that $\text{res}_B^A(\bar{U}) \cong U$ and $\text{res}_B^A(\bar{V}) \cong V$. We also choose a homomorphism $r : A \longrightarrow B$ that is a retraction to the inclusion. Then

$$\text{res}_B^A(r^*(x) \cdot a_{\bar{U}} \cdot t_{\bar{V}}) = x \cdot a_U \cdot t_V,$$

and we have shown (i) for A . To prove (ii) we smash the cofiber sequence of based A -spaces

$$A/K_+ \longrightarrow S^0 \longrightarrow S^\lambda \longrightarrow A/K_+ \wedge S^1$$

with S^V and apply A -equivariant E -homology, yielding a long exact sequence:

$$\dots \longrightarrow E_k^A(S^V) \xrightarrow{a_\lambda} E_k^A(S^{V \oplus \lambda}) \xrightarrow{\partial} E_{k-1}^A(S^V \wedge A/K_+) \longrightarrow \dots$$

The Wirthmüller isomorphism identifies the group $E_{k-1}^A(S^V \wedge A/K_+)$ with $E_{k-1}^K(S^{V|_K})$. Under this identification, the boundary map ∂ becomes the restriction homomorphism $\text{res}_K^A : E_k^A(S^{V \oplus \lambda}) \longrightarrow E_{k-1}^K(S^{V|_K})$, which is surjective by (i). So the long exact sequence decomposes into short exact sequences, showing (ii).

We prove (iii) by induction on $\dim(W) - \dim(W^A)$. If A acts trivially on W , the E_*^A -module $E_*^A(S^W)$ is free of rank 1 with basis the class t_W , so the claim holds. If W acts nontrivially on W , we write $W = V \oplus \lambda$ for an A -representation V and a nontrivial A -character λ , with kernel K . By part (i), the restriction map $\text{res}_K^A : E_{m-1}^A(S^V) \longrightarrow E_{m-1}^K(S^{V|_K})$ is surjective. So given $x \in E_k^A(S^W)$, there is a class $z \in E_{m-1}^A(S^V)$ such that $\text{res}_K^A(z) = \text{res}_K^A(x)$. Then

$$\text{res}_K^A(x + z \cdot t_\lambda) = \text{res}_K^A(x) + \text{res}_K^A(z) = 0.$$

Part (ii) provides a class $y \in E_k^A(S^V)$ such that $y \cdot a_\lambda = x + z \cdot t_\lambda$. By the inductive hypothesis, the classes y and z are E_*^A -linear combinations of products of a -classes and t -classes. Hence the same is true for $x = y \cdot a_\lambda + z \cdot t_\lambda$, and we have shown (iii). \square

3. THE UNIVERSAL PROPERTY OF BREDON HOMOLOGY

In this section we discuss *Bredon el_2^{RO} -algebras*, i.e., orientable el_2^{RO} -algebras whose \mathbb{Z} -graded parts consist only of a copy of \mathbb{F}_2 in dimension zero, see Definition 3.1. As the name suggests, Bredon homology of representation spheres with constant mod 2 coefficients is an example of a Bredon el_2^{RO} -algebra. Our main result, Theorem 3.5, exhibits a universal property: every Bredon el_2^{RO} -algebra is initial among *additive* oriented el_2^{RO} -algebras. It follows that Bredon el_2^{RO} -algebras are unique up to unique isomorphism, a fact that is not entirely obvious, at least to the authors, from the definition.

Definition 3.1. A *Bredon el_2^{RO} -algebra* is an orientable el_2^{RO} -algebra H such that for every elementary abelian 2-group A and all $k \in \mathbb{Z}$ we have

$$H(A)_k = H(A, k) = \begin{cases} \mathbb{F}_2 & \text{for } k = 0, \text{ and} \\ 0 & \text{for } k \neq 0. \end{cases}$$

As the following example shows, Bredon el_2^{RO} -algebras exist. Theorem 3.5 below proves that Bredon el_2^{RO} -algebras are unique up to unique isomorphism.

For every el_2^{RO} -algebra X , any two choices of pre-Euler classes differ by multiplication with a unit in $X(C, 0)$. Furthermore, as we discussed in Remark 1.7, the difference between two choices of inverse Thom classes is an element of the form $a \cdot y$, for $y \in X(C, 1)$. If H is a Bredon el_2^{RO} -algebra, then $H(C, 0)^\times = \mathbb{F}_2^\times = \{1\}$ and $H(C, 1) = 0$. Therefore, the pre-Euler class and the inverse Thom class of a Bredon el_2^{RO} -algebra are unique.

Example 3.2 (The global Eilenberg-MacLane spectrum). We let $H\mathbb{F}_2$ denote the classical commutative orthogonal ring spectrum model for the Eilenberg-MacLane spectrum for the field \mathbb{F}_2 , as defined for example in [25, Construction 5.3.8]. By [25, Proposition 5.3.9], its restriction to finite groups is an Eilenberg-MacLane ring spectrum for the constant global functor with value \mathbb{F}_2 . So $H\mathbb{F}_2$ represents Bredon homology and cohomology with coefficients in the constant Mackey functor \mathbb{F}_2 .

Because the transfer $\text{tr}_1^C(1)$ is trivial in Bredon homology with constant mod 2 coefficients, $H\mathbb{F}_2$ is an orientable global ring spectrum. By Theorem 2.5, the pair $((H\mathbb{F}_2)^\sharp, a)$ is an orientable el_2^{RO} -algebra, and hence a Bredon el_2^{RO} -algebra in the sense of Definition 3.1.

Definition 3.3 (Additive oriented el_2^{RO} -algebras). An oriented el_2^{RO} -algebra (X, a, t) is *additive* if the class

$$a_1 t_\mu t_2 + t_1 a_\mu t_2 + t_1 t_\mu a_2$$

in $X(C \times C, 2 - (p_1 \oplus \mu \oplus p_2))$ is trivial. Here $p_1, p_2, \mu : C \times C \rightarrow C$ are the three nontrivial characters of the group $C \times C$, and we abbreviate a_{p_i} to a_i , and t_{p_i} to t_i .

The term ‘additive’ in the previous definition is motivated by the fact that the 2-torsion formal group law associated to an additive el_2^{RO} -algebra is the additive formal group law, see Example 4.8 below.

Example 3.4 (Bredon el_2^{RO} -algebras are additive). We let (X, a, t) be an oriented el_2^{RO} -algebra. The class $a_1 t_\mu t_2 + t_1 a_\mu t_2 + t_1 t_\mu a_2$ in $X(C \times C, 2 - (p_1 \oplus \mu \oplus p_2))$ is in the kernel of all three restriction maps from $C \times C$ to C . By the fundamental exactness property, this class

is divisible by each of a_1 , a_μ and a_2 . Proposition 1.5 (ii) then shows that the class is divisible by the product $a_1 a_\mu a_2$. So there is a class $z \in X(C \times C)_2$ such that

$$z \cdot a_1 a_\mu a_2 = a_1 t_\mu t_2 + t_1 a_\mu t_2 + t_1 t_\mu a_2 .$$

If the group $X(C \times C)_2$ is trivial, then the class z necessarily vanishes, and so (X, a, t) is additive. This is the case in particular for all Bredon el_2^{RO} -algebras.

The following main result of this section is the fact that every Bredon el_2^{RO} -algebra is initial among additive el_2^{RO} -algebras. In the proof we will start using the notation

$$A^\circ = \{ \lambda : A \longrightarrow C \mid \lambda \text{ is surjective} \}$$

for the set of nontrivial characters of an elementary abelian 2-group A .

Theorem 3.5 (Universal property of Bredon homology). *Let H be a Bredon el_2^{RO} -algebra, and let X be an orientable el_2^{RO} -algebra.*

- (i) *Evaluation at the unique inverse Thom class of H is a bijection between the set of morphisms of orientable el_2^{RO} -algebras from H to X , and the set of those inverse Thom classes t of X such that (X, a, t) is additive.*
- (ii) *Any two Bredon el_2^{RO} -algebras are uniquely isomorphic.*

Proof. (i) The unique orientation and inverse Thom class make (H, a, t) additive, so every morphism of orientable el_2^{RO} -algebras sends t to an inverse Thom class of an additive orientation.

Now we prove the uniqueness. Proposition 1.9 (ii) shows that $H(A, \star)$ is generated as an I_A -graded \mathbb{F}_2 -algebra by the classes a_λ and t_λ for all nontrivial A -characters, where t is the unique inverse Thom class; so every morphism $f : H \longrightarrow X$ of oriented el_2^{RO} -algebras is determined by its effect on the pre-Euler class a and the inverse Thom class t .

Now we show the existence of the morphism for the specific example $H^\# = (H\mathbb{F}_2)^\#$, the Bredon el_2^{RO} -algebra given by the Bredon homology of representation spheres, see Example 3.2. We will exploit the presentation of the rings $H^\#(A, \star)$ obtained in [18, Theorem 2.5]. We let (\bar{a}, \bar{t}) be an additive orientation of X . We let $T \subset A^\circ$ be any set of nontrivial A -characters whose product $\prod_{\lambda \in T} \lambda$ is the trivial character. We claim that the class

$$r(T) = \sum_{\lambda \in T} \bar{a}_\lambda \cdot \left(\prod_{\mu \in T \setminus \{\lambda\}} \bar{t}_\mu \right)$$

in $X(A, |T| - 1 - \bigoplus_{\lambda \in T} \lambda)$ equals 0. We prove this by induction over the cardinality of T . A non-empty subset of A° whose elements multiply to 1 must have at least 3 elements, so we start the induction for $T = \{\alpha, \beta, \gamma\}$ with $\alpha \cdot \beta \cdot \gamma = 1$. Then

$$\begin{aligned} r(\{\alpha, \beta, \gamma\}) &= \bar{a}_\alpha \bar{t}_\beta \bar{t}_\gamma + \bar{t}_\alpha \bar{a}_\beta \bar{t}_\gamma + \bar{t}_\alpha \bar{t}_\beta \bar{a}_\gamma \\ &= (\alpha, \gamma)^* (\bar{a}_1 \bar{t}_\mu \bar{t}_2 + \bar{t}_1 \bar{a}_\mu \bar{t}_2 + \bar{t}_1 \bar{t}_\mu \bar{a}_2) = 0 \end{aligned}$$

by the additivity hypothesis, where $(\alpha, \gamma) : A \longrightarrow C \times C$.

Now we suppose that the set T has at least 4 elements. We pick two distinct elements $\alpha \neq \beta$ from T , and we set $\gamma = \alpha \cdot \beta$. If $\gamma \in T$, then the sets $T \setminus \{\alpha, \beta, \gamma\}$ and $\{\alpha, \beta, \gamma\}$ both have fewer elements than T , and for both the product of their elements is the trivial character. So $r(T \setminus \{\alpha, \beta, \gamma\}) = 0$ and $r(\{\alpha, \beta, \gamma\}) = 0$ by induction. Thus

$$r(T) = \bar{t}_\alpha \bar{t}_\beta \bar{t}_\gamma \cdot r(T \setminus \{\alpha, \beta, \gamma\}) + r(\{\alpha, \beta, \gamma\}) \cdot \prod_{\mu \in T \setminus \{\alpha, \beta, \gamma\}} \bar{t}_\mu = 0 .$$

If $\gamma \notin T$, then the sets $(T \setminus \{\alpha, \beta\}) \cup \{\gamma\}$ and $\{\alpha, \beta, \gamma\}$ both have fewer elements than T , and for both the product of their elements is the trivial character. So $r((T \setminus \{\alpha, \beta\}) \cup \{\gamma\}) = 0$ and $r(\{\alpha, \beta, \gamma\}) = 0$ by induction. Thus

$$\bar{a}_\gamma \cdot r(T) = (\bar{a}_\alpha \bar{t}_\beta + \bar{t}_\alpha \bar{a}_\beta) \cdot r((T \setminus \{\alpha, \beta\}) \cup \{\gamma\}) + r(\{\alpha, \beta, \gamma\}) \cdot r(T \setminus \{\alpha, \beta\}) = 0.$$

Since multiplication by \bar{a}_γ is injective, we conclude that $r(T) = 0$. This completes the inductive step, and hence the proof of the claim.

By [18, Theorem 2.5], the I_A -graded ring $H^\sharp(A, \star)$ is generated by the classes a_λ and t_λ for all nontrivial A -characters λ , and the ideal of relations between these classes is generated by the polynomials $r(T)$ as T runs through all minimally dependent subsets of A° . Minimally dependent sets of nontrivial characters in particular have the property that their elements multiply to the trivial character. By the previous claim, all generating relations map to 0 in $X(A, \star)$. So there is a unique morphism of I_A -graded \mathbb{F}_2 -algebras

$$f(A, \star) : H^\sharp(A, \star) \longrightarrow X(A, \star)$$

such that $f(A, -\lambda)(a_\lambda) = \bar{a}_\lambda$ and $f(A, 1 - \lambda)(t_\lambda) = \bar{t}_\lambda$ for all $\lambda \in A^\circ$. For varying A , these homomorphisms are compatible with restriction along group homomorphisms, simply because this holds for a - and t -classes and these generate $H^\sharp(A, \star)$ as an \mathbb{F}_2 -algebra. We have thus constructed the desired morphism $f : H^\sharp \longrightarrow X$.

(ii) We let H be any Bredon el_2^{RO} -algebra. Then H has a unique orientation, which is moreover additive. So part (i) provides a unique morphism of orientable el_2^{RO} -algebras $f : H^\sharp \longrightarrow H$ that matches the orientations. This morphism is necessarily an isomorphism in integer degrees, and hence an isomorphism of orientable el_2^{RO} -algebras, see Proposition 1.10. \square

4. THE ASSOCIATED 2-TORSION FORMAL GROUP LAW

As we shall now explain, the underlying \mathbb{F}_2 -algebra of every oriented el_2^{RO} -algebra (X, a, t) naturally comes with a 2-torsion formal group law, obtained roughly speaking by expanding the Euler class e_μ of the multiplication $\mu : C \times C \longrightarrow C$ in terms of the Euler classes of the two projections. As we will see later, a lot of the structure of (X, a, t) can be recovered from this underlying 2-torsion formal group law and the geometric fixed point rings of X .

Construction 4.1 (Localization away from t). We define the localization away from t of an oriented el_2^{RO} -algebra (X, a, t) by inverting all inverse Thom classes. For every elementary abelian 2-group A , we let $(t^{-1}X)(A)$ be the integer graded part of the localization of the I_A -graded ring $X(A, \star)$ obtained by inverting all classes of the form t_λ for all nontrivial A -characters λ . Homogeneous elements of degree k of $(t^{-1}X)(A)$ are thus fractions of the form x/t_V for some A -representation V , and some $x \in X(A, k + |V| - V)$. These fractions then satisfy

$$x/t_V = (x \cdot t_W)/t_{V \oplus W} \quad \text{and} \quad x/t_V \cdot y/t_W = (x \cdot y)/t_{V \oplus W}.$$

The localizations away from t come with restriction homomorphisms $\alpha^* : (t^{-1}X)(A) \longrightarrow (t^{-1}X)(B)$ that are contravariantly functorial for group homomorphisms $\alpha : B \longrightarrow A$, and defined by the formula

$$\alpha^*(x/t_V) = \alpha^*(x)/t_{\alpha^*(V)}.$$

Definition 4.2 (Euler classes). Let (X, a, t) be an oriented el_2^{RO} -algebra. The *Euler class* of a nontrivial character λ of an elementary abelian 2-group A is the class

$$e_\lambda = a_\lambda/t_\lambda = \lambda^*(a/t) \in (t^{-1}X)(A)_{-1}$$

of degree -1 in the localization away from t of X .

For $i = 1, \dots, n$, we let $p_i : C^n \rightarrow C$ be the projection to the i -th factor. We write

$$e_i = e_{p_i} = p_i^*(a/t) \in (t^{-1}X)(C^n)_{-1}$$

for the Euler class of p_i .

Proposition 4.3. *Let (X, a, t) be an oriented el_2^{RO} -algebra and $n \geq 0$. The Euler classes e_1, \dots, e_n form a regular sequence in the graded ring $(t^{-1}X)(C^n)$ that generates the kernel of the restriction homomorphism*

$$\text{res}_1^{C^n} : (t^{-1}X)(C^n) \rightarrow X(1).$$

Proof. We argue by induction over n , beginning with $n = 0$ where there is nothing to show. Now we assume $n \geq 1$. Because localization is exact, the sequence

$$0 \rightarrow (t^{-1}X)(C^n)_{*+1} \xrightarrow{e_1 \cdot -} (t^{-1}X)(C^n)_* \xrightarrow{j^*} (t^{-1}X)(C^{n-1})_* \rightarrow 0$$

is exact, where $j = (1, -) : C^{n-1} \rightarrow C^n$ is the embedding as the last $n-1$ factors. So e_1 is a non zero-divisor, and the restriction homomorphism j^* identifies $(t^{-1}X)(C^n)/(e_1)$ with $(t^{-1}X)(C^{n-1})$. By induction hypothesis, the images of the classes e_1, \dots, e_{n-1} are a regular sequence in $(t^{-1}X)(C^{n-1})$ that generates the augmentation ideal. Since $j^*(e_i) = e_{i-1}$ for $2 \leq i \leq n$, this completes the inductive step. \square

A key tool in our arguments will involve expansions of elements of certain rings as power series in a distinguished regular element. We will now set up some theory around this in a systematic fashion. In the following we will use the term *graded ring* as short hand for ‘commutative \mathbb{Z} -graded ring’. The next proposition is well-known; since it is central for various of our arguments, we take the time to spell it out as a point of reference.

Proposition 4.4. *Let (e_1, \dots, e_n) be a regular sequence of homogeneous elements of a graded ring R . Let $s : S \rightarrow R$ be a graded ring homomorphism whose composite with the projection $R \rightarrow R/(e_1, \dots, e_n)$ is an isomorphism.*

- (i) *The completion of R at the ideal (e_1, \dots, e_n) is a power series algebra over S in the elements e_1, \dots, e_n .*
- (ii) *Let T be a graded ring that is complete with respect to a homogeneous ideal J . Then the map*

$$\begin{aligned} \{\rho \in \text{grRing}(R, T) : \rho(e_1, \dots, e_n) \subset J\} &\rightarrow \text{grRing}(S, T) \times J^n \\ \rho &\mapsto (\rho s, \rho(e_1), \dots, \rho(e_n)) \end{aligned}$$

is bijective.

Proof. We write $I = (e_1, \dots, e_n)$ for the ideal generated by the regular sequence. It is a classical fact that because of the regularity, the unique R/I -algebra homomorphism

$$(R/I)[x_1, \dots, x_n] \rightarrow \bigoplus_{k \geq 0} I^k / I^{k+1}$$

that sends x_i to $e_i + I^2$ is an isomorphism, see for example [8, §9.7, Théorème 1]. In particular, I^k / I^{k+1} is a free R/I -module on the residue classes of the monomials of exact degree k in e_1, \dots, e_n . Thus by induction on k , the underlying S -module of R/I^{k+1} is free on the residue classes of the monomials of degree at most k . So R/I^{k+1} is a truncated polynomial algebra over S , and the completion of R at I is a power series algebra over S on the classes e_1, \dots, e_n , giving (i). This power series algebra is free when it comes to morphisms with complete targets, yielding (ii). \square

Definition 4.5. Let (e_1, \dots, e_n) be a regular sequence of homogeneous elements of a commutative graded ring R , and let $s : S \rightarrow R$ be a ring homomorphism whose composite with the projection $R \rightarrow R/(e_1, \dots, e_n)$ is an isomorphism. The *power series expansion* of R at e_1, \dots, e_n is the unique morphism $d : R \rightarrow S[[e_1, \dots, e_n]]$ of graded S -algebras that sends e_i to e_i for all $1 \leq i \leq n$.

Construction 4.6. We let (X, a, t) be an oriented el_2^{RO} -algebra and $n \geq 0$. The inflation homomorphism $p^* : X(1) \rightarrow (t^{-1}X)(C^n)$ is a section to the restriction homomorphism $\text{res}_1^{C^n}$. Since the kernel of $\text{res}_1^{C^n}$ is generated by the regular sequence of Euler classes e_1, \dots, e_n , Proposition 4.4 provides a unique graded $X(1)$ -algebra homomorphism

$$d^n : (t^{-1}X)(C^n) \rightarrow X(1)[[e_1, \dots, e_n]]$$

that sends e_i to e_i for all $1 \leq i \leq n$, the *power series expansion* in the Euler classes e_1, \dots, e_n .

We recall that a *2-torsion formal group law* over a graded commutative ring R is a power series $F(x, y) \in R[[x, y]]$ that is homogeneous of degree -1 with respect to the grading $\deg(x) = \deg(y) = -1$, and that satisfies

$$F(x, 0) = x, \quad F(x, y) = F(y, x), \quad F(x, F(y, z)) = F(F(x, y), z) \quad \text{and} \quad F(x, x) = 0.$$

In the next theorem and in what follows, we shall write $\mu : C \times C \rightarrow C$ for the multiplication homomorphism.

Theorem 4.7. *Let (X, a, t) be an oriented el_2^{RO} -algebra. The image of the Euler class e_μ under the power series expansion*

$$d^2 : (t^{-1}X)(C \times C) \rightarrow X(1)[[e_1, e_2]]$$

is a 2-torsion formal group law over the graded ring $X(1)$, the formal group law of (X, a, t) .

Proof. We let $\alpha : C^n \rightarrow C^2$ be a group homomorphism, for some $n \geq 1$. We claim that the following diagram of graded $X(1)$ -algebras commutes:

$$\begin{array}{ccc} (t^{-1}X)(C^2) & \xrightarrow{\alpha^*} & (t^{-1}X)(C^n) \\ d^2 \downarrow & & \downarrow d^n \\ X(1)[[e_1, e_2]] & \xrightarrow{\alpha^\natural} & X(1)[[e_1, \dots, e_n]] \end{array}$$

Here α^\natural is the unique graded $X(1)$ -algebra homomorphism that satisfies $\alpha^\natural(e_i) = d^n(e_{p_i \alpha})$ for $i = 1, 2$. Indeed, both composites have the same effect on the regular sequence (e_1, e_2) that generates the augmentation ideal of $(t^{-1}X)(C \times C)$. So they coincide by Proposition 4.4 (ii). The commutativity of the diagram then yields

$$\alpha^\natural(F(e_1, e_2)) = \alpha^\natural(d^2(e_\mu)) = d^n(\alpha^*(e_\mu)).$$

We derive the four relations of a 2-torsion formal group law by specializing for five different choices of α . When α is the inclusion of the first factor $i_1 : C \rightarrow C \times C$, then $i_1^\natural : X(1)[[e_1, e_2]] \rightarrow X(1)[[e_1]]$ sets $e_2 = 0$, and $i_1^*(e_\mu) = e_1$, so we get

$$F(e_1, 0) = i_1^\natural(F(e_1, e_2)) = d^1(e_1) = e_1.$$

When α is the diagonal $\Delta : C \rightarrow C \times C$, then $\delta^\natural : X(1)[[e_1, e_2]] \rightarrow X(1)[[e_1]]$ sends both e_1 and e_2 to e_1 , and $\delta^*(e_\mu) = 0$, so we obtain

$$F(e_1, e_1) = \Delta^\natural(F(e_1, e_2)) = d^1(0) = 0.$$

When α is the involution $\tau : C \times C \rightarrow C \times C$ that interchanges the two factors, then $\tau^{\natural} : X(1)[[e_1, e_2]] \rightarrow X(1)[[e_1, e_2]]$ interchanges e_1 and e_2 , and $\tau(e_\mu) = e_\mu$; so we get the commutativity relation

$$F(e_2, e_1) = \tau^{\natural}(F(e_1, e_2)) = d^2(e_\mu) = F(e_1, e_2) .$$

For $\alpha = \mu \times C : C^3 \rightarrow C^2$, we have $p_1(\mu \times C) = \mu \circ p_{1,2}$, for $p_{1,2} : C^3 \rightarrow C^2$ the projection to the first two coordinates, and $p_2(\mu \times C) = p_3$. So the homomorphism $(\mu \times C)^{\natural} : X(1)[[e_1, e_2]] \rightarrow X(1)[[e_1, e_2, e_3]]$ sends e_1 to $d^3(p_{1,2}^*(e_\mu)) = d^2(e_\mu) = F(e_1, e_2)$, and it sends e_2 to e_3 . So we obtain

$$F(F(e_1, e_2), e_3) = (\mu \times C)^{\natural}(F(e_1, e_2)) = d^3((\mu \times C)^*(e_\mu)) .$$

The analogous argument for $\alpha = C \times \mu$ yields

$$F(e_1, F(e_2, e_3)) = d^3((C \times \mu)^*(e_\mu)) .$$

Because $(\mu \times C)^*(e_\mu) = (C \times \mu)^*(e_\mu)$ (the multiplication of C is associative), this establishes the associativity relation $F(F(e_1, e_2), e_3) = F(e_1, F(e_2, e_3))$. \square

Example 4.8 (Additive orientations yield additive formal group laws). Let (X, a, t) be an additive el_2^{RO} -algebra in the sense of Definition 3.3, i.e.,

$$a_1 t_\mu t_2 + t_1 a_\mu t_2 + t_1 t_\mu a_2 = 0$$

in $X(C \times C, 2 - (p_1 \oplus \mu \oplus p_2))$. Then

$$e_1 + e_\mu + e_2 = (a_1 t_\mu t_2 + t_1 a_\mu t_2 + t_1 t_\mu a_2) / t_1 t_\mu t_2 = 0 .$$

So

$$F(e_1, e_2) = d^2(e_\mu) = d^2(e_1 + e_2) = e_1 + e_2 ,$$

i.e., the formal group law is the additive formal group law.

5. GEOMETRIC FIXED POINTS

To establish the universal property of equivariant bordism, we shall make use of the *geometric fixed points* of an orientable el_2^{RO} -algebra, obtained by inverting all the a -classes of all nontrivial characters. For orientable el_2^{RO} -algebras that arise from global ring spectra, this construction generalizes the geometric fixed point construction for equivariant spectra, see Remark 5.3 below, thence the name.

We also discuss various ways in which the power series expansions of the previous section can be extended to Laurent series expansions on geometric fixed points. The main result is the integrality criterion of Theorem 5.8 that allows to detect when a class in the geometric fixed point ring $\Phi_*^A X$ arises from a class in $X(A)_*$. This criterion will be used in Theorem 5.14 to show how morphisms of oriented el_2^{RO} -algebras can be recovered from the induced morphisms of geometric fixed point rings.

Construction 5.1 (Geometric fixed points). We let (X, a) be an orientable el_2^{RO} -algebra. For every elementary abelian 2-group A , we define the ring of A -geometric fixed points $\Phi_*^A X$ as the integer graded part of the localization of the I_A -graded ring $X(A, \star)$ obtained by inverting all classes of the form a_λ for all nontrivial A -characters λ . Homogeneous elements in $\Phi_k^A X$ are thus fractions of the form x/a_V for some A -representation V with trivial fixed points and some $x \in X(A, k - V)$. These elements satisfy the relations

$$x/a_V = (x \cdot a_W)/a_{V \oplus W} \quad \text{and} \quad x/a_V \cdot y/a_W = (x \cdot y)/a_{V \oplus W}$$

for all A -representations W with $W^A = 0$. In an orientable el_2^{RO} -algebra, multiplication by the classes a_λ is injective; hence also the maps $-/a_V : X(A, k - V) \rightarrow \Phi_k^A X$ are injective.

The geometric fixed point rings enjoy *inflation homomorphisms*, i.e., they are contravariantly functorial for epimorphisms $\beta : B \rightarrow A$. Indeed, $\beta^*(a_\lambda) = a_{\lambda\beta}$, so the inflation homomorphism $\beta^* : X(A, \star) \rightarrow X(B, \star)$ induces a ring homomorphism $\beta^* : \Phi_*^A X \rightarrow \Phi_*^B X$ of localizations. In terms of the above presentation as fractions, we have

$$(5.2) \quad \beta^*(x/a_V) = \beta^*(x)/a_{\beta^*(V)} .$$

We emphasize a crucial difference between $\Phi^A X$, the localization away from a , and the localization $(t^{-1}X)(A)$ away from t discussed in Construction 4.1: the latter admit restriction homomorphisms for *all* group homomorphisms, the former only admit inflations, i.e., restriction along epimorphisms.

Remark 5.3 (Relation to geometric fixed points of equivariant spectra). As already mentioned above, the term ‘geometric fixed points’ for the localization of an orientable el_2^{RO} -algebra away from the a -classes is motivated by the topological examples. Indeed, for every compact Lie group G and every genuine G -spectrum R , a common definition of the geometric fixed point homotopy groups is as the reduced G -equivariant R -homology of a specific based G -space $\tilde{E}\mathcal{P}$, see for example [16, (3.18)]. Here $\tilde{E}\mathcal{P}$ is the unreduced suspension of a classifying G -space for the family of proper closed subgroups of G . A natural homomorphism $R_*^G \rightarrow R_*^G(\tilde{E}\mathcal{P})$ from the equivariant homotopy groups to the geometric fixed point homotopy groups is defined as the effect on $R_*^G(-)$ of the inclusion of cone points $S^0 \rightarrow \tilde{E}\mathcal{P}$. If R is a ring spectrum in the homotopy category of genuine G -spectra, this map is a morphism of graded rings, and it factors through an isomorphism from the localization of the $RO(G)$ -graded homotopy ring of R at the pre-Euler classes of all real G -representations V with $V^G = 0$, see [16, Proposition 3.20].

If R is a global ring spectrum and λ a nontrivial character of an elementary abelian 2-group A , then $a_\lambda \in R^\sharp(A, -\lambda) = R_0^A(S^\lambda)$ is precisely the pre-Euler class of the A -representation λ , so the geometric fixed point ring $\Phi_*^A R^\sharp$, in the sense of Construction 5.1, of the orientable el_2^{RO} -algebra (R^\sharp, a) is the localization of the $RO(A)$ -graded homotopy ring of R at the pre-Euler classes. Hence the two geometric fixed point rings $\Phi_*^A R^\sharp$ and $R_*^A(\tilde{E}\mathcal{P})$ of an orientable global ring spectrum are naturally isomorphic.

For a \mathbb{Z} -graded ring R , we now start using the notation

$$R((\theta)) = R[[\theta]][\theta^{-1}]$$

for the \mathbb{Z} -graded Laurent power series in a variable θ of degree -1 , the localization of the power series ring *away from* the element θ . So elements of $R((\theta))$ of degree k are power series $\sum_{i \in \mathbb{Z}} x_i \theta^i$ such that $x_i \in R_{k+i}$, and $x_i = 0$ for all almost all negative values of i .

Let (X, a, t) be an oriented el_2^{RO} -algebra, and let K be an elementary abelian 2-group. We will now define a Laurent power series expansion on the geometric fixed point ring

$$d_K : \Phi_*^{K \times C} X \rightarrow (\Phi_*^K X)((\theta)) .$$

These homomorphisms will be instrumental in detecting *effective* classes in the geometric fixed point rings, see Theorem 5.8, and in reconstructing morphisms of oriented el_2^{RO} -algebra from their effect on geometric fixed point rings, see Theorem 5.14.

Construction 5.4. We let (X, a, t) be an oriented el_2^{RO} -algebra, and we let K be an elementary abelian 2-group. The construction of the Laurent power series expansion requires certain

intermediate constructions that we summarize in the following commutative diagram:

$$(5.5) \quad \begin{array}{ccccc} & \Psi^K X & \longrightarrow & (\Psi^K X)[e_2^{-1}] & \longleftarrow & \Phi^{K \times C} X \\ & \downarrow \delta_K & & \downarrow \bar{\delta}_K & & \\ \Phi^K X & \xleftarrow{\theta=0} & (\Phi^K X)[[\theta]] & \longrightarrow & (\Phi^K X)((\theta)) & \longleftarrow & \Phi^{K \times C} X \end{array}$$

(Note: Curved arrows i_1^ and d_K connect $\Psi^K X$ to $\Phi^K X$ and $\Phi^{K \times C} X$ to $(\Phi^K X)((\theta))$ respectively.)*

We introduce the graded ring

$$(5.6) \quad \Psi^K X = X(K \times C, \star)[t_2^{-1}, a_V^{-1} : V^K = 0],$$

the integer graded part of the localization of the $I_{K \times C}$ -graded ring $X(K \times C, \star)$ at all a -classes of $(K \times C)$ -representations with trivial K -fixed points, and the class $t_2 = p_2^*(t)$ of the projection to the second factor. Inverting the classes a_V for all $V^K = 0$ is the same as inverting the classes a_ν for all non-zero characters of $K \times C$ *except* the projection to the second factor. For $\nu \neq p_2$, the restriction of ν to K is nontrivial. The restriction homomorphism

$$i_1^* : X(K \times C, \star) \longrightarrow X(K, \star)$$

along the embedding $i_1 : K \longrightarrow K \times C$ as the first factor sends the class t_2 to 1, and it sends the class a_V to the pre-Euler class of the restriction of V to K . So the restriction extends uniquely to localizations to a morphism of graded rings

$$i_1^* : \Psi^K X \longrightarrow \Phi^K X$$

that we denote by the same symbol. The inflation homomorphism

$$p_1^* : X(K, \star) \longrightarrow X(K \times C, \star)$$

along the projection $p_1 : K \times C \longrightarrow K$ extends uniquely to a homomorphism on the localization

$$p_1^* : \Phi^K X \longrightarrow \Psi^K X$$

that splits i_1^* . We write $a_2 = p_2^*(a)$ for the pre-Euler class of the projection to the second factor, and we set

$$e_2 = a_2/t_2 \in \Psi_{-1}^K X.$$

Because localization is exact, the sequence

$$0 \longrightarrow \Psi_{*+1}^K X \xrightarrow{e_2} \Psi_*^K X \xrightarrow{i_1^*} \Phi_*^K X \longrightarrow 0$$

is exact. Hence e_2 is regular in $\Psi_*^K X$, and the composite of $p_1^* : \Phi_*^K X \longrightarrow \Psi_*^K X$ and the projection $\Psi_*^K X \longrightarrow (\Psi_*^K X)/(e_2)$ is an isomorphism. We let

$$\delta_K : \Psi^K X \longrightarrow (\Phi^K X)[[\theta]]$$

denote the associated power series expansion in the sense of Definition 4.5. So δ_K is the unique morphism of graded $\Phi^K X$ -algebras such that $\delta_K(e_2) = \theta$. The homomorphism extends uniquely to the localizations to a homomorphism

$$\bar{\delta}_K : (\Psi^K X)[e_2^{-1}] \longrightarrow (\Phi^K X)((\theta)).$$

The ring $(\Psi^K X)[e_2^{-1}]$ is the localization of $X(K \times C, \star)$ obtained by inverting all a -classes and the class t_2 . Since $\Phi^{K \times C} X$ is a less drastic localization – formed by inverting all a -classes – it affords a localization morphism $\Phi^{K \times C} X \longrightarrow (\Psi^K X)[e_2^{-1}]$ that is localization away from t_2/a_2 . The composite with $\bar{\delta}_K$ is the homomorphism

$$d_K : \Phi^{K \times C} X \longrightarrow (\Phi_*^K X)((\theta));$$

it is then given by

$$d_K(x/(a_V a_2^n)) = \delta_K(x/(a_V t_2^n)) \cdot \theta^{-n},$$

for $x \in X(K \times C, k - (V \oplus np_2))$ with $V^K = 0$. We have now introduced all ring homomorphisms in the diagram (5.5).

We make the following definition:

Definition 5.7. Let X be an orientable el_2^{RO} -algebra and A an elementary abelian 2-group. A class in $\Phi_k^A X$ is *effective* if it lies in the image of the homomorphism $-/1 : X(A)_k \rightarrow \Phi_k^A X$.

For every orientable el_2^{RO} -algebra, the map $-/1 : X(A)_k \rightarrow \Phi_k^A X$ is injective. So if a class is effective, then its preimage in the integer graded subring is unique. Said differently, the maps $-/1 : X(A)_* \rightarrow \Phi_*^A X$ form an isomorphism from the \mathbb{Z} -graded part of $X(A, \star)$ to the subring of effective classes.

The next theorem establishes an effectivity criterion for geometric fixed point classes in terms of the homomorphisms $d_K : \Phi^{K \times C} X \rightarrow (\Phi^K X)((\theta))$. Effective classes in $\Phi^{K \times C} X$ in particular also lie in the ring $\Psi^K X$, and hence their Laurent power series expansion is contained in the subring $(\Phi^K X)[[\theta]]$, compare the diagram (5.5). Moreover, d_K is designed so that for $u \in X(K \times C)_*$, the constant coefficient is given by

$$d_K(u/1)(0) = i_1^*(u/1) = i_1^*(u)/1.$$

Theorem 5.8. Let (X, a, t) be an oriented el_2^{RO} -algebra, and let A be an elementary abelian 2-group. For every $y \in \Phi^A X$, the following two conditions are equivalent.

- (a) The class y is effective.
- (b) For every isomorphism $\alpha : K \times C \cong A$, the class $d_K(\alpha^*(y))$ in $(\Phi^K X)((\theta))$ is integral, i.e., contained in the subring $(\Phi^K X)[[\theta]]$.

Proof. (a) \implies (b) For every $u \in X(A)_k$, the class $\alpha^*(u/1) = \alpha^*(u)/1$ lies in $\Psi^K X$, so $d_K(\alpha^*(u/1))$ lies in $(\Phi^K X)[[\theta]]$, compare the commutative diagram (5.5).

(b) \implies (a) We write $y = x/a_V$ for some A -representation V with $V^A = 0$, some $k \in \mathbb{Z}$, and some $x \in X(A, k - V)$. We let λ be a nontrivial A -character, with kernel K . We choose an isomorphism $\alpha : K \times C \cong A$ such that $\lambda \circ \alpha = p_2$. We decompose $\alpha^*(V) = U \oplus np_2$ for some $(K \times C)$ -representation U with $U^K = 0$, and some $n \geq 0$. Then

$$d_K(\alpha^*(y)) = d_K(\alpha^*(x)/a_{\alpha^*(V)}) = d_K(\alpha^*(x)/a_U a_2^n) = \delta_K(\alpha^*(x)/a_U t_2^n) \cdot \theta^{-n}.$$

This class is integral by hypothesis, and $\delta_K(\alpha^*(x)/a_U t_2^n)$ is integral by definition. So we conclude that

$$\delta_K(\alpha^*(x)/a_U t_2^n) \in \theta^n \cdot (\Phi^K X)[[\theta]].$$

In other words, the first n coefficients of the power series $\delta_K(\alpha^*(x)/a_U t_2^n)$ vanish. Hence the class $\alpha^*(x)/a_U t_2^n$ is divisible by e_2^n in the ring $\Psi^K X$.

Since all the a -classes are non zero-divisors, we deduce that for some $m \geq 0$ the element $\alpha^*(x) \cdot t_2^m$ is divisible by a_2^n in $X(K \times C, \star)$. Thus the element $x \cdot (\alpha^{-1})^*(t_2^m) = x \cdot t_\lambda^m$ is divisible by $(\alpha^{-1})^*(a_2^n) = a_\lambda^n$ in $X(A, \star)$. Because $\text{res}_K^A(t_\lambda^m) = 1$, Proposition 1.5 (i) shows that already x is divisible by a_λ^n . Since n is the multiplicity of p_2 in $\alpha^*(V)$, it is also the multiplicity of $\lambda = (\alpha^{-1})^*(p_2)$ in V . We have thus shown that for every nontrivial A -character λ , the class x is divisible by a_{V_λ} . Proposition 1.5 (ii) then shows that x is divisible by a_V , say $x = z \cdot a_V$. Hence $y = x/a_V = z/1$ is effective. \square

Remark 5.9 (Relation to Boardman's work in equivariant bordism). For the two-element group C , the statement of Theorem 5.8 simplifies: an element $y \in \Phi^C X$ is effective if and only if the Laurent power series $d_1(y)$ is integral. For the oriented el_2^{RO} -algebra of equivariant bordism, Boardman defines a 'stable bordism J -homomorphism' $J : \Phi_*^C \mathcal{N} \rightarrow \mathcal{N}_*((\theta))$ in [7, §6], using different notation and geometric arguments. In this incarnation, the special case

$A = C$ of Theorem 5.8 was proved by Boardman in [7, Theorem 11]. In this sense, our Theorem 5.8 is a vast generalization of Boardman's result to elementary abelian 2-groups of arbitrary rank and general oriented el_2^{RO} -algebras.

For every oriented el_2^{RO} -algebra, the class $t_\mu/a_\mu \in \Phi^{C \times C} X$ also lies in the ring $\Psi^C X$, so that $d_C(t_\mu/a_\mu) = \delta_C(t_\mu/a_\mu)$, and this class is a power series (as opposed to a Laurent power series) in θ with coefficients in $\Phi^C X$.

Definition 5.10. Let (X, a, t) be an oriented el_2^{RO} -algebra. For $n \geq 0$, we define the classes $\beta_n \in \Phi_{n+1}^C X$ by

$$d_C(t_\mu/a_\mu) = \sum_{n \geq 0} \beta_n \cdot \theta^n .$$

For example, $\beta_0 = i_1^*(t_\mu/a_\mu) = t/a$. The next theorem might seem unmotivated right now, but it will enter crucially in the proof of our main result, Theorem 7.14. In part (ii), what matters is not the precise formula, but the fact that the classes $d_C(\mu^*(\beta_n))$ can be described purely in terms of the 2-torsion formal group law and the classes β_i themselves.

Theorem 5.11. Let (X, a, t) be an oriented el_2^{RO} -algebra with 2-torsion formal group law F .

- (i) The images of the classes β_n under the power series expansion $d_1 : \Phi^C X \rightarrow X(1)((\theta))$ satisfy the relation

$$F(\theta, \xi) \cdot \sum_{n \geq 0} d_1(\beta_n) \cdot \xi^n = 1$$

in the ring $X(1)((\theta))[[\xi]]$.

- (ii) The classes $\delta_C(\mu^*(\beta_n)) \in (\Phi^C X)[[\theta]]$ satisfy the relation

$$\sum_{n \geq 0} \delta_C(\mu^*(\beta_n)) \cdot \xi^n = \sum_{n \geq 0} \beta_n \cdot F(\theta, \xi)^n$$

in the ring $(\Phi^C X)[[\theta, \xi]]$.

Proof. (i) We specialize Construction 5.4 to $K = \{1\}$. Then $\Psi^1 X = X(C, \star)[t^{-1}] = (t^{-1}X)(C)$, with the potentially confusing caveat that the class that is generically called e_2 in $\Phi^1 K$ in Construction 5.4 is called e_1 in $(t^{-1}X)(C)$. Under these identifications, the homomorphism $\delta_1 : \Psi^1 X \rightarrow X(1)[[\theta]]$ coincides with $d^1 : (t^{-1}X)(C) \rightarrow X(1)[[e_1]]$ up to renaming of the variables. So for all $x \in X(C, * - k\sigma)$, the relation

$$(5.12) \quad d^1(x/t^k) \cdot e_1^{-k} = \delta_1(x/t^k)\langle e_1 \rangle \cdot e_1^{-k} = (\delta_1(x/t^k) \cdot \theta^{-k})\langle e_1 \rangle = d_1(x/a^k)\langle e_1 \rangle$$

holds in $X(1)((e_1))$. Here, and in what follows, we write $\langle e_1 \rangle$ to indicate a renaming of the variable from θ to e_1 .

The classes $d^2(e_1) = e_1$ and $d^2(e_\mu) = F(e_1, e_2)$ both become invertible in the ring $X(1)((e_1))[[e_2]]$, so the power series expansion $d^2 : (t^{-1}X)(C^2) \rightarrow X(1)[[e_1, e_2]]$ from Construction 4.6 extends uniquely to a morphism of graded $X(1)$ -algebras

$$d^2 : (t^{-1}X)(C^2)[e_1^{-1}, e_\mu^{-1}] \rightarrow X(1)((e_1))[[e_2]] ,$$

for which we use the same name.

The ring $(t^{-1}X)(C^2)[e_1^{-1}, e_\mu^{-1}]$ is the localization of $X(C^2, \star)$ obtained by inverting all t -classes and the classes a_1 and a_μ . Since $\Psi^C X$ is a less drastic localization – formed by inverting t_2 , a_1 and a_μ – it affords a localization morphism $\ell : \Psi^C X \rightarrow (t^{-1}X)(C^2)[e_1^{-1}, e_\mu^{-1}]$ that is localization away from t_1/a_1 and t_μ/a_μ . This morphism is explicitly given by

$$\ell(x/(a_1^k a_\mu^l t_2^m)) = (x/(t_1^k t_\mu^l t_2^m)) \cdot e_1^{-k} e_\mu^{-l} .$$

We claim that the following diagram of graded rings commutes:

$$\begin{array}{ccc}
\Psi^C X & \xrightarrow{\ell} & (t^{-1}X)(C^2)[e_1^{-1}, e_\mu^{-1}] \\
\delta_C \downarrow & & \downarrow d^2 \\
(\Phi^C X)[[\theta]] & \xrightarrow[\langle e_2 \rangle]{\cong} & (\Phi^C X)[[e_2]] \xrightarrow{d_1[[e_2]]} X(1)(\langle e_1 \rangle)[[e_2]]
\end{array}$$

Indeed, on the one hand, the rectangle commutes after precomposition with $p_1^* : \Phi^C X \rightarrow \Psi^C X$: the two composites send $p_1^*(x/a^k)$, for $x/a^k \in \Phi^C X$, to $d_1(x/a^k)\langle e_1 \rangle$ and $d^1(x/t^k) \cdot e_1^{-k}$, respectively, which agree by (5.12). On the other hand, both composites in the rectangle send the regular generator e_2 of the kernel of $i_1^* : \Psi^C X \rightarrow \Phi^C X$ to the class with the same name. So the rectangle commutes by Proposition 4.4 (ii). Now we can prove the claim:

$$\begin{aligned}
F(e_1, e_2) \cdot \sum_{n \geq 0} d_1(\beta_n)(e_1) \cdot e_2^n &= d^2(e_\mu) \cdot d_1[[e_2]](\delta_C(t_\mu/a_\mu)\langle e_2 \rangle) \\
&= d^2(e_\mu) \cdot d^2(\ell(t_\mu/a_\mu)) = d^2(e_\mu \cdot e_\mu^{-1}) = 1
\end{aligned}$$

This is the desired relation, up to renaming e_1 and e_2 to θ and ξ .

(ii) We consider the \mathbb{Z} -graded ring

$$\Xi(X) = X(C^3, \star)[a_1^{-1}, a_{12}^{-1}, a_{\bar{\mu}}^{-1}, t_2^{-1}, t_{23}^{-1}, t_3^{-1}],$$

the integer graded subring of the localization of the I_{C^3} -graded ring $X(C^3, \star)$ by inverting

- the a -classes for the projection to the first factor, the sum of the first two factors, and the total multiplication $\bar{\mu} : C^3 \rightarrow C$,
- and the t -classes for the sum of the last two factors and the projections to the second and to the third factor.

The restriction $i_{12}^* : X(C^3, \star) \rightarrow X(C^2, \star)$ to the first two factors satisfies

$$i_{12}^*(a_1) = a_1, \quad i_{12}^*(a_{12}) = i_{12}^*(a_{\bar{\mu}}) = a_\mu, \quad i_{12}^*(t_2) = i_{12}^*(t_{23}) = t_2 \quad \text{and} \quad i_{12}^*(t_3) = 1;$$

so it descends to a homomorphism of localizations

$$i_{12}^* : \Xi(X) \rightarrow \Psi^C X,$$

where $\Psi^C X$ was defined in (5.6). The morphisms

$$p_{12}^*, (C \times \mu)^*, (\mu \times C)^* : X(C^2, \star) \rightarrow X(C^3, \star)$$

satisfy

$$\begin{aligned}
p_{12}^*(a_1) &= (C \times \mu)^*(a_1) = a_1, & p_{12}^*(a_\mu) &= (\mu \times C)^*(a_1) = a_{12}, & p_{12}^*(t_2) &= t_2 \\
(C \times \mu)^*(a_\mu) &= a_{\bar{\mu}} & \text{and} & & (C \times \mu)^*(t_2) &= t_{23} \\
(\mu \times C)^*(a_\mu) &= a_{\bar{\mu}} & \text{and} & & (\mu \times C)^*(t_2) &= t_3.
\end{aligned}$$

So all three extend to the localizations and yield homomorphisms

$$p_{12}^*, (C \times \mu)^*, (\mu \times C)^* : \Psi^C X \rightarrow \Xi(X).$$

Because $(C \times \mu) \circ i_{12} = p_{12} \circ i_{12} = \text{Id}$, the first two of these homomorphisms are sections to i_{12}^* . Because localization is exact, the class $e_3 = a_3/t_3$ is a regular element of $\Xi(X)$ that generates the kernel of i_{12}^* . We let

$$\partial : \Xi(X) \rightarrow (\Psi^C X)[[\xi]]$$

be the power series expansion, in the sense of Definition 4.5, of $\Xi(X)$ with respect to the class e_3 and the section p_{12}^* . We will show that the three subdiagrams of the following large diagram of graded rings commute:

$$\begin{array}{ccccc}
& & \Psi^C X & \xrightarrow{\delta_C} & (\Phi^C X)[[\theta]] & & \\
& \nearrow^{\mu^*} & \downarrow^{(C \times \mu)^*} & & & \searrow^{F_*} & \\
\Phi^C X & & \Xi(X) & \xrightarrow{\partial} & (\Psi^C X)[[\xi]] & \xrightarrow{\delta_C[[\xi]]} & (\Phi^C X)[[\theta, \xi]] \\
& \searrow_{\mu^*} & \uparrow^{(\mu \times C)^*} & & \uparrow^{\mu^*[[\xi]]} & & \\
& & \Psi^C X & \xrightarrow{\delta_C} & (\Phi^C X)[[\theta]] & \xrightarrow[\cong]{\langle \xi \rangle} & (\Phi^C X)[[\xi]]
\end{array}$$

Here F_* is the graded $\Phi^C X$ -algebra map that sends θ to the formal group law $F(\theta, \xi)$. And we continue to use pointy brackets $\langle - \rangle$ to indicate isomorphisms that rename variables. The left subdiagram with target $\Xi(X)$ commutes by the relation $\mu \circ (C \times \mu) = \mu \circ (\mu \times C)$.

Now we show the commutativity of the upper subdiagram featuring F_* . We appeal to the uniqueness statement in Proposition 4.4 (ii). After precomposition with the homomorphism $p_1^* : \Phi^C X \rightarrow \Psi^C X$, both composites in the upper part become the homomorphism $- \cdot 1 : \Phi^C X \rightarrow (\Phi^C X)[[\theta, \xi]]$, the inclusion of the coefficient ring. We claim that both composites in the upper subdiagram also agree on e_2 . The inflation homomorphism

$$p_{23}^* : X(C^2, \star) \rightarrow X(C^3, \star)$$

along the projection to the last two factors satisfies

$$p_{23}^*(t_1) = t_2, \quad p_{23}^*(t_\mu) = t_{23}, \quad p_{23}^*(t_2) = t_3,$$

and thus passes to a homomorphism from the localization away from t of the source to $\Xi(X)$. The composite

$$(5.13) \quad (t^{-1}X)(C^2) \xrightarrow{p_{23}^*} \Xi(X) \xrightarrow{\partial} (\Psi^C X)[[\xi]] \xrightarrow{\delta_C[[\xi]]} (\Phi^C X)[[\theta, \xi]]$$

is a morphism of $X(1)_*$ -algebras, and it sends the classes e_1 and e_2 to θ and ξ , respectively. Since the regular sequence (e_1, e_2) generates the augmentation ideal of $(t^{-1}X)(C \times C)$, the uniqueness result from Proposition 4.4 (ii) shows that the composite (5.13) agrees with

$$(t^{-1}X)(C^2) \xrightarrow{d^2} X(1)[[e_1, e_2]] \xrightarrow[\cong]{\langle \theta, \xi \rangle} X(1)[[\theta, \xi]] \xrightarrow{p^*[[\theta, \xi]]} (\Phi^C X)[[\theta, \xi]].$$

Here $p^* : X(1) \rightarrow \Phi^C X$ is the inflation homomorphism. So these two homomorphisms also agree on the class e_μ . Because

$$p_2 \circ (C \times \mu) = \mu \circ p_{23} : C^3 \rightarrow C,$$

we have $(C \times \mu)^*(e_2) = p_{23}^*(e_\mu)$ in $\Xi(X)$. Thus

$$\begin{aligned}
\delta_C[[\xi]](\partial((C \times \mu)^*(e_2))) &= \delta_C[[\xi]](\partial(p_{23}^*(e_\mu))) = p^*[[\theta, \xi]](d^2(e_\mu)\langle \theta, \xi \rangle) \\
&= p^*[[\theta, \xi]](F(\theta, \xi)) = F_*(\theta) = F_*(d_C(e_2)).
\end{aligned}$$

Since e_2 is a regular element that generates the kernel of $i_1^* : \Psi^C X \rightarrow \Phi^C X$, the uniqueness result from Proposition 4.4 (ii) shows that the upper part of the diagram commutes.

It remains to show the commutativity of the lower subdiagram, and one more time we appeal to Proposition 4.4 (ii). This time both composites with the homomorphism $p_1^* : \Phi^C X \rightarrow \Psi^C X$ become the composite

$$\Phi^C X \xrightarrow{\mu^*} \Psi^C X \xrightarrow{-1} (\Psi^C X)[[\xi]],$$

and both composites send the regular element e_2 to ξ . So the lower part of the big diagram commutes, too.

We can now prove the desired relation:

$$\begin{aligned} \sum_{n \geq 0} d_C(\mu^*(\beta_n)) \cdot \xi^n &= (d_C \circ \mu^*)[\![\xi]\!] \left(\sum_{n \geq 0} \beta_n \cdot \xi^n \right) = (\delta_C \circ \mu^*)[\![\xi]\!] (\delta_C(t_\mu/a_\mu) \langle \xi \rangle) \\ &= F_*(\delta_C(t_\mu/a_\mu)) = F_* \left(\sum_{n \geq 0} \beta_n \cdot \theta^n \right) = \sum_{n \geq 0} \beta_n \cdot F(\theta, \xi)^n. \end{aligned}$$

The third equality is the commutativity of the big diagram, applied to the element t/a . \square

We conclude this section with a criterion for when a system of morphisms of geometric fixed point rings arises from a morphism of oriented el_2^{RO} -algebras. This will be used in the proof of our main result, Theorem 7.14, showing that equivariant bordism is an initial oriented el_2^{RO} -algebra.

We let el_2^{epi} denote the category whose objects are all elementary abelian 2-groups, and whose morphisms are surjective group homomorphisms. An el_2^{epi} -algebra is a contravariant functor from el_2^{epi} to the category of commutative \mathbb{Z} -graded \mathbb{F}_2 -algebras. For every oriented el_2^{RO} -algebra X , the collection of geometric fixed point rings $\Phi^A X$ supports inflation homomorphisms (5.2) along morphisms in el_2^{epi} , so they naturally form an el_2^{epi} -algebra $\Phi^\bullet X$. Geometric fixed points are often easier to compute than the el_2^{RO} -algebras themselves. Therefore the following theorem will be useful to construct morphisms of orientable el_2^{RO} -algebras through their geometric fixed points.

Theorem 5.14. *Let (X, a, t) and (Y, \bar{a}, \bar{t}) be oriented el_2^{RO} -algebras. Let $f^\bullet = \{f^A\}_A : \Phi^\bullet X \rightarrow \Phi^\bullet Y$ be a morphism of el_2^{epi} -algebras. Suppose that the following two conditions are satisfied:*

- (a) $f^C(t/a) = \bar{t}/\bar{a}$
- (b) *For every elementary abelian 2-group K , the following diagram commutes:*

$$\begin{array}{ccc} \Phi^{K \times C} X & \xrightarrow{f^{K \times C}} & \Phi^{K \times C} Y \\ d_K \downarrow & & \downarrow d_K \\ (\Phi^K X)((\theta)) & \xrightarrow{f^K((\theta))} & (\Phi^K Y)((\theta)) \end{array}$$

Then there exists a morphism $f : X \rightarrow Y$ of oriented el_2^{RO} -algebras such that $\Phi^A f = f^A$ for all elementary abelian 2-groups A .

Proof. We start by observing

$$f^A(t_\lambda/a_\lambda) = f^A(\lambda^*(t/a)) \stackrel{(b)}{=} \lambda^*(f^C(t/a)) \stackrel{(a)}{=} \lambda^*(\bar{t}/\bar{a}) = \bar{t}_\lambda/\bar{a}_\lambda$$

for all nontrivial characters λ of an elementary abelian 2-group A .

We recall from Definition 5.7 that a class $y \in \Phi_k^A X$ is effective if it lies in the image of the homomorphism $-/1 : X(A)_k \rightarrow \Phi_k^A X$, and that the maps $-/1 : X(A)_* \rightarrow \Phi_*^A X$ form an isomorphism from the \mathbb{Z} -graded part of $X(A)$ to the subring of effective classes.

Claim 1: for every elementary abelian 2-group A , the morphism $f^A : \Phi^A X \rightarrow \Phi^A Y$ takes effective classes to effective classes.

To prove this claim we let $y \in \Phi^A X$ be an effective class, and we let $\alpha : K \times C \cong A$ be an isomorphism. Because the f -maps commute with inflation along α , we have

$$d_K(\alpha^*(f^A(y))) = d_K(f^{K \times C}(\alpha^*(y))) \stackrel{(b)}{=} f^K((\theta))(d_K(\alpha^*(y))).$$

By Theorem 5.8, the class $d_K(\alpha^*(y))$ in $(\Phi^K Y)((\theta))$ is integral. Since $f^K((\theta)) : (\Phi^K X)((\theta)) \rightarrow (\Phi^K Y)((\theta))$ preserves integrality, the class $d_K(\alpha^*(f^A(y)))$ in $(\Phi^K Y)((\theta))$ is integral. Again by Theorem 5.8, the class $f^A(y)$ is effective.

Because the ring homomorphism f^A takes effective classes to effective classes, there are unique maps $f(A) : X(A)_* \rightarrow Y(A)_*$ that make the diagram

$$\begin{array}{ccc} X(A)_* & \xrightarrow{f(A)} & Y(A)_* \\ \downarrow -/1 & & \downarrow -/1 \\ \Phi_*^A X & \xrightarrow{f^A} & \Phi_*^A Y \end{array}$$

commute. These maps are homomorphisms of graded rings and commute with inflations because this is the case for the other three morphisms in the commutative diagram, and because the vertical maps are injective.

Claim 2: For every monomorphism $i : K \rightarrow A$ between elementary abelian 2-groups, the following square commutes:

$$\begin{array}{ccc} X(A)_* & \xrightarrow{f(A)} & Y(A)_* \\ \downarrow i^* & & \downarrow i^* \\ X(K)_* & \xrightarrow{f(K)} & Y(K)_* \end{array}$$

It suffices to show this when $i(K)$ has index 2 in A , and then we may assume without loss of generality that $A = K \times C$ and $i = i_1 : K \rightarrow K \times C$ is the embedding as the first summand. For $u \in X(K \times C)_*$ the relation

$$f^K((\theta))(d_K(u/1)) \stackrel{(b)}{=} d_K(f^{K \times C}(u/1)) = d_K(f(K \times C)(u)/1)$$

holds in the ring $(\Phi_*^K Y)((\theta))$. Since both sides of this equation are integral classes by Theorem 5.8, the relation already holds in the ring $(\Phi_*^K Y)[[\theta]]$; in particular, the constant terms of both sides agree, and we obtain

$$\begin{aligned} (f(K)(i_1^*(u))/1) &= f^K(i_1^*(u)/1) = f^K(d_K(u/1)(0)) \\ &= f^K((\theta))(d_K(u/1))(0) \\ &\stackrel{(b)}{=} d_K(f(K \times C)(u)/1)(0) = i_1^*(f(K \times C)(u))/1. \end{aligned}$$

Because the map $-/1$ is injective, this concludes the proof of Claim 2.

We can now define the desired morphism of el_2^{RO} -algebras $f : X \rightarrow Y$. We let W be a representation of an elementary abelian 2-group A with $W^A = 0$. By Proposition 1.9, every class $x \in X(A, k - W)$ is a homogeneous polynomial with coefficients in $X(A)_*$ of degree $k - W$ in the classes a_λ and t_λ , for all nontrivial A -characters λ . In other words, there is a homogeneous polynomial $g \in X(A)_*[\alpha_\lambda, \tau_\lambda]$ in variables α_λ and τ_λ of degree $-\lambda$ and $1 - \lambda$, respectively, such that $x = g(a_\lambda, t_\lambda)$, i.e., substituting the variables α_λ and τ_λ by the classes a_λ and t_λ yields x . We write $f(A)_\flat(g) \in Y(A)_*[\alpha_\lambda, \tau_\lambda]$ for the effect of applying the graded ring homomorphism $f(A) : X(A)_* \rightarrow Y(A)_*$ to the coefficients of the polynomial g . The classes \bar{a}_λ and \bar{t}_λ of the el_2^{RO} -algebra Y can then be substituted for the variables in $f(A)_\flat(g)$,

yielding a class in $Y(A, k - W)$. There is no reason why the polynomial g that presents the class x should be unique, so we shall need:

Claim 3: The class $f(A)_b(g)(\bar{a}_\lambda, \bar{t}_\lambda)$ in $Y(A, k - W)$ is independent of how $x = g(a_\lambda, t_\lambda)$ is presented as a polynomial in the classes a_λ and t_λ .

For every homogeneous polynomial $h \in Y(A)_*[\alpha_\lambda, \tau_\lambda]$ of the appropriate degree we have

$$h(\bar{a}_\lambda, \bar{t}_\lambda)/\bar{a}_W = h(1, \bar{t}_\lambda/\bar{a}_\lambda)$$

in $\Phi_k^A Y$, i.e., the class $h(\bar{a}_\lambda, \bar{t}_\lambda)/\bar{a}_W$ is obtained by substituting the variables α_λ and τ_λ by the multiplicative unit 1 and by $\bar{t}_\lambda/\bar{a}_\lambda$, respectively. For $h = f(A)_b(g)$, this yields

$$\begin{aligned} f(A)_b(g)(\bar{a}_\lambda, \bar{t}_\lambda)/\bar{a}_W &= f(A)_b(g)(1, \bar{t}_\lambda/\bar{a}_\lambda) = f(A)_b(g)(1, f^A(t_\lambda/a_\lambda)) \\ &= f^A(g(1, t_\lambda/a_\lambda)) = f^A(g(a_\lambda, t_\lambda)/a_W) = f^A(x/a_W). \end{aligned}$$

This geometric fixed point class is thus independent of how x is presented as a polynomial in the classes a_λ and t_λ . Since the map $-\bar{a}_W : Y(A, k - W) \rightarrow \Phi_k^A Y$ is injective, this proves Claim 3.

With Claim 3 established, we can now define a map $f(A, k - W) : X(A, k - W) \rightarrow Y(A, k - W)$ by

$$f(A, k - W)(x) = f(A)_b(g)(\bar{a}_\lambda, \bar{t}_\lambda),$$

where $g \in X(A)_*[\alpha_\lambda, \tau_\lambda]$ is any homogeneous polynomial of degree $k - W$ such that $x = g(a_\lambda, t_\lambda)$. With the independence of g at hand, it is then straightforward to show that these assignments define an I_A -graded ring homomorphism. And clearly, the map $f(A, -\lambda) : X(A, -\lambda) \rightarrow Y(A, -\lambda)$ sends a_λ to \bar{a}_λ , and the map $f(A, 1 - \lambda) : X(A, 1 - \lambda) \rightarrow Y(A, 1 - \lambda)$ sends t_λ to \bar{t}_λ .

Now we check that for varying A , these homomorphisms of I_A -graded rings are compatible with restriction along homomorphisms $\beta : B \rightarrow A$ of elementary abelian 2-groups. We already argued above that the relation $\beta^* \circ f(A, \star) = f(B, \star) \circ \beta^*$ holds on the integer graded subrings. The I_A -graded ring $X(A, \star)$ is generated by its integer graded subring and the classes a_λ and t_λ for all nontrivial A -characters λ , by Proposition 1.9 (ii). So it suffices to show that the ring homomorphisms $\beta^* \circ f(A, \star)$ and $f(B, \star) \circ \beta^*$ also agree on the classes a_λ and t_λ . We distinguish two cases. If the B -character $\lambda\beta$ is nontrivial, then $\beta^*(a_\lambda) = a_{\lambda\beta}$, and we obtain

$$\beta^*(f(A, -\lambda)(a_\lambda)) = \beta^*(\bar{a}_\lambda) = \bar{a}_{\lambda\beta} = f(B, -(\lambda\beta))(a_{\lambda\beta}) = f(B, -(\lambda\beta))(\beta^*(a_\lambda)).$$

The argument for the t -classes is the same, mutatis mutandis. If the B -character $\lambda\beta$ is trivial, we need to argue differently. In this case $\beta^*(a_\lambda) = 0$ in X , and $\beta^*(\bar{a}_\lambda) = 0$ in Y , and the relation $\beta^*(f(A, -\lambda)(a_\lambda)) = f(B, -1)(\beta^*(a_\lambda))$ holds as both sides are zero. Also in this case, $\beta^*(t_\lambda) = 1$ in X and $\beta^*(\bar{t}_\lambda) = 1$ in Y , because t and \bar{t} are inverse Thom classes. Hence $\beta^*(f(A, 1 - \lambda)(t_\lambda))$ and $f(B, 0)(\beta^*(t_\lambda))$ are both the multiplicative unit 1. This finishes the proof. \square

6. EQUIVARIANT BORDISM

In this section we introduce the oriented el_2^{RO} -algebra \mathcal{N} of *equivariant bordism*, and we collect various basic properties. The main result of this paper is the fact that \mathcal{N} is an initial oriented el_2^{RO} -algebra, see Theorem 7.14 below.

Construction 6.1. Given a finite group G and a based G -space X , we write $\tilde{\mathcal{N}}_k^G(X)$ for the k -th reduced G -equivariant bordism group of X . We refer to [29, §2] or [25, Section 6.2] for a detailed background on equivariant bordism.

For an elementary abelian 2-group A , we write a given grading $m \in I_A$ as $m = k - V$ for some integer k and an A -representation V with trivial fixed points. Then we set

$$\mathcal{N}(A, m) = \mathcal{N}(A, k - V) = \tilde{\mathcal{N}}_k^A(S^V),$$

the reduced A -equivariant bordism group of dimension k of the onepoint compactification S^V . A key point is that this definition is independent up to preferred isomorphism of the choice of V . Indeed, an isomorphism of A -representations $\psi : V \rightarrow W$ induces an equivariant homeomorphism $S^\psi : S^V \rightarrow S^W$, and hence an isomorphism of bordism groups

$$\mathcal{N}_k^A(S^\psi) : \tilde{\mathcal{N}}_k^A(S^V) \rightarrow \tilde{\mathcal{N}}_k^A(S^W).$$

Theorem 2.3 (iii) and the fact that equivariant bordism is represented by an oriented global ring spectrum (Example 6.2) guarantee that the isomorphism does not depend on the choice of ψ . Equivariant bordism is concentrated in non-negative integer degrees, so the group $\mathcal{N}(A, k - V)$ is trivial whenever k is negative.

To define the functoriality we recall the suspension isomorphism in equivariant bordism. A specific class $d \in \tilde{\mathcal{N}}_1^G(S^1)$ is represented by the identity of S^1 . The suspension homomorphism

$$- \wedge d : \tilde{\mathcal{N}}_k^G(X) \rightarrow \tilde{\mathcal{N}}_{k+1}^G(X \wedge S^1)$$

is given by reduced product with the class d ; it is an isomorphism, for example by [25, Proposition 6.2.11].

Now we let $\alpha : (B, \alpha^*(m)) \rightarrow (A, m)$ be a morphism in the category el_2^{RO} . We choose an A -representation V and a B -representation W so that $m = k - V$ and $\alpha^*(m) = k - \alpha^*(V) = l - W$ for integers $k \geq l$. There is then an isomorphism of B -representations

$$\psi : \alpha^*(V) \xrightarrow{\cong} W \oplus \mathbb{R}^{k-l}.$$

We define

$$\alpha^* : \mathcal{N}(A, m) \rightarrow \mathcal{N}(B, \alpha^*(m))$$

as the composite

$$\tilde{\mathcal{N}}_k^A(S^V) \xrightarrow{\alpha^*} \tilde{\mathcal{N}}_k^B(S^{\alpha^*(V)}) \xrightarrow[\cong]{\tilde{\mathcal{N}}_k^B(S^\psi)} \tilde{\mathcal{N}}_k^B(S^{W \oplus \mathbb{R}^{k-l}}) \xleftarrow[\cong]{- \wedge d^{k-l}} \tilde{\mathcal{N}}_l^B(S^W).$$

The first homomorphism is restriction of the action on manifolds and bordisms along the homomorphism α . The homomorphism decorated $- \wedge d^{k-l}$ is an iterated suspension isomorphism. Again by Theorem 2.3 (iii), the definition of α^* does not depend on the choice of ψ ; this independence is also needed in verifying that the assignment is compatible with composition.

The pre-Euler class

$$a \in \mathcal{N}(C, -\sigma) = \tilde{\mathcal{N}}_0^C(S^\sigma)$$

is represented by the inclusion of the isolated fixed point $\{0\} \rightarrow S^\sigma$. The inverse Thom class

$$t \in \mathcal{N}(C, 1 - \sigma) = \tilde{\mathcal{N}}_1^C(S^\sigma)$$

is represented by the identity of S^σ , with equivariant smooth structure through its identification with $S(\mathbb{R} \oplus \sigma)$ by the stereographic projection.

Example 6.2 (Equivariant bordism versus the global Thom spectrum \mathbf{mO}). The global Thom spectrum \mathbf{mO} is defined in [25, Example 6.1.24]; the underlying G -equivariant homotopy type of \mathbf{mO} has been considered before, often in different language and notation, for example in [31, Section 5]. The spectrum \mathbf{mO} is the orthogonal spectrum whose V -th term is

$$\mathbf{mO}(V) = Th(Gr_{|V|}(V \oplus \mathbb{R}^\infty)),$$

the Thom space of the tautological vector bundle of $|V|$ -dimensional subspaces of $V \oplus \mathbb{R}^\infty$. We refer to [25, Example 6.1.24] for the definition of the structure maps and the E_∞ -multiplication on \mathbf{mO} .

For every compact Lie group G , the equivariant Pontryagin-Thom construction provides a natural transformation of G -equivariant homology theories from G -equivariant bordism to the G -homology theory represented by the global Thom spectrum \mathbf{mO} , see [25, Construction 6.2.27]. If the group G is a product of a finite group and a torus, then the equivariant Pontryagin-Thom construction is an isomorphism. This results goes back to Wasserman [33], a homotopy theoretic proof for finite groups was given by tom Dieck [31, Satz 5], and a proof in the general case can be found in [25, Theorem 6.2.33]. The reader should beware that the equivariant Pontryagin-Thom construction is provably *not* an isomorphism beyond this class of groups; for $G = SU(2)$, the homotopy theoretic transfer from the maximal torus normalizer to $SU(2)$ yields a class in $\pi_0^{SU(2)}(\mathbf{mO})$ that is not in the image, see [25, Remark 6.2.34].

In any case, the Thom-Pontryagin construction is in particular an isomorphism

$$\Theta^A(X) : \tilde{\mathcal{N}}_*^A(X) \xrightarrow{\cong} \mathbf{mO}_*^A(X)$$

for every elementary abelian 2-group A and every based A -CW-complex X . These isomorphisms are furthermore compatible with products and restriction along group homomorphisms, compare [25, Theorem 6.2.31]. Specializing to the case where $X = S^V$ is the one-point compactification of an A -representation shows that the Thom-Pontryagin construction is an isomorphism of el_2^{RO} -algebras

$$\Theta : \mathcal{N} \xrightarrow{\cong} \mathbf{mO}^\sharp.$$

Both in \mathcal{N} and in \mathbf{mO}^\sharp , the pre-Euler class a is obtained by applying the map of based C -spaces $S^0 \rightarrow S^\sigma$ to the multiplicative unit in the C -equivariant coefficients. Since the Thom-Pontryagin construction is natural in based equivariant maps, the pre-Euler classes of \mathcal{N} and \mathbf{mO}^\sharp match up.

Several times in this paper we will need the identification of the geometric fixed points of equivariant bordism in terms of fixed point data. This technique, like several others in the subject, goes back to the pioneering work of Conner and Floyd.

Construction 6.3 (Geometric fixed points of equivariant bordism). We recall the description of the geometric fixed point rings of equivariant bordism in terms of non-equivariant bordism groups. We will need it for elementary abelian 2-groups A , but this fixed point description works in the generality of compact Lie groups, see for example [25, Proposition 6.2.16]. The result takes the form of an isomorphism of graded rings:

$$(6.4) \quad \varphi_A : \Phi_n^A \mathcal{N} \xrightarrow{\cong} \bigoplus_{j \geq 0} \mathcal{N}_{n-j}(Gr_j^{A,\perp})$$

Here $Gr_j^{A,\perp}$ is the moduli space of j -dimensional A -representations with trivial fixed points, and $\mathcal{N}_{n-j}(Gr_j^{A,\perp})$ is its non-equivariant bordism group. More formally, $Gr_j^{A,\perp}$ is the A -fixed point space of the Grassmannian of j -planes in the orthogonal complement of the A -fixed subspace of a complete A -universe. The multiplication on the target arises from the maps

$$Gr_i^{A,\perp} \times Gr_j^{A,\perp} \longrightarrow Gr_{i+j}^{A,\perp}$$

that classify the direct sum of A -representations with trivial fixed points.

Roughly speaking, the isomorphism (6.4) is defined as follows. Elements of $\Phi_n^A \mathcal{N}$ are fractions x/a_V , where $x \in \tilde{\mathcal{N}}_n^A(S^V)$ for some $n \geq 0$, and some A -representation V with trivial fixed points. We let (M, h) represent the class x , so that M is an n -dimensional smooth closed A -manifold, and $h : M \rightarrow S^V$ is a continuous A -map; such pairs are taken up to bordism and

modulo the subgroup of pairs where h is constant at the basepoint. Because the A -action is smooth, the fixed set M^A is a disjoint union of regularly embedded smooth submanifolds of M of varying dimensions. The A -space S^V has only two fixed points 0 and ∞ , and only the fixed points over the non-basepoint 0 will play a role. The Conner–Floyd map (6.4) sends the bordism class $[M, h]$ to $\sum_{j \geq 0} [M^{(j)}, \nu_j]$, where $M^{(j)}$ is the union of the $(n - j)$ -dimensional components of $M^A \cap h^{-1}(0)$, and where $\nu_j : M^{(j)} \rightarrow Gr_j^{A, \perp}$ classifies the A -equivariant normal bundle of $M^{(j)}$ inside M . We refer to [25, Proposition 6.2.16] for more details on the construction and for a proof that the Conner–Floyd map is an isomorphism.

The next theorem collects various key properties of equivariant bordism that are relevant for this paper.

Theorem 6.5.

- (i) *The pair (\mathcal{N}, a) is an orientable el_2^{RO} -algebra with a unique inverse Thom class t .*
- (ii) *The 2-torsion formal group law of the oriented el_2^{RO} -algebra (\mathcal{N}, a, t) is an initial 2-torsion formal group law.*
- (iii) *For every elementary abelian 2-group A , the morphism of \mathcal{N}_* -algebras*

$$\bigotimes_{\mathcal{N}_*}^{\lambda \in A^\circ} \Phi_*^C \mathcal{N} \longrightarrow \Phi_*^A \mathcal{N}$$

that multiplies the inflation homomorphisms for all nontrivial A -characters is an isomorphism.

Proof. (i) As we detailed in Example 6.2, equivariant bordism is represented by the orientable global ring spectrum \mathbf{mO} . Part (i) is thus a special case of Theorem 2.5. By the general theory, the set of inverse Thom classes of (\mathcal{N}, a) is a torsor over the bordism group \mathcal{N}_1^C ; since this group is trivial, the inverse Thom class is unique.

(ii) We reduce the claim to Quillen’s theorem [24], saying that the formal group law associated to the preferred, classical real orientation of the non-equivariant Thom spectrum MO is initial. To this end we recall Quillen’s result in a form tailored to our purpose.

We let BO denote the classifying space of the infinite orthogonal group. The Thom spectrum construction assigns to a continuous map $f : X \rightarrow BO$ from a CW-complex a Thom spectrum $Th(f)$, making it a functor

$$Th : Ho(Top/BO) \rightarrow \mathcal{SH}$$

from the homotopy category of spaces over BO to the stable homotopy category of spectra. The functoriality equips the Thom spectrum with a morphism of spectra $u(f) : Th(f) \rightarrow Th(\mathrm{Id}_{BO}) = MO$ that is a tautological Thom class in MO -cohomology, in the sense that the group of stable maps $[Th(f), MO] = \mathcal{SH}(Th(f), MO)$ is free of rank 1 as a module, under the Thom diagonal, over the ring $MO^0(X) = [\Sigma_+^\infty X, MO]$, with $u(f)$ as generator.

In this language, the preferred real orientation of the spectrum MO arises as the Euler class of the continuous map $\beta(\sigma - \mathbb{R}) : BC \rightarrow BO$ that classifies the group homomorphism that sends the generator of the group C to the infinite diagonal matrix with entries $(-1, 1, 1, \dots)$. Some readers might prefer to think of continuous maps to BO as classifying virtual vector bundles of rank 0; our name ‘ $\beta(\sigma - \mathbb{R})$ ’ for the above map reflects that it classifies the virtual vector bundle over BC associated to the virtual C -representation $\sigma - \mathbb{R}$. The Thom spectrum of $\beta(\sigma - \mathbb{R})$ is then equivalent to the desuspension of the reduced suspension spectrum of the space $S^\sigma \wedge_C EC_+$ (which happens to be homotopy equivalent to BC), and its Thom class $u(\beta(\sigma - \mathbb{R}))$ lies in the group

$$[Th(\beta(\sigma - \mathbb{R})), MO] = [S^{-1} \wedge \Sigma^\infty S^\sigma \wedge_C EC_+, MO] = \widetilde{MO}^1(S^\sigma \wedge_C EC_+).$$

Its Euler class $e_\tau \in MO^1(BC)$, the restriction of the Thom class along the zero section $BC_+ \rightarrow S^\sigma \wedge_C EC_+$, is the preferred real orientation of MO .

By the theory of real-oriented cohomology theories, the ring $MO^*(BC \times BC)$ is then a power series algebra over MO^* in the classes $p_1^*(e_\tau)$ and $p_2^*(e_\tau)$, and expanding the class $(B\mu)^*(e_\tau)$ in $p_1^*(e_\tau)$ and $p_2^*(e_\tau)$ yields a 2-variable power series with coefficients in MO^* that is a 2-torsion formal group law. Quillen's theorem [24, Theorem 3] says that this 2-torsion formal group law is initial.

Now we use the global Thom ring spectrum \mathbf{MO} defined in [25, Example 6.1.7] that represents stable equivariant bordism. For every elementary abelian 2-group A , the equivariant homology theory represented by the global ring spectrum \mathbf{MO} is the localization of $\mathbf{mO}_*^A(-)$ at the inverse Thom classes, compare [25, Corollary 6.1.35]. The localization maps $\mathbf{mO}_*^A(-) \rightarrow \mathbf{MO}_*^A(-)$ at all representation spheres define a morphism of oriented el_2^{RO} -algebras $(\mathbf{mO}^\sharp, a, t) \rightarrow (\mathbf{MO}^\sharp, a, t)$. Here we denote the image of the unique inverse Thom class of \mathbf{mO}^\sharp by the same letter $t \in \mathbf{MO}_1^C(S^\sigma)$; we alert the reader that in the orientable el_2^{RO} -algebra \mathbf{MO}^\sharp , the inverse Thom class is far from unique. The previous morphism exhibits \mathbf{MO}^\sharp as the localization away from t of \mathbf{mO}^\sharp . In particular, this morphism is an isomorphism of underlying graded \mathbb{F}_2 -algebras, and so it necessarily takes the 2-torsion formal group law of \mathbf{mO}^\sharp to the 2-torsion formal group law of \mathbf{MO}^\sharp . So it suffices to show that the 2-torsion formal group law of the oriented el_2^{RO} -algebra $(\mathbf{MO}^\sharp, a, t)$ is an initial 2-torsion formal group law.

By Theorem 6.1.7 and Remark 6.1.20 of [25], the class t in $\mathbf{MO}_1^C(S^\sigma)$ is an $RO(C)$ -graded unit. The inverse of t is the 'shifted Thom class' $u = \bar{\sigma}_{C,\sigma} \in \mathbf{MO}_C^1(S^\sigma)$ discussed in [25, Remark 6.1.20]. We claim that the bundling homomorphism

$$(6.6) \quad \beta : \widetilde{\mathbf{MO}}_C^1(S^\sigma) \rightarrow \widetilde{MO}^1(S^\sigma \wedge_C EC_+)$$

takes the C -equivariant Thom class u to the non-equivariant Thom class $u(\beta(\sigma - \mathbb{R}))$ discussed above. To prove this we refer to the C -equivariant Thom spectrum formalism, a general reference for which is [23, Chapter X, §3]. We write \mathbf{BO}_C for the classifying C -space for virtual C -equivariant vector bundles of rank 0. Much like in the non-equivariant situation, the equivariant Thom construction turns C -equivariant maps $X \rightarrow \mathbf{BO}_C$ into genuine C -spectra endowed with a morphism to \mathbf{MO}_C , the underlying C -spectrum of the global spectrum \mathbf{MO} . An explicit representative for the class u in $\mathbf{MO}_C^1(S^\sigma)$ is the C -equivariant map

$$\begin{aligned} S^{\sigma \oplus \sigma} &\rightarrow Th(Gr_2(\sigma \oplus \mathbb{R} \oplus \sigma \oplus \mathbb{R})) = \mathbf{MO}(\sigma \oplus \mathbb{R}) \\ (x, y) &\mapsto ((x, 0, y, 0), \sigma \oplus 0 \oplus \sigma \oplus 0) \end{aligned}$$

So $u : S^{\sigma-1} \rightarrow \mathbf{MO}_C$ is the C -equivariant Thom class of the C -equivariant map $* \rightarrow \mathbf{BO}_C$ that classifies the virtual C -representation $2\sigma - (\sigma \oplus \mathbb{R}) = \sigma - \mathbb{R}$. The unstable and stable bundling maps take equivariant Thom spectra to non-equivariant Thom spectra, see [23, X Corollary 6.3]. When applied to the C -map $\sigma - \mathbb{R} : * \rightarrow \mathbf{BO}_C$, this identifies the image of $u : S^{\sigma-1} = Th(\sigma - \mathbb{R}) \rightarrow \mathbf{MO}_C$ under the stable bundling map (6.6) with the non-equivariant stable map

$$u(\beta(\sigma - \mathbb{R})) : S^{\sigma-1} \wedge_C EC_+ = Th(\beta(\sigma - \mathbb{R})) \rightarrow MO$$

obtained by thomifying $\beta(\sigma - \mathbb{R}) : BC \rightarrow BO$. This proves the claim that $\beta(u) = u(\beta(\sigma - \mathbb{R}))$. Naturality of the bundling homomorphisms for the C -equivariant fixed point inclusion $a : S^0 \rightarrow S^\sigma$ then shows that the bundling homomorphism

$$\beta : \mathbf{MO}_{-1}^C = \mathbf{MO}_C^1(S^0) \rightarrow \widetilde{MO}^1(S^0 \wedge_C EC_+) = MO^1(BC)$$

takes the C -equivariant Euler class $e = a/t = a \cdot u$ of the oriented el_2^{RO} -algebra $(\mathbf{MO}^\sharp, a, t)$ to the Euler class e_τ that defines the preferred classical orientation of MO .

Now we can wrap up. The formal group law of \mathbf{MO}^\sharp is defined by expanding the class $\mu^*(e) = \mu^*(a/t)$ as a power series in $p_1^*(e)$ and $p_2^*(e)$. And the formal group law in Quillen's work is the expansion of $\mu^*(e_\tau)$ as a power series in $p_1^*(e_\tau)$ and $p_2^*(e_\tau)$. Since the bundling maps are compatible with restriction along group homomorphisms and take e to e_τ , the 2-torsion formal group law of the oriented el_2^{RO} -algebra $(\mathbf{MO}^\sharp, a, t)$ is the preferred 2-torsion formal group law over MO^* that features in Quillen's theorem [24]. So we have shown claim (ii).

(iii) We consider the following commutative square of graded commutative \mathcal{N}_* -algebras:

$$\begin{array}{ccc} \bigotimes_{\mathcal{N}_*}^{\lambda \in A^\circ} \Phi_*^C \mathcal{N} & \longrightarrow & \Phi_*^A \mathcal{N} \\ \otimes \varphi_C \downarrow \cong & & \cong \downarrow \varphi_A \\ \bigotimes_{\mathcal{N}_*}^{\lambda \in A^\circ} \left(\bigoplus_{j \geq 0} \mathcal{N}_{*-j}(Gr_j^{C,\perp}) \right) & \longrightarrow & \bigoplus_{j \geq 0} \mathcal{N}_{n-j}(Gr_j^{A,\perp}) \end{array}$$

The left vertical map is a tensor product of copies of the Conner–Floyd fixed point isomorphism (6.4). The right vertical map is the isomorphism (6.4) for the group A . The lower horizontal map is also induced by inflation along characters. More precisely, the restriction to the tensor factor indexed by (λ, j) is induced by the continuous map $Gr_j^{C,j} \rightarrow Gr_j^{A,\perp}$ that identifies σ^∞ with the λ -isotypical summand of the complete A -universe, and then takes Grassmannians of j -planes.

Since the vertical maps in the commutative square are isomorphisms, we are reduced to showing that the lower horizontal map is an isomorphism. This, in turn, is a combination of two facts. Firstly, the nontrivial irreducible representations of A are all 1-dimensional, and parameterized by the set A° of nontrivial A -characters. So the isotypical decomposition of A -invariant subspaces becomes a homeomorphism

$$\coprod_{\sum i_\lambda = j} \left(\prod_{\lambda \in A^\circ} Gr_{i_\lambda} \right) \cong Gr_j^{A,\perp}.$$

The coproduct is indexed by A° -tuples of natural numbers i_λ that add up to j . Secondly, all non-equivariant bordism groups of spaces are free as modules over the non-equivariant bordism ring, see for example [12, Theorem 8.3]. So the Künneth isomorphism identifies the graded bordism groups of a product with the tensor product of the bordism groups of the factors. If we take the grading shift by j on the j -plane summands into account, these two facts combine into the statement that the lower horizontal map in the square is an isomorphism. This completes the proof. \square

7. THE UNIVERSAL PROPERTY OF EQUIVARIANT BORDISM

In this section we prove our main result Theorem 7.14, saying that equivariant bordism is an initial oriented el_2^{RO} -algebra. We will use Theorem 5.14, so we need a firm grasp on the geometric fixed point rings of equivariant bordism. It is well-known that the C -geometric fixed point ring $\Phi^C \mathcal{N}$ of equivariant bordism is a polynomial algebra over the non-equivariant bordism ring in infinitely many generators, one in every positive degree, see for example [13, Lemma 25.1]. A key ingredient for the proof of Theorem 7.14 is the fact that the specific classes β_n which were introduced purely in terms of the structure as an oriented el_2^{RO} -algebra in Definition 5.10 can be taken as such polynomial generators, see Theorem 7.6. Unfortunately, we do not know a direct argument to prove this fact, so we introduce auxiliary classes $\zeta_n \in \mathcal{N}_{n+1}^C$ represented by explicit C -manifolds in Construction 7.5. Then we show that the classes ζ_n for

$n \geq 1$, together with $t/a = \beta_0$ form polynomial generators for $\Phi^C \mathcal{N}$, and that β_n is congruent to ζ_n modulo $(\beta_0, \dots, \beta_{n-1})$, see Theorem 7.6 and Theorem 7.12.

Construction 7.1. We define a group G_n as the set \mathbb{Z}^n endowed with a group structure by

$$(7.2) \quad (x_1, \dots, x_n) \cdot (y_1, \dots, y_n) = (x_1 + y_1, (-1)^{y_1} \cdot x_2 + y_2, \dots, (-1)^{y_{n-1}} \cdot x_n + y_n) .$$

In other words, $G_1 = \mathbb{Z}$, and $G_n = G_{n-1} \times \mathbb{Z}$, the semidirect product formed with respect to the G_{n-1} -action on \mathbb{Z} via the sign of the last coordinate. We define a right action of the group G_n on \mathbb{R}^n by the same formula (7.2), but now with the elements x_i being real numbers as opposed to integers. This action by affine linear transformations is free, properly discontinuous and cocompact. So the orbit space \mathbb{R}^n/G_n is simultaneously a smooth closed n -manifold and a classifying space for the group G_n . The mod 2 cohomology of \mathbb{R}^n/G_n , and hence the mod 2 group cohomology of G_n , is thus a Poincaré duality algebra with duality class in dimension n . For example, $\mathbb{R}/G_1 = \mathbb{R}/\mathbb{Z}$ is a circle, and \mathbb{R}^2/G_2 is a Klein bottle. For $n \geq 2$, the manifold \mathbb{R}^n/G_n is not orientable.

The following calculation of the mod 2 cohomology algebra of the group G_n , or equivalently of the manifold \mathbb{R}^n/G_n , will enter into the proof that the classes ζ_n are polynomial generators of $\Phi_*^C \mathcal{N}$, compare Theorem 7.6. We expect that this calculation is well-known, but we were unable to find a reference. The cases $n = 1$ and $n = 2$ are classical, being the cohomology algebras of the circle and the Klein bottle. For $i = 1, \dots, n$, we denote by

$$p_i : G_n \longrightarrow \mathbb{F}_2$$

the projection to the i -th coordinate of G_n , taken modulo 2. We view these homomorphisms as cohomology classes in $H^1(G_n; \mathbb{F}_2) = \text{Hom}(G, \mathbb{F}_2)$.

Proposition 7.3. *The cohomology algebra $H^*(G_n; \mathbb{F}_2)$ has an \mathbb{F}_2 -basis consisting of the classes*

$$p_{i_1} \cdots p_{i_m}$$

for all $1 \leq i_1 < i_2 < \cdots < i_m \leq n$. The multiplicative structure is determined by the relations

$$p_i^2 = \begin{cases} 0 & \text{for } i = 1, \text{ and} \\ p_{i-1} \cdot p_i & \text{for } 2 \leq i \leq n. \end{cases}$$

In particular, $p_n^n = p_1 \cdot p_2 \cdots p_n$, and this class is a generator of $H^n(G_n; \mathbb{F}_2)$.

Proof. We argue by induction on n ; for $G_1 = \mathbb{Z}$, the result is well-known. Now we suppose that $n \geq 2$. We consider the cohomology algebra of G_n as module over the cohomology algebra of G_{n-1} by restriction along the projection

$$q : G_n = G_{n-1} \times \mathbb{Z} \longrightarrow G_{n-1}, \quad q(y_1, \dots, y_n) = (y_1, \dots, y_{n-1}) .$$

The mod 2 cohomology of \mathbb{Z} is concentrated in dimensions 0 and 1, where it is 1-dimensional. So the action of G_{n-1} on the mod 2 cohomology of \mathbb{Z} through p_{n-1} is necessarily trivial, and the Lyndon-Hochschild-Serre spectral sequence degenerates into a short exact sequence of graded $H^*(G_{n-1}; \mathbb{F}_2)$ -modules

$$0 \longrightarrow H^*(G_{n-1}; \mathbb{F}_2) \xrightarrow{q^*} H^*(G_n; \mathbb{F}_2) \xrightarrow{\delta} H^{*-1}(G_{n-1}; \mathbb{F}_2) \longrightarrow 0 .$$

Since $H^1(G_n; \mathbb{F}_2) = \text{Hom}(G_n, \mathbb{F}_2)$ is the direct sum of $q^*(H^1(G_{n-1}; \mathbb{F}_2))$ and the span of the homomorphism $p_n : G_n \longrightarrow \mathbb{F}_2$, the connecting homomorphism must satisfy $\delta(p_n) = 1$. Hence $H^*(G_n; \mathbb{F}_2)$ is free as a $H^*(G_{n-1}; \mathbb{F}_2)$ -module on the classes 1 and p_n . So the claim about the \mathbb{F}_2 -basis and those multiplicative relations that do not involve p_n follow by induction.

To prove the remaining multiplicative relation $p_n^2 = p_{n-1} \cdot p_n$, we use naturality for the group homomorphism

$$h : G_n \longrightarrow \mathbb{F}_2 \times \mathbb{Z}/4, \quad h(y_1, \dots, y_n) = (y_{n-1} + 2\mathbb{Z}, y_n + 4\mathbb{Z})$$

to the semidirect product of \mathbb{F}_2 acting by sign on $\mathbb{Z}/4$, also known as the dihedral group of order 8. The cohomology ring of this dihedral group is well-known, see for example [1, Chapter IV, Theorem 2.7]; we only need the information that the two homomorphisms

$$\alpha, \beta : \mathbb{F}_2 \times \mathbb{Z}/4 \longrightarrow \mathbb{F}_2$$

given by $\alpha(x, y + 4\mathbb{Z}) = x + y + 2\mathbb{Z}$ and $\beta(x, y + 4\mathbb{Z}) = y + 2\mathbb{Z}$ satisfy the cup product relation $\alpha \cdot \beta = 0$. Because $\alpha \circ h = p_{n-1} + p_n$ and $\beta \circ h = p_n$, naturality yields the desired multiplicative relation

$$p_{n-1} \cdot p_n + p_n^2 = (p_{n-1} + p_n) \cdot p_n = h^*(\alpha \cdot \beta) = 0. \quad \square$$

The next proposition gives a criterion for recognizing C -equivariant bordism classes as polynomial generators of $\Phi_*^C \mathcal{N}$. It is very similar to [7, Lemma 4] which gives a characteristic number criterion to detect when elements of $\mathcal{N}_*(BO)$ can serve as polynomial generators. This result ought to be well-known to experts, but we do not know of a reference in this form.

Proposition 7.4. *Let M be a smooth closed $(n+1)$ -dimensional C -manifold. Let F be the union of the n -dimensional components of M^C , and let $\nu : F \longrightarrow \mathbb{R}P^\infty$ classify the normal line bundle of F inside M . Suppose that $\nu_*[F] \neq 0$ in $H_n(\mathbb{R}P^\infty, \mathbb{F}_2)$, where $[F] \in H_n(F; \mathbb{F}_2)$ is the mod 2 fundamental class. Then $[M]/1$ is indecomposable in $\Phi_*^C \mathcal{N}$ as an algebra over \mathcal{N}_* .*

Proof. By the Conner–Floyd isomorphism (6.4)

$$\varphi_C : \Phi_*^C \mathcal{N} \xrightarrow{\cong} \bigoplus_{j \geq 0} \mathcal{N}_{*-j}(Gr_j),$$

the class $[M]/1$ is indecomposable in $\Phi_*^C \mathcal{N}$ if and only if the projection of $\varphi_C[M]$ to the linear summand $j = 1$ in the direct sum decomposition can serve as a generator in an \mathcal{N}_* -basis of $\mathcal{N}_*(\mathbb{R}P^\infty)$. By the geometric description of φ_C , the projection to the linear summand is represented by the pair (F, ν) consisting of the n -dimensional fixed point components and the map classifying their normal line bundles inside M . For every space X , the map

$$\mathcal{N}_*(X) \longrightarrow H_*(X; \mathbb{F}_2), \quad [f : W \longrightarrow X] \longmapsto f_*[W],$$

passes to an isomorphism

$$\mathcal{N}_*(X) \otimes_{\mathcal{N}_*} \mathbb{F}_2 \cong H_*(X; \mathbb{F}_2).$$

So for $X = \mathbb{R}P^\infty = Gr_1$, a pair (W, f) consisting of a smooth closed n -manifold W and a continuous map $f : W \longrightarrow \mathbb{R}P^\infty$ represents a generator in an \mathcal{N}_* -basis of $\mathcal{N}_*(\mathbb{R}P^\infty)$ if and only if $f_*[W]$ is the nonzero element of $H_n(\mathbb{R}P^\infty; \mathbb{F}_2)$. Altogether, this proves the proposition. \square

Construction 7.5 (The ζ -classes). We define a left action of the group G_{n-1} on the projective space $\mathbb{R}P^2$ by

$$(s_1, \dots, s_{n-1}) \cdot [x : y : z] = [x : y : (-1)^{s_{n-1}} \cdot z].$$

By taking balanced product with \mathbb{R}^{n-1} we obtain a smooth closed $(n+1)$ -manifold

$$W_{n+1} = \mathbb{R}^{n-1} \times_{G_{n-1}} \mathbb{R}P^2.$$

We define a smooth involution $\tau : W_{n+1} \longrightarrow W_{n+1}$ by

$$\tau[s_1, \dots, s_{n-1}, [x : y : z]] = [s_1, \dots, s_{n-1}, [-x : y : z]].$$

We write

$$\zeta_n = [W_{n+1}, \tau]/1 \in \Phi_{n+1}^C \mathcal{N}$$

for the bordism class of the C -manifold (W_{n+1}, τ) , or rather the effective class in the geometric fixed point ring that it represents.

Theorem 7.6. *The classes ζ_n for $n \geq 1$ and the class t/a form a set of polynomial generators of the C -geometric fixed points of equivariant bordism as an algebra over \mathcal{N}_* .*

Proof. We verify the criterion of Proposition 7.4. The fixed points of the involution $\tau : W_{n+1} \rightarrow W_{n+1}$ are

$$\begin{aligned} (W_{n+1})^C &= \mathbb{R}^{n-1} \times_{G_{n-1}} (\mathbb{R}P^2)^C \\ &= \mathbb{R}^{n-1} \times_{G_{n-1}} (P(0 \oplus \mathbb{R}^2) \cup \{\mathbb{R} \oplus 0 \oplus 0\}) \\ &\cong (\mathbb{R}^{n-1} \times_{G_{n-1}} P(0 \oplus \mathbb{R}^2)) \cup \mathbb{R}^{n-1}/G_{n-1}. \end{aligned}$$

This fixed point manifold has two connected components, one of dimension n and one of dimension $n-1$. For our present purposes, we only care about the n -dimensional component

$$\mathbb{R}^{n-1} \times_{G_{n-1}} P(0 \oplus \mathbb{R}^2),$$

and we can ignore the other one. To apply Proposition 7.4, we need to identify the normal line bundle of this component.

We let $p_n : G_n \rightarrow \{\pm 1\}$ denote the group homomorphism given by $p_n(s_1, \dots, s_n) = (-1)^{s_n}$. We embed the total space of the line bundle over \mathbb{R}^n/G_n associated to p_n into the manifold W_{n+1} by

$$\begin{aligned} \mathbb{R}^n \times_{G_n} p_n^*(\sigma) &\longrightarrow \mathbb{R}^{n-1} \times_{G_{n-1}} P(\mathbb{R}^3) = W_{n+1} \\ [s_1, \dots, s_n; u] &\longmapsto [s_1, \dots, s_{n-1}, [u : \sin(s_n \cdot \pi) : \cos(s_n \cdot \pi)]] . \end{aligned}$$

This map is a smooth open embedding with image $\mathbb{R}^{n-1} \times_{G_{n-1}} (\mathbb{R}P^2 \setminus \{\mathbb{R} \oplus 0 \oplus 0\})$, and it identifies the zero section \mathbb{R}^n/G_n in the source with the n -dimensional fixed point component $\mathbb{R}^{n-1} \times_{G_{n-1}} P(0 \oplus \mathbb{R}^2)$. So this embedding witnesses that the normal line bundle of $\mathbb{R}^{n-1} \times_{G_{n-1}} P(0 \oplus \mathbb{R}^2)$ inside W_{n+1} is the line bundle over \mathbb{R}^n/G_n associated to the character p_n .

Proposition 7.4 thus shows that the bordism class of W_{n+1} can serve as one of the polynomial generators for $\Phi_*^C \mathcal{N}$, provided we show that the map $\mathbb{R}^n/G_n \rightarrow \mathbb{R}P^\infty$ that classifies the line bundle associated to p_n is nonzero in mod 2 homology $H_n(-; \mathbb{F}_2)$. Or equivalently, the classifying map is nonzero in mod 2 cohomology $H^n(-; \mathbb{F}_2)$. Since the n -manifold \mathbb{R}^n/G_n is also a classifying space for the group G_n , we may show that the group homomorphism p_n induces a nontrivial map in mod 2 group cohomology in dimension n . By Proposition 7.3, the group cohomology class in $H^1(G_n; \mathbb{F}_2)$ represented by p_n has the property that its n -th cup power is the generator of the top cohomology group $H^n(G_n; \mathbb{F}_2)$. This concludes the argument. \square

Our next aim is to relate the classes β_n introduced in Definition 5.10 as the coefficients of the power series $d_C(t_\mu/a_\mu)$ to the geometrically defined classes ζ_n . To this end, we shall make use of the ‘division operator’.

Construction 7.7 (Division operator). We let (X, a, t) be an oriented el_2^{RO} -algebra. For an elementary abelian 2-group K and $k \in \mathbb{Z}$, we define an additive map

$$(7.8) \quad \Gamma : X(K \times C)_k \longrightarrow X(K \times C)_{k+1};$$

we refer to Γ as a *division operator*. We write $i_1 : K \rightarrow K \times C$ and $p_1 : K \times C \rightarrow K$ for the embedding of the first factor, and the projection to the first factor, respectively. For $u \in X(K \times C)_k$, we have the relation

$$i_1^*(t_2 \cdot (u + p_1^*(i_1^*(u)))) = i_1^*(u) + i_1^*(p_1^*(i_1^*(u))) = 0.$$

So there is a unique class $\Gamma(u) \in X(K \times C)_{k+1}$ such that

$$(7.9) \quad a_2 \cdot \Gamma(u) = t_2 \cdot (u + p_1^*(i_1^*(u)))$$

in the group $X(K \times C, (k+1) - p_2)$.

The following proposition relates the division operator to the power series expansion $d_K : \Phi^{K \times C} X \rightarrow (\Phi^K X)(\theta)$ that we introduced in Construction 5.4.

Proposition 7.10. *Let (X, a, t) be an oriented el_2^{RO} -algebra. For every elementary abelian 2-group K and all homogeneous elements $u \in X(K \times C)$, the power series expansion of $u/1$ is given by the formula*

$$d_K(u/1) = \sum_{n \geq 0} i_1^*(\Gamma^n(u))/1 \cdot \theta^n .$$

Proof. The class $u/1$ belongs to $\Psi^K X$, so the Laurent power series $d_K(u/1)$ is contained in the integral subring $(\Phi_*^K X)[[\theta]]$, by construction. We will show that the formula for $d_K(u/1)$ holds in the ring $(\Phi_*^K X)[[\theta]]$ modulo θ^m , for all $m \geq 0$. We argue by induction on m , starting with $m = 0$, where there is nothing to show. Now we suppose that $m \geq 1$. By the inductive hypothesis, $d_K(\Gamma(u)/1)$ is congruent modulo θ^{m-1} to

$$\sum_{n \geq 0} i_1^*(\Gamma^n(\Gamma(u)))/1 \cdot \theta^n .$$

Hence

$$\begin{aligned} d_K(u/1) \stackrel{(7.9)}{=} & d_K((p_1^*(i_1^*(u)) + e_2 \cdot \Gamma(u))/1) \\ &= i_1^*(u)/1 + \theta \cdot d_K(\Gamma(u)/1) \\ &\equiv i^*(u)/1 + \theta \cdot \sum_{n \geq 0} i_1^*(\Gamma^n(\Gamma(u)))/1 \cdot \theta \quad \text{modulo } \theta^m \\ &= \sum_{n \geq 0} i_1^*(\Gamma^n(u))/1 \cdot \theta^n \end{aligned}$$

This completes the inductive step, and hence the proof. \square

For equivariant bordism, the division operator $\Gamma : \mathcal{N}_k^{K \times C} \rightarrow \mathcal{N}_{k+1}^{K \times C}$ has a geometric interpretation that we now recall. Variations and special cases of the following proposition have been used in the literature on equivariant bordism, so we make no claim to originality. For example, the special case $K = 1$ is treated in [13, Lemma 25.3].

Proposition 7.11. *Let K be an elementary abelian 2-group. Let M be a k -dimensional smooth closed $(K \times C)$ -manifold. The class $\Gamma[M]$ is represented by the $(k+1)$ -manifold*

$$\mathbb{R} \times_{\mathbb{Z}} M = (\mathbb{R} \times M)/(x, m) \sim (x+1, (-1, 1) \cdot m)$$

with $(K \times C)$ -action by

$$(\kappa, \kappa') \cdot [x, m] = [\kappa' \cdot x, (\kappa, 1) \cdot m] .$$

Proof. The class $t_2 \cdot ([M] + p_1^*(i_1^*[M]))$ is represented by the $(K \times C)$ -manifold $S^{p_2} \times (M \amalg p_1^*(i_1^*M))$ with the first projection as reference map to S^{p_2} . The class $a_2 \cdot [\mathbb{R} \times_{\mathbb{Z}} M]$ is represented by the $(K \times C)$ -manifold $\mathbb{R} \times_{\mathbb{Z}} M$ with the constant map with value 0 as reference map to S^{p_2} . Since all a -classes are regular, the map $\tilde{\mathcal{N}}_*^{K \times C}(S^{p_2}) \rightarrow \Phi_*^{K \times C} \mathcal{N}$ is injective. Moreover, the fixed point map (6.4)

$$\varphi_{K \times C} : \Phi_*^{K \times C} \mathcal{N} \xrightarrow{\cong} \bigoplus_{j \geq 0} \mathcal{N}_{*-j}(Gr_j^{K \times C, \perp})$$

is bijective. So it suffices to show that the two representatives for classes in $\tilde{\mathcal{N}}_{k+1}^{K \times C}(S^{p_2})$ have the same fixed point invariants.

The $(K \times C)$ -fixed points of the mapping torus $\mathbb{R} \times_{\mathbb{Z}} M$ decompose into two parts:

$$(\mathbb{R} \times_{\mathbb{Z}} M)^{K \times C} \cong M^{K \times C} \cup M^{K \times 1}$$

The first summand is embedded via

$$M^{K \times C} \longrightarrow \mathbb{R} \times_{\mathbb{Z}} M, \quad x \longmapsto [1/2, x];$$

its normal bundle is isomorphic to the Whitney sum of the normal bundle of $M^{K \times C}$ inside M and the trivial line bundle associated with the representation p_2 . The second summand is embedded via

$$M^{K \times 1} \longrightarrow \mathbb{R} \times_{\mathbb{Z}} M, \quad x \longmapsto [0, x];$$

its normal bundle is isomorphic to the Whitney sum of the inflation along $p_1 : K \times C \longrightarrow K$ of the normal bundle of $M^{K \times 1}$ inside M , and the trivial line bundle associated with the representation p_2 . The $(K \times C)$ -fixed points of $S^{p_2} \times (M \amalg p_1^*(i_1^*M))$ over $0 \in S^{p_2}$ are

$$\{0\} \times (M \amalg p_1^*(i_1^*M))^{K \times C} \cong M^{K \times C} \cup M^{K \times 1};$$

these are diffeomorphic fixed points with isomorphic normal bundles. This proves the claim. \square

Theorem 7.12.

(i) In the oriented el_2^{RO} -algebra (\mathcal{N}, a, t) , the classes β_n and ζ_n are related by the formula

$$\zeta_n = \beta_n + \beta_0 \beta_{n-1}$$

in $\Phi_{n+1}^C \mathcal{N}$, for $n \geq 1$.

(ii) The classes β_n for $n \geq 0$ form a set of polynomial generators of the C -geometric fixed points of equivariant bordism as an algebra over \mathcal{N}_* .

Proof. (i) We write $P \in \mathcal{N}_2^{C \times C}$ for the bordism class of the projective space $\mathbb{R}P^2$ with $(C \times C)$ -action by

$$(\kappa, \kappa') \cdot [x : y : z] = [\kappa'x : y : \kappa z].$$

By Proposition 7.11 and induction on n , the class $\Gamma^n(P)$ in $\mathcal{N}_{n+2}^{C \times C}$ is represented by

$$\mathbb{R} \times_{\mathbb{Z}} (\mathbb{R}^{n-1} \times_{G_{n-1}} \mathbb{R}P^2) \cong \mathbb{R}^n \times_{G_n} \mathbb{R}P^2,$$

where G_n acts on $\mathbb{R}P^2$ by

$$(s_1, \dots, s_n) \cdot [x : y : z] = [x : y : (-1)^{s_n} z],$$

and where $C \times C$ acts on the balanced product by

$$(\kappa, \kappa') \cdot [s_1, \dots, s_n, [x : y : z]] = [\kappa s_1, s_2, \dots, s_n, [\kappa'x : y : z]].$$

Restricting the action along $i_1 : C \longrightarrow C \times C$ shows that $i_1^*(\Gamma^n(P))$ is represented by (W_{n+2}, τ) , i.e., $i_1^*(\Gamma^n(P)) = \zeta_{n+1}$.

Now we claim that the relation

$$(7.13) \quad P/1 = t_\mu/a_\mu \cdot t_2/a_2 + t_1/a_1 \cdot t_2/a_2 + t_1/a_1 \cdot t_\mu/a_\mu$$

holds in the group $\Phi_2^{C \times C} \mathcal{N}$. To prove this, we verify that the two classes have the same image under the fixed point isomorphism $\varphi_{C \times C}$, defined in (6.4), that records the fixed points and their normal data. If λ is a character of a finite group, and V a G -representation that does not contain λ , then λ is an isolated fixed point in the projective space $P(V \oplus \lambda)$. Moreover, the normal G -representation at this fixed point is isomorphic to $V \otimes \lambda$. The given $(C \times C)$ -action on $\mathbb{R}P^2$ is that of the projective space of $p_2 \oplus \mathbb{R} \oplus p_1$. It has three isolated fixed points, with normal representations $p_1 \oplus \mu$, $p_1 \oplus p_2$, and $\mu \oplus p_2$, respectively.

For a nontrivial A -character λ , the class $t_\lambda \in \widetilde{\mathcal{N}}_1^A(S^\lambda)$ is represented by the identity of S^λ , which has an isolated fixed point over the non-basepoint, with normal representation λ . So the fixed point isomorphism (6.4) takes the class $t_\lambda/a_\lambda \in \Phi_1^A \mathcal{N}$ to the element of $\mathcal{N}_0(Gr_1^{A,\perp})$ in the summand for $j = 1$ represented by λ , considered as a point in $Gr_1^{A,\perp}$. So the fixed point invariant of $P/1$ is

$$\begin{aligned} \varphi_{C \times C}(P/1) &= [\mu] \cdot [p_2] + [p_1] \cdot [p_2] + [p_1] \cdot [\mu] \\ &= \varphi_{C \times C}(t_\mu/a_\mu \cdot t_2/a_2 + t_1/a_1 \cdot t_2/a_2 + t_1/a_1 \cdot t_\mu/a_\mu). \end{aligned}$$

This proves the relation (7.13).

Now we apply the Laurent power series expansion $d_C : \Phi^{C \times C} \mathcal{N} \rightarrow (\Phi^C \mathcal{N})((\theta))$ to (7.13). The ring $\Phi^{C \times C} \mathcal{N}$ is an algebra over $\Phi^C \mathcal{N}$ via the inflation p_1^* , so $d_C(t_1/a_1) = d_C(p_1^*(t/a)) = t/a = \beta_0$. Moreover, $d_C(t_2/a_2) = \theta^{-1}$. So

$$\begin{aligned} d_C(P/1) &= d_C(t_\mu/a_\mu \cdot t_2/a_2 + t_1/a_1 \cdot t_2/a_2 + t_1/a_1 \cdot t_\mu/a_\mu) \\ &= (d_C(t_\mu/a_\mu) + \beta_0) \cdot \theta^{-1} + \beta_0 \cdot d_C(t_\mu/a_\mu) = \sum_{n \geq 0} (\beta_{n+1} + \beta_0 \beta_n) \cdot \theta^n. \end{aligned}$$

Proposition 7.10 yields the relation

$$d_C(P/1) = \sum_{n \geq 0} i_1^*(\Gamma^n(P))/1 \cdot \theta^n = \sum_{n \geq 0} \zeta_{n+1} \cdot \theta^n.$$

Comparing coefficients of θ^n yields $\zeta_{n+1} = \beta_{n+1} + \beta_0 \beta_n$ for all $n \geq 0$, as claimed.

(ii) The classes $t/a = \beta_0$ and $\zeta_n/1$ for $n \geq 1$ are polynomial \mathcal{N}_* -algebra generators of $\Phi^C \mathcal{N}$ by Theorem 7.6. Since β_n is congruent to ζ_n modulo $(\beta_0, \dots, \beta_{n-1})$, by part (i), also the classes β_n for $n \geq 0$ form polynomial generators for $\Phi_*^C \mathcal{N}$ as an \mathcal{N}_* -algebra. \square

We are now ready to prove the main theorem of this paper.

Theorem 7.14 (Universal property of equivariant bordism). *The oriented el_2^{RO} -algebra \mathcal{N} of equivariant bordism is an initial oriented el_2^{RO} -algebra.*

Proof. We need to show that there is a unique morphism of oriented el_2^{RO} -algebras from (\mathcal{N}, a, t) to any given oriented el_2^{RO} -algebra (Y, \bar{a}, \bar{t}) . In accordance with the notation for pre-Euler and inverse Thom classes, we shall write $\bar{\beta}_n$ for the β -classes for the theory (Y, \bar{a}, \bar{t}) .

If $f : \mathcal{N} \rightarrow Y$ is a morphism of oriented el_2^{RO} -algebras, then $f(1) : \mathcal{N}_* \rightarrow Y(1)_*$ must take the formal group law of (\mathcal{N}, a, t) to that of (Y, \bar{a}, \bar{t}) . The formal group of (\mathcal{N}, a, t) is initial by Theorem 6.5 (ii), so $f(1)$ is uniquely determined by this property. For an elementary abelian 2-group A , we make the geometric fixed point ring $\Phi_*^A Y$ into an \mathcal{N}_* -algebra via the composite of $f(1) : \mathcal{N}_* \rightarrow Y(1)_*$ and the inflation homomorphism $p_A^* : Y(1)_* \rightarrow \Phi_*^A Y$. The morphism $\Phi^C f : \Phi^C \mathcal{N} \rightarrow \Phi^C Y$ satisfies $(\Phi^C f) \circ p_C^* = p_C^* \circ f(1)$, so it is a morphism of graded \mathcal{N}_* -algebras. Since the β -classes are defined intrinsically via the structure of oriented el_2^{RO} -algebras, the relation $(\Phi^C f)(\beta_n) = \bar{\beta}_n$ holds. By Theorem 7.12 (ii), the classes β_n generate $\Phi_*^C \mathcal{N}$ as an \mathcal{N}_* -algebra, so the morphism $\Phi^C f$ is uniquely determined.

Now we let A be any elementary abelian 2-group. Then for every nontrivial A -character λ , the relation $(\Phi^A f) \circ \lambda^* = \lambda^* \circ (\Phi^C f)$ holds. Since $\Phi^C f$ is uniquely determined and $\Phi^A \mathcal{N}$ is generated as an \mathcal{N}_* -algebra by the images of the homomorphisms $\lambda^* : \Phi^C \mathcal{N} \rightarrow \Phi^A \mathcal{N}$, by Theorem 6.5 (iii), also the morphism $\Phi^A f$ is uniquely determined. Since the maps $-/a_V : Y(A, k - V) \rightarrow \Phi_k^A Y$ are injective, the morphism $\Phi^A f$ uniquely determines the morphism of I_A -graded rings $f(A, \star) : \mathcal{N}(A, \star) \rightarrow Y(A, \star)$. This concludes the proof of uniqueness.

To construct a morphism of oriented el_2^{RO} -algebras from (\mathcal{N}, a, t) to (Y, \bar{a}, \bar{t}) , we turn the uniqueness argument around. Since the 2-torsion formal group law of the oriented el_2^{RO} -algebra

(\mathcal{N}, a, t) is initial, there is a unique homomorphism of graded \mathbb{F}_2 -algebras $f^1: \mathcal{N}_* \rightarrow Y(1)_*$ that classifies the 2-torsion formal group law of (Y, \bar{a}, \bar{t}) . In other words, f^1 is the unique morphism that when applied to the coefficients of the 2-torsion formal group law of (\mathcal{N}, a, t) yields the one of (Y, \bar{a}, \bar{t}) .

Again we make the geometric fixed point ring $\Phi^A Y$ into an \mathcal{N} -algebra via the composite of $f^1: \mathcal{N}_* \rightarrow Y(1)_*$ and the inflation homomorphism $p_A^*: Y(1)_* \rightarrow \Phi^A Y$. By Theorem 7.6, the geometric fixed points $\Phi_*^C \mathcal{N}$ are polynomial over \mathcal{N}_* on the classes β_n . So there is a unique \mathcal{N}_* -algebra homomorphism

$$f^C : \Phi^C \mathcal{N} \rightarrow \Phi^C Y$$

that sends the class β_n in $\Phi^C \mathcal{N}$ to the class $\bar{\beta}_n$ in $\Phi^C Y$, for all $n \geq 0$.

Now we let λ be a nontrivial character of an elementary abelian 2-group A . Because $f^C \circ p_C^* = p_C^* \circ f^1$, the composite of

$$\mathcal{N}_* \xrightarrow{p_C^*} \Phi^C \mathcal{N} \xrightarrow{f^C} \Phi^C Y \xrightarrow{\lambda^*} \Phi^A Y$$

equals $p_A^* \circ f^1: \mathcal{N}_* \rightarrow \Phi^A Y$. In particular, the composite $\lambda^* \circ f^C \circ p_C^*$ is independent of λ . By Theorem 6.5 (iii), $\Phi^A \mathcal{N}$ is the coproduct, in the category of commutative graded \mathcal{N}_* -algebras, of copies of $\Phi^C \mathcal{N}$, indexed by the set A° of nontrivial characters. So there is a unique morphism of graded rings

$$f^A : \Phi^A \mathcal{N} \rightarrow \Phi^A Y$$

such that $f^A \circ \lambda^* = \lambda^* \circ f^C$ for all nontrivial A -characters.

We will now show that the collection of morphisms $f^\bullet = \{f^A : \Phi^A \mathcal{N} \rightarrow \Phi^A Y\}_A$ satisfies the hypotheses of Theorem 5.14. To show that the morphisms are compatible with inflations, we let $\alpha: B \rightarrow A$ be any epimorphism between elementary abelian 2-groups. Then

$$f^B \circ \alpha^* \circ \lambda^* = f^B \circ (\lambda \alpha)^* = (\lambda \alpha)^* \circ f^C = \alpha^* \circ \lambda^* \circ f^C = \alpha^* \circ f^A \circ \lambda^*$$

for every nontrivial A -character λ . By Theorem 6.5 (iii), the images of the homomorphisms $\lambda^*: \Phi^C \mathcal{N} \rightarrow \Phi^A \mathcal{N}$ generate $\Phi^A \mathcal{N}$ as an \mathcal{N}_* -algebra, so this proves that $f^B \circ \alpha^* = \alpha^* \circ f^A$.

Condition (a) of Theorem 5.14 holds because $f^C(t/a) = f^C(\beta_0) = \bar{\beta}_0 = \bar{t}/\bar{a}$. Verifying condition (b) is the most involved part of this proof, and we establish it through a sequence of claims.

Claim 1: The following diagram commutes:

$$\begin{array}{ccc} \Phi^C \mathcal{N} & \xrightarrow{d_1} & \mathcal{N}_*((\theta)) \\ f^C \downarrow & & \downarrow f^1((\theta)) \\ \Phi^C Y & \xrightarrow{d_1} & Y(1)((\theta)) \end{array}$$

Theorem 5.11 (i) provides the relation

$$F_{\mathcal{N}}(\theta, \xi) \cdot \sum_{n \geq 0} d_1(\beta_n) \cdot \xi^n = 1$$

in the ring $\mathcal{N}((\theta))[[\xi]]$. We apply the homomorphism $f^1((\theta))[[\xi]]$ to this equation and exploit that the pushforward of the formal group law $F_{\mathcal{N}}$ along $f^1: \mathcal{N}_* \rightarrow Y(1)_*$ is F_Y . This yields

$$F_Y(\theta, \xi) \cdot \sum_{n \geq 0} f^1((\theta))(d_1(\beta_n)) \cdot \xi^n = 1 = F_Y(\theta, \xi) \cdot \sum_{n \geq 0} d_1(\bar{\beta}_n) \cdot \xi^n$$

in the ring $Y(1)((\theta))[[\xi]]$. The second equality is Theorem 5.11 (i) for (Y, \bar{a}, \bar{t}) . Since multiplicative inverses are unique, the two infinite sums on both sides must be equal. Comparing

coefficients of ξ^n yields $f^1((\theta))(d_1(\beta_n)) = d_1(\bar{\beta}_n) = d_1(f^C(\beta_n))$. Since the classes β_n generate $\Phi^C \mathcal{N}$ as an \mathcal{N}_* -algebra, this proves that the diagram of Claim 1 indeed commutes.

Claim 2: The following diagram commutes:

$$\begin{array}{ccccc} \Phi^C \mathcal{N} & \xrightarrow{\mu^*} & \Phi^{C \times C} \mathcal{N} & \xrightarrow{d_C} & (\Phi^C \mathcal{N})(\theta) \\ f^C \downarrow & & & & \downarrow f^C((\theta)) \\ \Phi^C Y & \xrightarrow{\mu^*} & \Phi^{C \times C} Y & \xrightarrow{d_C} & (\Phi^C Y)(\theta) \end{array}$$

Here, as before, $\mu : C \times C \rightarrow C$ is the multiplication. Theorem 5.11 (ii) provides the relation

$$\sum_{n \geq 0} d_C(\mu^*(\beta_n)) \cdot \xi^n = \sum_{n \geq 0} \beta_n \cdot F_{\mathcal{N}}(\theta, \xi)^n$$

in the ring $(\Phi^C \mathcal{N})[[\theta, \xi]]$. We apply the homomorphism $f^C[[\theta, \xi]]$ to this equation and exploit that the pushforward of the formal group law $F_{\mathcal{N}}$ along $f^1 : \mathcal{N}_* \rightarrow Y(1)_*$ is F_Y . This yields

$$\begin{aligned} \sum_{n \geq 0} f^C[[\theta]](d_C(\mu^*(\beta_n))) \cdot \xi^n &= \sum_{n \geq 0} f^C(\beta_n) \cdot F_Y(\theta, \xi)^n \\ &= \sum_{n \geq 0} \bar{\beta}_n \cdot F_Y(\theta, \xi)^n = \sum_{n \geq 0} d_C(\mu^*(\bar{\beta}_n)) \cdot \xi^n . \end{aligned}$$

The third equality is Theorem 5.11 (ii) for (Y, \bar{a}, \bar{t}) . Comparing coefficients of ξ^n yields

$$f^C[[\theta]](d_C(\mu^*(\beta_n))) = d_C(\mu^*(\bar{\beta}_n)) = d_C(\mu^*(f^C(\beta_n))) .$$

Because the classes β_n generate $\Phi_*^C \mathcal{N}$ as an \mathcal{N}_* -algebra, this proves that the diagram of Claim 2 indeed commutes.

Claim 3: For every elementary abelian 2-group K and every nontrivial character $\lambda : K \times C \rightarrow C$, the following diagram commutes:

$$\begin{array}{ccccc} \Phi^C \mathcal{N} & \xrightarrow{\lambda^*} & \Phi^{K \times C} \mathcal{N} & \xrightarrow{d_K} & (\Phi^K \mathcal{N})(\theta) \\ f^C \downarrow & & & & \downarrow f^K((\theta)) \\ \Phi^C Y & \xrightarrow{\lambda^*} & \Phi^{K \times C} Y & \xrightarrow{d_K} & (\Phi^K Y)(\theta) \end{array}$$

We distinguish three cases, each based on a different kind of naturality argument. Firstly, if $\lambda = p_2$ is the projection to the second factor, then $d_K \circ \lambda^* = p_K^*((\theta)) \circ d_1$ by naturality of the power series expansions d_K in the group K . So

$$\begin{aligned} f^K((\theta)) \circ d_K \circ \lambda^* &= f^K((\theta)) \circ p_K^*((\theta)) \circ d_1 = p_K^*((\theta)) \circ f^1((\theta)) \circ d_1 \\ \text{Claim 1} &= p_K^*((\theta)) \circ d_1 \circ f^C = d_K \circ \lambda^* \circ f^C . \end{aligned}$$

Secondly, we suppose that $\lambda = \nu \circ p_1$ for some nontrivial K -character $\nu : K \rightarrow C$. By design, the composite $d_K \circ p_1^* : \Phi^K \mathcal{N} \rightarrow \Phi^{K \times C} \mathcal{N}$ is the inclusion $\iota : \Phi^K \mathcal{N} \rightarrow (\Phi^K \mathcal{N})(\theta)$ as the constant Laurent power series. So

$$d_K \circ \lambda^* = d_K \circ p_1^* \circ \nu^* = \iota \circ \nu^* ,$$

and hence

$$f^K((\theta)) \circ d_K \circ \lambda^* = f^K((\theta)) \circ \iota \circ \nu^* = \iota \circ f^K \circ \nu^* = \iota \circ \nu^* \circ f^C = d_K \circ \lambda^* \circ f^C .$$

The third and final case is when $\lambda = \mu \circ (\nu \times C)$ for a nontrivial K -character ν . Then

$$d_K \circ \lambda^* = d_K \circ (\nu \times C)^* \circ \mu^* = \nu^*((\theta)) \circ d_C \circ \mu^*$$

by naturality of the power series expansions d_K in the group K . So

$$f^K((\theta)) \circ d_K \circ \lambda^* = f^K((\theta)) \circ \nu^*((\theta)) \circ d_C \circ \mu^* = \nu^*((\theta)) \circ f^C((\theta)) \circ d_C \circ \mu^*$$

$$\text{Claim 2} = \nu^*((\theta)) \circ d_C \circ \mu^* \circ f^C = d_K \circ \lambda^* \circ f^C .$$

Now we can prove condition (b) of Theorem 5.14. For any nontrivial character $\lambda : K \times C \rightarrow C$, Claim 3 and the fact that $f^{K \times C} \circ \lambda^* = \lambda^* \circ f^C$ show that the two homomorphisms

$$f^K((\theta)) \circ d_K, d_K \circ f^{K \times C} : \Phi^{K \times C} \mathcal{N} \rightarrow (\Phi^K Y)((\theta))$$

agree after precomposition with $\lambda^* : \Phi^C \mathcal{N} \rightarrow \Phi^{K \times C} \mathcal{N}$. By Theorem 6.5 (iii), the images of these homomorphisms λ^* generate $\Phi^{K \times C} \mathcal{N}$ as an \mathcal{N}_* -algebra, so this proves the desired relation $f^K((\theta)) \circ d_K = d_K \circ f^{K \times C}$. \square

8. INVERTIBLY ORIENTED el_2^{RO} -ALGEBRAS AND GLOBAL 2-TORSION GROUP LAWS

In this section we explain the relationship between the oriented el_2^{RO} -algebras discussed in this paper and the global 2-torsion group laws in the sense of the first author's paper [17]. The bottom line is that global 2-torsion group laws 'are' oriented el_2^{RO} -algebras whose inverse Thom class is invertible. Because inverse Thom classes can be universally inverted as described in Construction 4.1, this exhibits the category of global 2-torsion group laws as a reflective subcategory of the category of oriented el_2^{RO} -algebras.

Definition 8.1 (Invertible inverse Thom classes). An inverse Thom class t of an orientable el_2^{RO} -algebra X is *invertible* if for every nontrivial character λ of an elementary abelian 2-group A and all $m \in I_A$, the multiplication map

$$t_\lambda \cdot - : X(A, m) \rightarrow X(A, m + 1 - \lambda)$$

is an isomorphism. An *invertibly oriented* el_2^{RO} -algebra is an oriented el_2^{RO} -algebra whose inverse Thom class is invertible.

An invertible inverse Thom class essentially collapses the I_A -gradings to integer gradings: for every elementary abelian 2-group A , the I_A -graded ring $X(A, \star)$ is a polynomial ring over its \mathbb{Z} -graded subring in the variables t_λ for $\lambda \in A^\circ$:

$$X(A, \star) = X(A)_* [t_\lambda : \lambda \in A^\circ] .$$

In the presence of an invertible inverse Thom class t , one can therefore translate the structure and properties of an orientable el_2^{RO} -algebra into structure and properties of the el_2 -algebra made up by the integer graded subrings. For orientable el_2^{RO} -algebras arising from global ring spectra, invertibility in the sense of Definition 8.1 is equivalent to invertibility in the $RO(C)$ -graded homotopy ring, see Proposition 8.4.

We recall from [17, Definition 4.1] that a *global 2-torsion group law* is an $\text{el}_2\text{-}\mathbb{F}_2$ -algebra \mathcal{G} equipped with a *coordinate*, i.e., a class $e \in \mathcal{G}(C)$ such that for every nontrivial character λ of an elementary abelian 2-group A , the sequence

$$0 \rightarrow \mathcal{G}(A)_{*+1} \xrightarrow{e_\lambda \cdot -} \mathcal{G}(A)_* \xrightarrow{\text{res}_K^A} \mathcal{G}(K)_* \rightarrow 0$$

is exact, where $e_\lambda = \lambda^*(e)$ and K is the kernel of λ .

As described in Construction 4.1, we can turn an oriented el_2^{RO} -algebra (X, a, t) into an $\text{el}_2\text{-}\mathbb{F}_2$ -algebra $t^{-1}X$ by inverting all t -classes. When $A = 1$ is a trivial group, we have $(t^{-1}X)(1)_* = X(1)_*$.

Theorem 8.2.

- (i) For every oriented el_2^{RO} -algebra (X, a, t) , the el_2 -algebra $t^{-1}X$ is a global 2-torsion group law with respect to the coordinate $e = a/t$.
- (ii) The functor

$$t^{-1} : (\text{oriented } \mathrm{el}_2^{RO}\text{-algebras}) \longrightarrow (\text{global 2-torsion group laws})$$

$$(X, a, t) \longmapsto (t^{-1}X, a/t)$$

has a fully faithful right adjoint. The essential image of the right adjoint is the subcategory of invertibly oriented el_2^{RO} -algebras.

Proof. (i) We have $e_\lambda = \lambda^*(a/t) = a_\lambda/t_\lambda$ in $(t^{-1}X)(A)_{-1}$. Because localization is exact, the short exact sequences

$$0 \longrightarrow X(A, m+1) \xrightarrow{a_\lambda \cdot -} X(A, m+1-\lambda) \xrightarrow{\mathrm{res}_K^A} X(K, \mathrm{res}_K^A(m)) \longrightarrow 0$$

for varying $m \in I_A$ become a short exact sequence

$$0 \longrightarrow (t^{-1}X)(A)_{|m|+1} \xrightarrow{e_\lambda \cdot -} (t^{-1}X)(A)_{|m|} \xrightarrow{\mathrm{res}_K^A} (t^{-1}X)(K)_{|m|} \longrightarrow 0 .$$

This verifies the defining exactness property of a global 2-torsion group law.

- (ii) The right adjoint

$$\psi : (\text{global 2-torsion group laws}) \longrightarrow (\text{oriented } \mathrm{el}_2^{RO}\text{-algebras})$$

to t^{-1} is obtained as follows. Given any global 2-torsion group law (\mathcal{G}, e) , we define an el_2^{RO} -algebra $\mathcal{G}[t]$ at A by

$$\mathcal{G}[t](A, \star) = \mathcal{G}(A)[t_\lambda : \lambda \in A^\circ] .$$

Here each t_λ is a polynomial variable in degree $(A, 1 - \lambda)$, for each nontrivial A -character λ . If $\alpha : B \rightarrow A$ is a homomorphism between elementary abelian 2-groups, we define $\beta^* : X(A, \star) \rightarrow X(B, \star)$ as the given restriction homomorphism $\beta^* : \mathcal{G}(A) \rightarrow \mathcal{G}(B)$ on coefficient rings, and by setting

$$\beta^*(t_\lambda) = \begin{cases} t_{\lambda\beta} & \text{if } \lambda\beta \text{ is nontrivial, and} \\ 1 & \text{if } \lambda\beta \text{ is trivial.} \end{cases}$$

We omit the straightforward verification that these data define an el_2^{RO} -algebra. By design, the polynomial generator t in $\mathcal{G}[t](C, \star) = \mathcal{G}(C)[t]$ is an invertible inverse Thom class of this el_2^{RO} -algebra. Moreover, the exactness property of the orientation e shows that the class

$$a = e \cdot t \in \mathcal{G}(C)[t]_{-\sigma} = \mathcal{G}[t](C, -\sigma) .$$

is a pre-Euler class, so $\psi(\mathcal{G}, e) = (\mathcal{G}[t], et, t)$ is an invertibly oriented el_2^{RO} -algebra.

The unit of the adjunction $\eta_X : (X, a, t) \rightarrow \psi(t^{-1}X, a/t) = ((t^{-1}X)[t], (a/t) \cdot t, t)$ is defined at an elementary abelian 2-group A by

$$\eta_X(A, k - V)(x) = (x/t_V) \cdot t_V ,$$

where $k \in \mathbb{Z}$, and V is an A -representation with trivial fixed points. We omit the straightforward verification that this indeed defines a morphism of oriented el_2^{RO} -algebras. Moreover, if the inverse Thom class t is invertible, then η_X is an isomorphism.

Since the orientation of $\psi(\mathcal{G}, e)$ is invertible, localization away from t has no effect on the underlying rings, and the restriction of $\mathcal{G}[t]$ to the integer graded subrings returns the original el_2 -algebra \mathcal{G} , with its original coordinate $(et)/t = e$. More formally: the functor ψ is right inverse to the functor t^{-1} , and a natural isomorphism $\mathcal{G} \rightarrow t^{-1}(\psi(\mathcal{G}, e))$ is given by the

identifications of the rings $\mathcal{G}(A)$ as the integer graded subrings of $\mathcal{G}(A)[t]$. We omit the verification that the inverse isomorphism $\epsilon_{\mathcal{G}} : t^{-1}(\psi(\mathcal{G}, e)) \rightarrow \mathcal{G}$ and the natural transformation $\eta : \text{Id} \rightarrow \psi \circ t^{-1}$ satisfy the triangle equalities of an adjunction. In any adjunction for which the counit is a natural isomorphism, the right adjoint is fully faithful.

The right adjoint ψ takes values in invertibly oriented el_2^{RO} -algebras. And if (X, a, t) is an invertibly oriented el_2^{RO} -algebra, then the adjunction unit $(X, a, t) \rightarrow \psi(t^{-1}X, a/t)$ is an isomorphism at all integer graded subrings, and hence an isomorphism of el_2^{RO} -algebras. Thus (X, a, t) is in the essential image of the right adjoint. \square

Remark 8.3. As explained by the first author in [17], every global 2-torsion group law has a 2-torsion formal group law over its underlying ring. Every oriented el_2^{RO} -algebra also comes with a formal group law over its underlying ring, as explained in Theorem 4.7. Comparing definitions reveals that for every oriented el_2^{RO} -algebra (X, a, t) , the formal group law of $(t^{-1}X, a/t)$ defined in [17] is the one that we introduced in Theorem 4.7.

The formal group law construction from a global 2-torsion group law also admits a fully faithful right adjoint; the right adjoint embeds the category of 2-torsion formal group laws as the full subcategory of *complete* global 2-torsion group laws, compare [17, Example 4.7]. Altogether, the category of 2-torsion formal group laws embeds fully faithfully into the category of oriented el_2^{RO} -algebras, as the ones that are both invertibly oriented and complete. The following diagram of categories and functors summarizes the situation:

$$\begin{array}{ccc}
 \text{(2-torsion formal group laws)} & \xrightarrow[\cong]{\psi} & \text{(complete invertibly oriented } \text{el}_2^{RO}\text{-algebras)} \\
 \uparrow \int \downarrow \dashv & & \uparrow \int \downarrow \dashv \\
 \text{(global 2-torsion group laws)} & \xrightarrow[\cong]{\psi} & \text{(invertibly oriented } \text{el}_2^{RO}\text{-algebras)} \\
 & & \uparrow \int \downarrow \dashv \\
 & & \text{(oriented } \text{el}_2^{RO}\text{-algebras)}
 \end{array}$$

In Definition 8.1 we defined when an inverse Thom class of an el_2^{RO} -algebra is invertible. For global ring spectra, invertibility is equivalent to invertibility in the $RO(C)$ -graded homotopy ring:

Proposition 8.4. *For every oriented global ring spectrum (E, t) , the following conditions are equivalent.*

- (a) *The class t is an $RO(C)$ -graded unit, i.e., there exists a class $s \in \pi_{\sigma_{-1}}^C(E)$ such that $s \cdot t = 1$.*
- (b) *The class t is an invertible as an inverse Thom class of the orientable el_2^{RO} -algebra (E^{\sharp}, a) .*

Proof. (a) \implies (b) We let $\lambda : A \rightarrow C$ be any nontrivial character of an elementary abelian 2-group. Then $\lambda^*(s) \in E_{\lambda_{-1}}^A = \pi_{\lambda_{-1}}^A(E)$ is a homotopy class such that

$$\lambda^*(s) \cdot t_{\lambda} = \lambda^*(s \cdot t) = 1.$$

We write $n \in I_A$ as $n = k - V$ for $k \in \mathbb{Z}$ and some A -representation V with trivial fixed points. Then the multiplication map

$$- \cdot \lambda^*(s) : E_{k+1}^A(S^{V \oplus \lambda}) \rightarrow E_k^A(S^V)$$

is inverse to

$$- \cdot t_{\lambda} : E^{\sharp}(A, n) = E_k^A(S^V) \rightarrow E_{k+1}^A(S^{V \oplus \lambda}) = E^{\sharp}(A, n + 1 - \lambda).$$

In particular, multiplication by t_λ is bijective.

(b) \implies (a) We consider the commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
\pi_0^C(E) & \xrightarrow{\text{res}_1^C} & \pi_0(E) & \xrightarrow{\text{tr}_1^C} & \pi_{\sigma-1}^C(E) & \xrightarrow{a} & \pi_{-1}^C(E) & \xrightarrow{\text{res}_1^C} & \pi_{-1}(E) \\
\downarrow -t & & \parallel & & \downarrow -t & & \downarrow -t & & \parallel \\
\pi_{1-\sigma}^C(E) & \xrightarrow{\text{res}_1^C} & \pi_0(E) & \xrightarrow{\text{tr}_1^C} & \pi_0^C(E) & \xrightarrow{a} & \pi_{-\sigma}^C(E) & \xrightarrow{\text{res}_1^C} & \pi_{-1}(E)
\end{array}$$

Of the three maps labeled as multiplication by t , the outer two lie in the range graded by the submonoid I_C of $RO(C)$ that is encoded by the el_2^{RO} -algebra $E^\#$. Hence those are isomorphisms by the invertibility hypothesis (a). By the five lemma, the middle map is thus an isomorphism, too. So the multiplicative unit 1 in $\pi_0^C(E)$ has an inverse, which is the desired $RO(C)$ -graded inverse of t . \square

Construction 8.5 (Stable equivariant bordism). For a compact Lie group G , *stable equivariant bordism* is a G -equivariant homology theory $\mathfrak{N}_*^{G:S}$ first considered by Bröcker and Hook [9, §2]. It is a specific localization of G -equivariant bordism at geometrically defined classes associated to G -representations. If we restrict attention to elementary abelian 2-groups A , then the localization that defines stable equivariant bordism is exactly inverting the inverse Thom classes for the oriented el_2^{RO} -algebra \mathcal{N} of equivariant bordism:

$$\mathfrak{N}_*^{A:S} = (t^{-1}\mathcal{N})(A)_*$$

Equivariant bordism \mathcal{N} is an initial oriented el_2^{RO} -algebra by Theorem 7.14; and the localization morphism $\mathcal{N} \rightarrow t^{-1}\mathcal{N}$ is initial among morphisms of oriented el_2^{RO} -algebras that send the unique orientation of \mathcal{N} to an invertible inverse Thom class. Together, these universal properties immediately yield the following corollary about stable equivariant bordism:

Corollary 8.6 (Universal property of stable equivariant bordism). *Stable equivariant bordism $(t^{-1}\mathcal{N}, a, t)$ is an initial invertibly oriented el_2^{RO} -algebra. The global 2-torsion group law of stable equivariant bordism is an initial global 2-torsion group law.*

Combining Corollary 8.6 with the main result of Bröcker and Hook [9] yields an independent proof of a recent result of the first author, namely that the global 2-torsion group law carried by the global Thom spectrum \mathbf{MO} is initial, see [17, Theorem D]. Indeed, Theorem 4.1 of [9] says that for every compact Lie group G , stable equivariant bordism is isomorphic to the equivariant homology theory represented by the global Thom spectrum \mathbf{MO} of [25, Example 6.1.7]; see also [25, Remark 6.2.38] for a different proof. The underlying G -spectra of the global object \mathbf{MO} are the real analogues of tom Dieck's ‘homotopical equivariant bordism’ spectra [32], and they have been much studied since the 70’s. So via [9, Theorem 4.1], our Corollary 8.6 becomes the statement that the global 2-torsion group law associated to \mathbf{MO} is initial; in other words, we recover Theorem D of [17]. We want to emphasize that this earlier result of the first author was a key motivation for the present project, and some of our present arguments are inspired by arguments from [17]. Our proof here differs from the one in [17] in that we make no use of equivariant formal groups.

Example 8.7 (Mod 2 global Borel homology). The forgetful functor from the global stable homotopy category to the non-equivariant stable homotopy category has a lax symmetric monoidal right adjoint, the *global Borel* functor $b : \mathcal{SH} \rightarrow \mathcal{GH}$, see Theorem 4.5.1 and Construction 4.5.21 of [25]. For every spectrum R and every compact Lie group G , the underlying G -spectrum of the global Borel spectrum bR represents G -equivariant Borel R -cohomology,

i.e., R -cohomology of the homotopy orbit construction, thence the name. Because the Borel functor is lax symmetric monoidal, it takes non-equivariant homotopy ring spectra to global ring spectra.

In particular, the global Borel spectrum $b(H\mathbb{F}_2)$ of the non-equivariant mod 2 Eilenberg-MacLane spectrum is a global ring spectrum. The adjunction unit $\eta : H\mathbb{F}_2 \rightarrow b(H\mathbb{F}_2)$ from the mod 2 global Eilenberg-MacLane spectrum is a morphism of global ring spectra. Since $H\mathbb{F}_2$ is orientable, so is $b(H\mathbb{F}_2)$. The group $\pi_1^C(b(H\mathbb{F}_2))$ is isomorphic to $H^{-1}(BC, \mathbb{F}_2)$, and hence trivial. So the orientable global ring spectrum $b(H\mathbb{F}_2)$ has a unique inverse Thom class, which is necessarily the image of the unique inverse Thom class of $H\mathbb{F}_2$ under the morphism $\eta : H\mathbb{F}_2 \rightarrow b(H\mathbb{F}_2)$. Since the unique inverse Thom class of $H\mathbb{F}_2$ is additive, the unique inverse Thom class of $b(H\mathbb{F}_2)$ is additive, too.

All real vector bundles are orientable in mod 2 cohomology. In particular, there is a Thom class in $H^1(S^\sigma \wedge_C EC_+; \mathbb{F}_2)$ for the line bundle over BC associated to the sign representation. By the Thom isomorphism, the image of this Thom class under the isomorphism

$$H^1(S^\sigma \wedge_C EC_+; \mathbb{F}_2) \cong \pi_{\sigma-1}^C(b(H\mathbb{F}_2))$$

is an $RO(C)$ -graded inverse to the unique inverse Thom class. So the unique inverse Thom class of $b(H\mathbb{F}_2)$ is invertible, by Proposition 8.4.

Theorem 8.8 (Universal property of mod 2 Borel homology). *Let Y be an orientable el_2^{RO} -algebra. Evaluation at the unique inverse Thom class of the global Borel spectrum $b(H\mathbb{F}_2)$ is a bijection between the set of morphisms of orientable el_2^{RO} -algebras from $b(H\mathbb{F}_2)^\sharp$ to Y , and the set of orientations of Y that are both additive and invertible.*

Proof. We claim that the morphism of oriented el_2^{RO} -algebras

$$\eta^\sharp : H = ((H\mathbb{F}_2)^\sharp, a, t) \rightarrow (b(H\mathbb{F}_2)^\sharp, a, t)$$

exhibits the target as the localization of the source away from its unique inverse Thom class; more precisely: the induced morphism of global 2-torsion group laws $t^{-1}(\eta^\sharp)$ obtained by localization away from t is an isomorphism. To prove this, we write $b(H\mathbb{F}_2)^\natural$ for the el_2 -algebra made from the integer graded homotopy rings of the global Borel spectrum $b(H\mathbb{F}_2)$. Because t acts invertibly in the global Borel theory, we must show that the unique extension

$$\tilde{\eta} : t^{-1}H \rightarrow b(H\mathbb{F}_2)^\natural$$

of η^\sharp is an isomorphism of el_2 -algebras. To this end, it suffices to show that $\tilde{\eta}$ is an isomorphism in integer gradings for the elementary abelian 2-groups C^n for all $n \geq 0$. The I_A -graded ring $H(A, \star)$ is generated as an \mathbb{F}_2 -algebra by the classes a_λ and t_λ for all nontrivial A -characters λ , by Proposition 1.9. So the integer graded ring $(t^{-1}H)(A)_*$ is generated by the classes $e_\lambda = a_\lambda/t_\lambda$ for all nontrivial A -characters λ . If three such characters satisfy a multiplicative relation $\alpha \cdot \beta \cdot \gamma = 1$, then additivity of the orientation yields the relation

$$a_\alpha t_\beta t_\gamma + t_\alpha a_\beta t_\gamma + t_\alpha t_\beta a_\gamma = 0.$$

Dividing by $t_\alpha t_\beta t_\gamma$ yields the relation $e_\alpha + e_\beta + e_\gamma = 0$ in $(t^{-1}H)(A)_{-1}$. In particular, for all $n \geq 0$, the integer graded ring $(t^{-1}H)(C^n)$ is generated by the Euler classes e_1, \dots, e_n of the projections $p_i : C^n \rightarrow C$ to the n factors.

On the other hand, $b(H\mathbb{F}_2)^\natural(C^n)$ is the mod 2 cohomology ring of $B(C^n)$, which is an \mathbb{F}_2 -polynomial algebra on the corresponding Euler classes e_1, \dots, e_n of the projections. So the map $\tilde{\eta}(C^n) : (t^{-1}H)(C^n) \rightarrow b(H\mathbb{F}_2)^\natural(C^n)$ is surjective. Since the Euler classes e_1, \dots, e_n do not satisfy any nontrivial polynomial relations in the target, the Euler classes do not satisfy any nontrivial polynomial relations in the source either. So both rings are \mathbb{F}_2 -polynomial algebras on the Euler classes e_1, \dots, e_n , and the map is an isomorphism.

Now we can conclude the argument. The source of the morphism η^\sharp is a Bredon el_2^{RO} -algebra in the sense of Definition 3.1; so by Theorem 3.5, it is initial among oriented el_2^{RO} -algebras equipped with an additive orientation. Since the target of η^\sharp is a localization of the source away from the inverse Thom class t , the target is an initial oriented el_2^{RO} -algebra whose orientation is both additive and invertible. \square

9. EPILOGUE: REPROVING CLASSICAL RESULTS ON EQUIVARIANT BORDISM

In this final section we describe the relationship between the framework we develop in this paper and previous work on equivariant bordism rings, specifically that by Conner–Floyd [12], Alexander [2], and Firsching [15]. We show that these classical results can be readily deduced from our formalism, and in fact generalize to a large class of oriented el_2^{RO} -algebras.

9.1. The Conner–Floyd exact sequence. We start with a generalization of the Conner–Floyd exact sequence [12, Theorem 28.1] for bordism of involutions. For this, we let (X, a, t) be an oriented el_2^{RO} -algebra, and A an elementary abelian 2-group. As we used many times throughout this paper, the map $-/1 : X(A)_* \rightarrow \Phi_*^A X$ is injective. For $A = C$, it is furthermore straightforward to describe the structure of the cokernel.

Proposition 9.1. *For every oriented el_2^{RO} -algebra (X, a, t) the cokernel of the monomorphism $-/1 : X(C)_* \rightarrow \Phi^C X$ is free as an $X(1)_*$ -module with basis the classes $(t/a)^n$ for $n \geq 1$.*

Proof. The fundamental exact sequence and induction on n shows that $X(C, * - n\sigma)$ is the direct sum of the image of $\cdot a^n : X(C)_* \rightarrow X(C, * - n\sigma)$ and a free graded $X(1)_*$ -module spanned by $a^{n-i}t^i$ for $i = 1, \dots, n$. Since $\Phi_*^C X$ is the colimit of these groups along multiplication by a , the claim follows by passage to colimit in n . \square

Proposition 9.1 also shows that if $\Phi^C X$ is free as a graded module over $X(1)_*$, then $X(C)_*$ is projective as a graded $X(1)_*$ -module. If, in addition, the \mathbb{F}_2 -algebra $X(1)_*$ is graded-connected, then this implies that $X(C)_*$ is even free as a graded $X(1)_*$ -module.

9.2. Alexander’s theorem. It was already observed by Conner and Floyd as a consequence of their exact sequence [12, Theorem 28.1] that the C -equivariant bordism ring is free as a graded module over the non-equivariant bordism ring. Alexander provided an explicit \mathcal{N}_* -basis of \mathcal{N}_*^C in [2, Theorem 1.2]. We generalize Alexander’s result to certain kinds of oriented el_2^{RO} -algebras, with a different, and arguably simpler, proof.

We recall the division operator $\Gamma : X(C)_k \rightarrow X(C)_{k+1}$ from construction (7.8), which is characterized by the equation $a \cdot \Gamma(x) = t \cdot x + \epsilon^*(x)$. Here $\epsilon : C \rightarrow C$ is the trivial homomorphism, so that ϵ^* is the composition of restriction to the trivial group, followed by inflation.

Proposition 9.2. *Let (X, a, t) be an oriented el_2^{RO} -algebra. Suppose that $X(1)_*$ is graded-connected as an \mathbb{F}_2 -algebra. Let $(b_i)_{i \in I}$ be a family of homogeneous elements of $X(C)_*$ of positive degrees. Then the following conditions are equivalent:*

- (a) *The classes 1 and $\Gamma^n(b_i)$ for $n \geq 0$ and $i \in I$ form a basis of $X(C)_*$ as a graded module over $X(1)_*$.*
- (b) *The classes 1 and $b_i/1$ for $i \in I$ form a basis of $\Phi_*^C X$ as a graded module over the polynomial ring $X(1)_*[t/a]$.*

Proof. Condition (a) and Proposition 9.1 together imply that the classes $(t/a)^n$ and $\Gamma^n(b_i)/1$ for $n \geq 0$ and $i \in I$ form a basis of $X(C)_*$ as a graded module over $X(1)_*$. The defining

relation of the division operator yields

$$\Gamma(x)/1 = (t/a) \cdot x/1 + \epsilon(x) \cdot (t/a) .$$

If the dimension of the class x is positive, this shows that $\Gamma(x)/1$ is congruent to $(t/a) \cdot x/1$ modulo the graded ideal of positive degree elements in $X(1)$. By induction on n we see that $\Gamma^n(x)/1$ is congruent to $(t/a)^n \cdot x/1$ modulo the ideal of positive degree elements in $X(1)$. In particular, the class $\Gamma^n(b_i)/1$ is congruent to $(t/a)^n \cdot b_i$. Since $X(C)_*$ is bounded below and $X(1)_*$ is a graded-connected \mathbb{F}_2 -algebra, this shows that the classes $(t/a)^n$ and $(t/a)^n \cdot b_i/1$ for $n \geq 0$ and $i \in I$ also form a basis of $X(C)_*$ as a graded module over $X(1)_*$. Hence the classes 1 and $b_i/1$ for $i \in I$ form a basis of $X(C)_*$ as a graded module over $X(1)_*[t/a]$, i.e., condition (b) holds. The other implication is proved by reversing the argument. \square

Corollary 9.3. *Let (X, a, t) be an oriented el_2^{RO} -algebra. Suppose that $X(1)_*$ is graded-connected as an \mathbb{F}_2 -algebra. Let $(y_i)_{i \in I}$ be a family of homogeneous elements of $X(C)_*$ of positive degree such that $\Phi_*^C X$ is a polynomial algebra over $X(1)_*$ on the classes $(y_i)_{i \in I}$ and the class t/a . Then the classes 1 and the classes $\Gamma^n(y_{i_1} \cdots y_{i_r})$ for all $n \geq 0$, $r \geq 1$ and $i_1, \dots, i_r \in I$ form a basis of $X(C)_*$ as a graded module over $X(1)_*$.*

Proof. We let \mathcal{B} be the set of all monomials of degree at least one in the classes $y_i/1$ for all $i \in I$. The hypothesis that $\Phi_*^C X$ is polynomial over $X(1)_*$ on the classes $(y_i)_{i \in I}$ and t/a is equivalent to the property that \mathcal{B} is a basis of $\Phi_*^C X$ as module over $X(1)_*[t/a]$. Proposition 9.2 (a) then yields the claim. \square

For equivariant bordism, we obtain Alexander's \mathcal{N}_* -basis of \mathcal{N}_*^C by letting $y_i \in \mathcal{N}_i^C$ be the bordism class of projective space $P(\sigma \oplus \mathbb{R}^i)$, for $i = 2, 3, \dots$. The C -fixed points of $P(\sigma \oplus \mathbb{R}^i)$ have two components: an isolated fixed point $P(\sigma)$, and the space $\mathbb{R}P^{i-1}$ with normal bundle the tautological line bundle. So Proposition 7.4 shows that the classes $y_i/1$ for $i \geq 1$ and a/t form a set of polynomial generators of $\Phi_*^C \mathcal{N}$, as desired. So Corollary 9.3 applies, and we recover [2, Theorem 1.2].

We can also apply Corollary 9.3 to the classes $\zeta_i \in \mathcal{N}_{i+1}^C$ from Construction 7.5, for $i \geq 1$. This way we obtain a different \mathcal{N}_* -basis of \mathcal{N}_*^C , given by 1 and the classes $\Gamma^n(\zeta_{i_1} \cdots \zeta_{i_r})$ for all $n \geq 0$, $r \geq 1$ and $i_1, \dots, i_r \geq 1$.

9.3. Firsching's pullback square. Finally we explain how a generalization of the main results of Firsching's paper [15] drops out of our theory. In the following theorem we will write

$$x_\lambda = t_\lambda/a_\lambda \in \Phi_1^A X ,$$

where λ is any nontrivial A -character. And we write $(\Phi_*^A X)[x_\lambda^{-1}]$ for the localization of the geometric fixed point ring away from the classes x_λ for all nontrivial A -characters λ .

Theorem 9.4. *Let (X, a, t) be an oriented el_2^{RO} -algebra. Let A be an elementary abelian 2-group such that for some (hence any) index two subgroup K of A and every nontrivial K -character μ , the class $x_\mu = t_\mu/a_\mu$ is a regular element of the ring $\Phi_*^K X$. Then the following square is a pullback of \mathbb{Z} -graded rings:*

$$\begin{array}{ccc} X(A)_* & \xrightarrow{-/1} & \Phi_*^A X \\ \downarrow & & \downarrow \\ (t^{-1}X)(A)_* & \xrightarrow{z/t_V \mapsto z/a_V \cdot x_V^{-1}} & (\Phi_*^A X)[x_\lambda^{-1} : \lambda \in A^\circ] \end{array}$$

Proof. We claim that for every index two subgroup K of A and every K -character μ , the class t_μ is a regular element in the I_K -graded ring $X(K, \star)$. Indeed, suppose that $y \cdot t_\mu = 0$ for some $y \in X(K, k - V)$ with $k \in \mathbb{Z}$ and $V^K = 0$. Then

$$0 = (y \cdot t_\mu)/(a_V \cdot a_\mu) = (y/a_V) \cdot (t_\mu/a_\mu)$$

in $\Phi_*^A X$. So $y/a_V = 0$ by the regularity hypothesis. Because the map $-/a_V$ is injective, this implies $y = 0$. Because all the classes t_μ for $\mu \in K^\circ$ are regular, we also deduce that for every A -representation U , the class $\text{res}_K^A(t_U) = t_{U|_K}$ is a regular element in $X(K, \star)$.

Now we turn to the proof of the theorem. Since the horizontal maps in the square are injective, we only need to show the following claim: let $y \in \Phi_k^A X$ be a class whose image in $(\Phi_*^A X)[x_\lambda^{-1}]$ is in the image of $(t^{-1}X)(A)_k$. Then the class y is effective. We suppose that $y = y'/a_W$ for some A -representation W with trivial fixed points, and $y' \in X(A, k - W)$. By assumption there is another A -representation V with trivial fixed points and a class $z \in X(A, k - V)$ such that $(z/a_V) \cdot x_V^{-1} = y = y'/a_W$ in the localization $(\Phi_*^A X)[x_\lambda^{-1}]$. Hence there is another A -representation U with trivial fixed points such that

$$(a_W \cdot z \cdot t_U)/a_{W \oplus V \oplus U} = (z/a_V) \cdot x_U = (y'/a_W) \cdot x_V \cdot x_U = (y' \cdot t_V \cdot t_U)/a_{W \oplus V \oplus U}$$

in the geometric fixed point ring $\Phi_*^A X$. Since $-/a_{W \oplus V \oplus U}$ is injective, this forces

$$z \cdot t_U \cdot a_W = y' \cdot t_{V \oplus U}$$

in $X(A, k - (W \oplus V \oplus U))$. Now we let λ be an A -character with kernel K , and we let m be the multiplicity of λ in W . Then the left hand side of the previous equation is divisible by a_λ^m . Because the class $\text{res}_K^A(t_{V \oplus U})$ is regular in $X(K, \star)$, Proposition 1.5 (i) shows that also y' is divisible by a_λ^m . Proposition 1.5 (ii) then shows y' is divisible by a_W , i.e., $y' = y'' \cdot a_W$ for a unique $y'' \in X(A, k)$. Thus the class $y = y'/a_W = y''/1$ is effective. \square

For the oriented el_2^{RO} -algebra \mathcal{N} of equivariant bordism, all the geometric fixed point rings $\Phi^K \mathcal{N}$ are polynomial algebras over the non-equivariant bordism ring \mathcal{N}_* , and hence they do not contain any zero-divisors. So Theorem 9.4 shows that for every elementary abelian 2-group A , the followings square is a pullback:

$$\begin{array}{ccc} \mathcal{N}_*^A & \longrightarrow & \Phi_*^A \mathcal{N} \\ \downarrow & & \downarrow \\ \mathfrak{N}_*^{A:S} & \longrightarrow & (\Phi_*^A \mathcal{N})[x_\lambda^{-1} : \lambda \in A^\circ] \end{array}$$

Here $\mathfrak{N}_*^{A:S} = t^{-1}\mathcal{N}$ is stable A -equivariant bordism, see Construction 8.5. Firsching writes e_λ^{-1} for x_λ , and he uses the A -homology theory represented by the global Thom spectrum **MO** instead of $\mathfrak{N}_*^{A:S}$; these equivariant homology theories are isomorphic by [9, Theorem 4.1]. So modulo notation and the identification of $\Phi_*^A \mathcal{N}$ with a polynomial \mathcal{N}_* -algebra, this is the main result Theorem 3.18 of [15].

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