Summer term 2025 Prof. Dr. Stefan Schwede
Dr. Tobias Lenz

Exercises for **Topology II**
Sheet 4

Exercise 1 (*15 points*)**.** Show the following relative version of the topological Künneth theorem:

Theorem. *Let (X, A) and (Y, B) be pairs of spaces. Then for every $n \in \mathbb{N}$ the exterior product map*

$$\bigoplus_{p+q=n} H_p(X, A; \mathbb{Z}) \otimes H_q(Y, B; \mathbb{Z}) \to H_n(X \times Y, X \times B \cup A \times Y; \mathbb{Z})$$

is split injective, and the cokernel is naturally isomorphic to

$$\bigoplus_{p+q=n-1} \mathrm{Tor}(H_p(X, A; \mathbb{Z}), H_q(Y, B; \mathbb{Z})).$$

Hint. Use that the Eilenberg–Zilber theorem was proved for general tensor products of simplicial abelian groups in the lecture.

Exercise 2 (*15 points*)**.** Let $K = \big([0,1] \times [0,1]\big)/_{\sim}$ denote the Klein bottle. Use the Künneth theorem to compute the singular homology groups $H_*(K \times \mathbb{R}\mathrm{P}^2; \mathbb{Z})$ and $H_*(K \wedge \mathbb{R}\mathrm{P}^2; \mathbb{Z})$; here we choose the class of one of the corners as the basepoint of K, while the basepoint of $\mathbb{R}\mathrm{P}^2$ is the point $[1 : 0 : 0]$.

Exercise 3 (*10 points*)**.** Let p be a prime. Further let

- A be an abelian group which is *p-power torsion*, i.e. for every element $a \in A$ there exists an $n \in \mathbb{N}$ such that $p^n \cdot a = 0$, and
- B be an abelian group on which p acts invertibly, i.e. every element b of B is uniquely divisible by p.

1. Show that both the tensor product $A \otimes B$ and the Tor-group $\mathrm{Tor}(A, B)$ are the trivial group.
2. Let (X, x_0) and (Y, y_0) be path-connected pointed topological spaces such that for all $m > 0$ the group $H_m(X; \mathbb{Z})$ is p-power torsion while p acts invertibly on $H_m(Y; \mathbb{Z})$. Show that the inclusion
$$X \vee Y \hookrightarrow X \times Y$$
induces an isomorphism on all integral homology groups.

∗ **Exercise 4** (*5 points*)**.** *This is marked as a bonus exercise not because it is particularly challenging, but rather because it is purely algebraic.*

Let k be a field. The *exterior algebra* $\Lambda(k)$ over k is defined to be the commutative ring $k[\epsilon]/\epsilon^2$. We define a $\Lambda(k)$-module structure on k by setting $\epsilon \cdot x = 0$ for all $x \in k$.

1. Construct a projective resolution P_* of k over $\Lambda(k)$ and compute the homology groups of the levelwise tensor product $P_* \otimes_{\Lambda(k)} k$, i.e. the Tor groups $\mathrm{Tor}_*^{\Lambda(k)}(k, k)$ as defined last term.
2. Conclude that $\Lambda(k)$ is not of global dimension ≤ 1, i.e. there exists a submodule of a projective module that is not projective.

Please hand in your solutions by **May 12, 2pm** via eCampus.