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Exercises for **Topology II**

Sheet 3

Exercise 1 (*14 points*)**.** In the lecture the following addendum to the acyclic models theorem was stated without proof:

Theorem. *Let $\mathcal{C}$ be a category, and let $F, G\colon \mathcal{C} \to \mathrm{Ch}_+$ two functors to the category of non-negatively graded chain complexes. Let $\psi_0\colon F_0 \to G_0$ be a natural transformation of functors from $\mathcal{C}$ to* Ab*. Suppose that for all $n \in \mathbb{N}$, F_n is isomorphic to a sum of represented functors $\mathbb{Z}[\mathcal{C}(c, -)]$ for some family of $\mathcal{C}$-objects, such that $H_n(G(c)) = 0$ for all c in the family and $n > 0$. Then there is a natural transformation $\psi\colon F \to G$ that is ψ_0 in degree 0.*

1. Prove this theorem.
2. Show that any two such extensions are naturally chain homotopic.

Exercise 2 (*12 points*)**.**

1. Use the theorem from the previous exercise to construct chain maps

$$m\colon C_*(X;\mathbb{Z}) \otimes C_*(Y;\mathbb{Z}) \to C_*(X \times Y;\mathbb{Z})$$

which are natural in all simplicial sets X and Y and are given by the product map

$$m\colon \mathbb{Z}[X_0] \otimes \mathbb{Z}[Y_0] \longrightarrow \mathbb{Z}[X_0 \times Y_0]$$
$$\left(\sum_i r_i x_i\right) \otimes \left(\sum_j s_j y_j\right) \longmapsto \sum_{i,j} r_i \otimes s_j (x_i, y_j)$$

in chain dimension 0.

2. Show that any such m is associative up to natural chain homotopy, i.e. the two composites

$$C_*(X;\mathbb{Z}) \otimes C_*(Y;\mathbb{Z}) \otimes C_*(Z;\mathbb{Z}) \xrightarrow{m\otimes \mathrm{id}} C_*(X \times Y;\mathbb{Z}) \otimes C_*(Z;\mathbb{Z}) \xrightarrow{m} C_*(X \times Y \times Z;\mathbb{Z})$$

and

$$C_*(X;\mathbb{Z}) \otimes C_*(Y;\mathbb{Z}) \otimes C_*(Z;\mathbb{Z}) \xrightarrow{\mathrm{id}\otimes m} C_*(X;\mathbb{Z}) \otimes C_*(Y \times Z;\mathbb{Z}) \xrightarrow{m} C_*(X \times Y \times Z;\mathbb{Z})$$

are naturally chain homotopic.

3. Show that any such m is naturally chain homotopic to the Eilenberg–Zilber map defined in the lecture.

please turn over

Construction. Recall the Bockstein homomorphism on homology associated to a short exact sequence of abelian groups $0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$. There is an analogous version for cohomology: For every space X, the sequence

$$0 \to C^*(X; A) \xrightarrow{i_*} C^*(X; B) \xrightarrow{p_*} C^*(X; C) \to 0$$

is short exact, where i_* sends a function $f\colon \mathcal{S}(X)_n \to A$ to the composite $i \circ f\colon \mathcal{S}(X)_n \to B$, and similarly for p_*. Hence there is an induced long exact sequence in cohomology groups of the form

$$\cdots \to H^{n-1}(X; C) \xrightarrow{\beta} H^n(X; A) \xrightarrow{i_*} H^n(X; B) \xrightarrow{p_*} H^n(X; C) \xrightarrow{\beta} \ldots$$

The boundary map β is called the *Bockstein homomorphism*. Explicitly, β sends a class $[c]$ of a cocycle $c \in C^n(X; C)$ to the class $[i_*^{-1}(\partial^n(c'))]$, where c' is a lift of c along the surjection $C^n(X; B) \xrightarrow{p_*} C^n(X; C)$, $\partial^n(c')$ is its image under the nth differential, and $i_*^{-1}(\partial^n(c'))$ is the unique preimage under the embedding $C^{n+1}(X; A) \xrightarrow{i_*} C^{n+1}(X; B)$ (which can be shown to be a cocycle).

Exercise 3 (*14 points*).

1. Consider the Bockstein $H^1(X; \mathbb{Z}/2) \to H^2(X; \mathbb{Z}/2)$ associated to the short exact sequence $0 \to \mathbb{Z}/2 \xrightarrow{2} \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0$. Show that β is given by the 'cup-square,' i.e., it sends a class $x \in H^1(X; \mathbb{Z}/2)$ to $x^2 = x \cup x \in H^2(X; \mathbb{Z}/2)$.

   Hint. Consider the explicit formula for the Alexander–Whitney map in degree 2.

2. Use this to show that the cup product $H^1(\mathbb{R}\mathrm{P}^2; \mathbb{Z}/2) \times H^1(\mathbb{R}\mathrm{P}^2; \mathbb{Z}/2) \to H^2(\mathbb{R}\mathrm{P}^2; \mathbb{Z}/2)$ is non-trivial.

∗ **Exercise 4** (*15 bonus points*). Let X and Y be CW-complexes, at least one of which is locally compact, so that $X \times Y$ carries an induced CW-structure with n-skeleton given by

$$\bigcup_{i=0}^{n} X_i \times Y_{n-i} \subseteq X \times Y.$$

We further choose a commutative ring R, and consider the maps

$$\alpha_{i,n-i}\colon H_i(X_i, X_{i-1}; R) \otimes_R H_{n-i}(Y_{n-i}, Y_{n-i-1}; R) \xrightarrow{\nabla} H_n(X_i \times Y_{n-i}, X_{i-1} \times Y_{n-i} \cup X_i \times Y_{n-i-1}; R)$$
$$\longrightarrow H_n((X \times Y)_n, (X \times Y)_{n-1}; R)$$

where ∇ is the relative Eilenberg–Zilber map on homology.

1. Show that the $\alpha_{i,n-i}$ assemble to a map of chain complexes

   $$\alpha\colon C_*^{\mathrm{cell}}(X; R) \otimes_R C_*^{\mathrm{cell}}(Y; R) \to C_*^{\mathrm{cell}}(X \times Y; R).$$

   Hint. Use naturality to reduce to the case where X and Y are disks.

2. Show that α is an *isomorphism* of chain complexes.

   Hint. Again reduce to the case of disks and use that the Eilenberg–Zilber map induces an isomorphism

   $$H^i(D^i, S^{i-1}; R) \otimes_R H^{n-i}(D^{n-i}, S^{n-i-1}; R) \xrightarrow{\cong} H^n(D^i \times D^{n-i}, S^{i-1} \times D^{n-i} \cup D^i \times S^{n-i-1}; R).$$

3. We now assume in addition that X and Y have only finitely many cells in each dimension (in fact it would be enough to assume this for only one of X and Y). Use the above to construct an isomorphism

   $$C^*_{\mathrm{cell}}(X \times Y; R) \xrightarrow{\cong} C^*_{\mathrm{cell}}(X; R) \otimes_R C^*_{\mathrm{cell}}(Y; R)$$

   of cochain complexes.