# A GLOBAL-EQUIVARIANT SEGAL–BECKER SPLITTING, EXPLICIT BRAUER INDUCTION, AND GLOBAL ADAMS OPERATIONS

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Abstract. We prove a splitting result in global equivariant homotopy theory that is a simultaneous refinement of the Segal–Becker splitting and its equivariant generalizations, and of the explicit Brauer induction of Boltje and Symonds. We show that the morphism of ultra-commutative ring spectra from $\Sigma^\infty_+ B_{\mathrm{gl}}U(1)$ to the global K-theory spectrum that classifyies the tautological $U(1)$-representation admits a section on underlying global infinite loop spaces that is a 1-fold global loop map. We prove that this global Segal–Becker splitting induces the Boltje–Symonds explicit Brauer induction on equivariant homotopy groups, and that it induces the classical Segal–Becker splittings on equivariant cohomology theories. As an application we rigidify the unstable Adams operations in equivariant K-theory to global self-maps of the global space $\mathbf{BUP}$.

## Contents



## Introduction

The purpose of this paper is to prove a splitting result in global equivariant homotopy theory that is a simultaneous refinement of the Segal–Becker splitting [4, 24] and its equivariant generalizations [8, 11], and of the explicit Brauer induction of Boltje [5] and Symonds [28]. The main player is the morphism of ultra-commutative ring spectra $\eta\colon \Sigma^\infty_+\mathbf{P} \longrightarrow \mathbf{KU}$ from the unreduced suspension spectrum of the global space $\mathbf{P}$, a multiplicative model of the global classfying space of $U(1)$ and a global refinement of $\mathbb{C}P^\infty$, to the global K-theory spectrum that classifies the tautological $U(1)$-representation. We construct a section to the morphism of underlying global infinite loop spaces $\Omega^\bullet(\eta)\colon \Omega^\bullet(\Sigma^\infty_+\mathbf{P}) \longrightarrow \Omega^\bullet(\mathbf{KU})$ that is a 1-fold global loop map. We prove that this *global Segal–Becker splitting* induces the Boltje–Symonds explicit Brauer induction on equivariant homotopy groups, and that it induces the classical Segal–Becker splittings on equivariant cohomology theories. As an application we rigidify the unstable Adams operations in equivariant K-theory to global loop self-maps of the representing global space $\mathbf{BUP}$.

To put our results into perspective, we give a brief review of the history of the Segal–Becker splitting and of 'explicit Brauer induction'. The Segal–Becker splitting is the statement that the morphism $\Sigma^\infty_+\mathbb{C}P^\infty \longrightarrow$

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$KU$ from the suspension spectrum of infinite complex projective space to the complex K-theory spectrum that classifies the tautological line bundle over $\mathbb{C}P^\infty$ has a section after passing to infinite loop spaces. The theorem implies in particular that the transformation of degree 0 cohomology theories on spaces induced by the morphism is a split epimorphism, and several papers on the subject state the splitting in this form. The original splitting theorem was proved by Segal in [24] for complex K-theory. Becker provided a different proof in [4] that also provides a splitting for real and symplectic K-theory. Segal's construction produces $p$-complete maps for every prime $p$ that are then assembled via an arithmetic square; all other sources on the subject use transfers in some incarnation to construct the relevant splitting. It seems to be well-known that the section to $\Omega^\infty(\Sigma^\infty_+\mathbb{C}P^\infty) \longrightarrow \Omega^\infty KU \simeq \mathbb{Z}\times BU$ can be arranged as a loop map, but it does *not* deloop twice; we recall an argument in Remark 4.9.

In the paper [16], Nagata, Nishida and Toda prove a version of Segal's splitting for Real K-theory in the sense of Atiyah [2], i.e., the K-theory made from complex vector bundle over spaces with involutions, equipped with a fiberwise conjugate-linear involution. A different proof of the Real splitting was given by Kono [14]. On underlying non-equivariant spaces this recovers Segal splitting for complex K-theory; taking $C$-fixed points yields Becker's splitting for real K-theory. A version of the Segal–Becker splitting in motivic homotopy theory, for algebraic K-theory in place of topological K-theory, is provided in [13].

An equivariant generalization of the Segal–Becker splitting for finite groups $G$ was obtained by Iriye and Kono [11, Theorem 1]. Iriye and Kono also remark in [11, Theorem 1'] that their method works in the same way for Real-equivariant K-theory, for finite groups with involution. The most comprehensive discussion of an equivariant Segal–Becker splitting is in Crabb's paper [8]; Crabb proves the Segal–Becker splitting for compact Lie groups (and not just for finite groups), and he shows that his splittings are natural for restriction along continuous homomorphisms between compact Lie groups. Compatible equivariant results for all compact Lie groups are always highly suggestive of an underlying globally-equivariant statement, and this is precisely what had originally motivated the work in this paper.

Now we give a more detailed outline of our own results. The non-equivariant morphism $\Sigma^\infty_+\mathbb{C}P^\infty \longrightarrow KU$ that classifies the tautological line bundle over $\mathbb{C}P^\infty$ has a particularly nice and prominent global-equivariant refinement, a morphism of global spectra

$$\eta \ : \ \Sigma^\infty_+\mathbf{P} \ \longrightarrow \ \mathbf{KU} \ .$$

The global K-theory spectrum $\mathbf{KU}$ was introduced by Joachim [12], see also [18, Construction 6.4.9]; for every compact Lie group $G$, the underlying genuine $G$-spectrum of $\mathbf{KU}$ represents $G$-equivariant complex K-theory, see [12, Theorem 4.4] or [18, Corollary 6.4.23]. The global space $\mathbf{P}$ is is a specific global refinement of $\mathbb{C}P^\infty$, made from projective spaces of complex inner product spaces, see Construction 3.1. The underlying $G$-equivariant homotopy type of $\mathbf{P}$ is that of the projective space of a complete complex $G$-universe. It is a global classifying space, in the sense of [18, Definition 1.1.27], for the circle group $U(1)$ that we shall denote by $T$ throughout this paper; so its unreduced global suspension spectrum $\Sigma^\infty_+\mathbf{P}$ represents the functor $\pi_0^T$ on the global stable homotopy category, compare [18, Theorem 4.4.3]. The morphism $\eta$ is extremely highly structured, and has a range of marvelous properties. It is a morphism of ultra-commutative ring spectra that sends the universal element in $\pi_0^T(\Sigma^\infty_+\mathbf{P})$ to the class of the tautological $T$-representation in $\pi_0^T(\mathbf{KU}) \cong R(T)$. As a morphism of ultra-commutative ring spectra, the effect of $\eta$ on equivariant homotopy groups is not only compatible with restriction, inflations and transfers, but also with products, multiplicative power operations and norms. In [21], the author establishes a global refinement and generalization of Snaith's celebrated theorem [25, 26], saying that $KU$ can be obtained from $\Sigma^\infty_+\mathbb{C}P^\infty$ by 'inverting the Bott class': the morphism $\eta\colon \Sigma^\infty_+\mathbf{P} \longrightarrow \mathbf{KU}$ is initial in the $\infty$-category of ultra-commutative ring spectra among morphisms from $\Sigma^\infty_+\mathbf{P}$ that invert a specific family of representation-graded equivariant homotopy classes in $\pi^{U(n)}_{\nu_n}(\Sigma^\infty_+\mathbf{P})$, *pre-Bott classes*, for all $n \geq 1$.

The global space $\mathbf{U}$ is a specific global refinement of the infinite unitary group, made from the unitary groups of all hermitian inner product spaces, see Construction 1.2. The underlying $G$-equivariant homotopy type of $\mathbf{U}$ is that of the unitary group of a complete complex $G$-universe. The global space $\mathbf{U}$ features in a global refinement of Bott periodicity, a global equivalence $\Omega^2\mathbf{U} \sim \mathbf{U}$ that encodes equivariant Bott periodicity for all compact Lie groups at once, see [18, Theorem 2.5.41]. In Construction 4.1 we use the global stable splitting of $\Sigma^\infty_+\mathbf{U}$ from [19, Theorem 4.10] to construct a morphism

$$d \,:\, \mathbf{U} \longrightarrow \Omega^\bullet(\Sigma^\infty_+\mathbf{P}\wedge S^1)$$

in the unstable global homotopy category, our deloop of the global Segal–Becker splitting. The splitting property is our first main result, to be proved as Theorem 4.2:

**Theorem A.** *The composite*

$$\mathbf{U} \xrightarrow{\ d\ } \Omega^\bullet(\Sigma^\infty_+\mathbf{P}\wedge S^1) \xrightarrow{\Omega^\bullet(\eta\wedge S^1)} \Omega^\bullet(\mathbf{KU}\wedge S^1)$$

*is a global equivalence.*

We emphasize that the composite $\Omega^\bullet(\eta\wedge S^1)\circ d$ of Theorem A is not just any global equivalence, but it coincides with the 'preferred infinite delooping' of $\mathbf{U}$, i.e., the global equivalence

$$\mathbf{U} \xrightarrow{\ \sim\ } \Omega^\bullet(\mathrm{sh}\,\mathbf{KU})$$

established in [18, Theorem 6.4.21], up to a natural global equivalence $\mathbf{KU}\wedge S^1 \sim \mathrm{sh}\,\mathbf{KU}$. In fact, all the work in proving Theorem A goes into showing precisely this. At the heart of the argument is a subtle connection between the global stable splitting and the preferred delooping of $\mathbf{U}$, two features that are a priori unrelated. As we show in Theorem 3.10, the adjoint $\Sigma^\infty\mathbf{U} \longrightarrow \mathrm{sh}\,\mathbf{KU}$ of the preferred infinite delooping annihilates the higher terms of the stable global splitting (1.1).

The global space $\mathbf{BUP}$ is a global refinement of the space $\mathbb{Z}\times BU$, and $\mathbf{BUP}$ represents equivariant K-theory, see [18, Theorem 2.4.10]. Just as $\mathbb{Z}\times BU$ is the infinite loop space of the topological K-theory spectrum $KU$, the global space $\mathbf{BUP}$ 'is' the global infinite loop space of $\mathbf{KU}$, see [18, Remark 6.4.22]. By looping the morphism $d\colon \mathbf{U} \longrightarrow \Omega^\bullet(\Sigma^\infty_+\mathbf{P}\wedge S^1)$ and composing with the global Bott periodicity equivalence $\mathbf{BUP} \sim \Omega\mathbf{U}$ from [18, Theorem 2.5.41], we obtain another morphism in the unstable global homotopy category

$$c \,:\, \mathbf{BUP} \longrightarrow \Omega^\bullet(\Sigma^\infty_+\mathbf{P})\ ,$$

the *global Segal–Becker splitting*. By design, the morphism $c$ is a global loop map. Its splitting property, to be proved as Corollary 4.8, is an easy consequence of Theorem A:

**Theorem B.** *The composite*

$$\mathbf{BUP} \xrightarrow{\ c\ } \Omega^\bullet(\Sigma^\infty_+\mathbf{P}) \xrightarrow{\Omega^\bullet(\eta)} \Omega^\bullet(\mathbf{KU})$$

*is a global equivalence.*

A morphism that admits a section tends to admit many different sections. Our next two results justify that the global Segal–Becker splitting $c\colon \mathbf{BUP} \longrightarrow \Omega^\bullet(\Sigma^\infty_+\mathbf{P})$ is the 'correct' splitting to $\Omega^\bullet(\eta)$, by showing that its effect on equivariant homotopy groups and on equivariant cohomology theories recover certain classical and much studied algebraic sections. On equivariant homotopy groups, the global Segal–Becker splitting induces the so-called *explicit Brauer induction* of Boltje [5] and Symonds [28]. By Brauer's theorem [6, Theorem I] the complex representation ring of a finite group is generated, as an abelian group, by representations that are induced from 1-dimensional representations of subgroups. Segal generalized this result to compact Lie groups in [23, Proposition 3.11 (ii)], where 'induction' refers to the smooth induction. We write $\mathbf{A}(T,G)$ for the free abelian group with basis the symbols $[H,\chi]$, where $H$ runs over all conjugacy classes of closed subgroup of $G$ with finite Weyl group, and $\chi\colon H \longrightarrow T = U(1)$ runs over all characters of $H$. The Brauer–Segal theorem can then be paraphrased as the fact that the group homomorphism

$$\mathbf{A}(T,G) \longrightarrow R(G)$$

that sends $[H,\chi]$ to $\mathrm{tr}_H^G(\chi)$ is surjective. An *explicit Brauer induction* is an 'explicit' section to this map, possibly with naturality properties as the group $G$ varies. The first explicit Brauer induction was Snaith's formula [27, Theorem (2.16)]; however, Snaith's maps are not additive and not compatible with restriction to subgroups. Not much later Boltje [5] specified a different explicit Brauer induction formula by purely algebraic means; Symonds [28] gave a topological interpretation of Boltje's construction. The Boltje–Symonds maps are additive and natural for restriction along group homomorphisms; the maps are not (and in fact cannot be) in general compatible with transfers. We prove the following result as Theorem 5.7:

**Theorem C.** *For every compact Lie group $G$, the composite*

$$R(G) \;\cong\; \pi_0^G(\mathbf{BUP}) \xrightarrow{\pi_0^G(c)} \pi_0^G(\Sigma_+^\infty \mathbf{P}) \;\cong\; \mathbf{A}(T,G)$$

*coincides with the Boltje–Symonds explicit Brauer induction.*

Our fourth main result justifies the name 'global Segal–Becker splitting' for the morphism $c\colon \mathbf{BUP} \longrightarrow \Omega^\bullet(\Sigma_+^\infty\mathbf{P})$ by showing that it refines the classical equivariant Segal–Becker splittings. The latter are defined at the level of equivariant cohomology theories, and we review them in Construction 5.8. We show the following in Theorem 5.15:

**Theorem D.** *Let $G$ be a compact Lie group and $A$ a finite $G$-CW-complex. Then the composite*

$$\mathbf{K}_G(A) \cong [A,\mathbf{BUP}]^G \xrightarrow{[A,c]^G} [A,\Omega^\bullet(\Sigma_+^\infty\mathbf{P})]^G$$

*coincides with the $G$-equivariant Segal–Becker splitting $\vartheta_{G,A}$ defined in* (5.9).

The isomorphism between the $G$-equivariant K-group $\mathbf{K}_G(A)$ and the group $[A,\mathbf{BUP}]^G$ will be recalled in Construction 5.13. The group $[A,\Omega^\bullet(\Sigma_+^\infty\mathbf{P})]^G$ is isomorphic to the group of morphisms from $\Sigma_+^\infty A$ to $\Sigma_+^\infty\mathbf{P}(\mathcal{U}_G)$ is the $G$-equivariant stable homotopy category, where $\mathbf{P}(\mathcal{U}_G)$ is a classifying $G$-space for $G$-equivariant complex line bundles. Theorem C is in fact a special case of Theorem D, but we prove Theorem C first, and then deduce Theorem D using the global functoriality.

As an application of the global Segal–Becker splitting, we construct global rigidifications of the unstable Adams operations on equivariant K-theory. In (6.4) we define the *$n$-th global Adams operation*

$$\psi^n \;:\; \mathbf{BUP} \;\longrightarrow\; \mathbf{BUP}\ ,$$

for $n \geq 1$. These global Adams operations are morphisms in the the unstable global homotopy category that arise as global loop maps, the deloopings being certain endomorphisms of $\mathbf{U}$. The following result, proved as Theorem 6.7, justifies the name of the global Adams operations:

**Theorem E.** *For every compact Lie group $G$ and every finite $G$-CW-complex $A$, the following square commutes:*

$$\begin{tikzcd} \mathbf{K}_G(A) \arrow[r, "\psi^n"] \arrow[d, "\cong"'] & \mathbf{K}_G(A) \arrow[d, "\cong"] \\ {[A,\mathbf{BUP}]^G} \arrow[r, "{[A,\psi^n]^G}"'] & {[A,\mathbf{BUP}]^G} \end{tikzcd}$$

*The upper horizontal map in the diagram is the $n$-th classical Adams operation on equivariant K-theory.*

**Conventions.** We use the models of [18] to represent unstable and stable global homotopy types. So global spaces are represented by orthogonal spaces, relative to the notion of global equivalence introduced in [18, Definition 1.1.2]. Our global Segal–Becker splitting and its deloop will thus be morphisms in the unstable global homotopy category, i.e., the localization of the category of orthogonal spaces at the class of global equivalences. Similarly, global spectra are represented by orthogonal spectra, relative to the notion of global equivalence introduced in [18, Definition 4.1.3].

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## 1. The global stable splitting of $\mathbf{U}$

The construction of our deloop of the global Segal–Becker splitting depends crucially on a stable splitting of the global space $\mathbf{U}$. In [19], the author constructs certain morphisms $s_k\colon \Sigma^\infty(\mathbf{Gr}_k^{\mathbb{C}})^{\mathrm{ad}(k)} \longrightarrow \Sigma^\infty_+\mathbf{U}$ in the global stable homotopy category such that the combined morphism

$$\sum s_k \ : \ \bigvee_{k\geq 0} \Sigma^\infty(\mathbf{Gr}_k^{\mathbb{C}})^{\mathrm{ad}(k)} \ \xrightarrow{\ \sim\ } \ \Sigma^\infty_+\mathbf{U} \tag{1.1}$$

is a global equivalence, see [19, Theorem 4.10]. The splitting (1.1) is a global-equivariant refinement of Miller's stable splitting [15] of the infinite unitary group. The orthogonal space $\mathbf{Gr}_k^{\mathbb{C}}$ is made from Grassmannians of complex $k$-planes; see [18, Example 2.3.16]. And $(\mathbf{Gr}_k^{\mathbb{C}})^{\mathrm{ad}(k)}$ denotes the global Thom space over $\mathbf{Gr}_k^{\mathbb{C}}$ associated with the adjoint representation $\mathrm{ad}(k)$ of $U(k)$; see [19, Example 3.12]. In this section we review the construction of the splitting morphisms, and then translate the splitting (1.1) into an interpretation of the group $[\![\Sigma^\infty\mathbf{U}, X]\!]$ of stable global morphisms in terms of $\mathrm{ad}(k)$-graded $U(k)$-equivariant homotopy groups of $X$, see Theorem 1.13. Moreover, we establish two geometric fixed point relations for the splitting that we need later, see Theorems 1.16 and 1.21.

**Construction 1.2** (The ultra-commutative monoid $\mathbf{U}$)**.** We recall the ultra-commutative monoid $\mathbf{U}$ made from unitary groups, compare [18, Example 2..37]. We write

$$V_{\mathbb{C}} \ = \ \mathbb{C}\otimes_{\mathbb{R}} V$$

for the complexification of a euclidean inner product space $V$. The euclidean inner product $\langle -,-\rangle$ on $V$ induces a hermitian inner product $(-,-)$ on $V_{\mathbb{C}}$, defined as the unique sesquilinear form that satisfies $(1\otimes v, 1\otimes w) = \langle v,w\rangle$ for all $v,w\in V$. The value of the orthogonal space $\mathbf{U}$ on $V$ is

$$\mathbf{U}(V) \ = \ U(V_{\mathbb{C}}) \ ,$$

the unitary group of the complexification of $V$. The complexification of every $\mathbb{R}$-linear isometric embedding $\varphi\colon V\longrightarrow W$ preserves the hermitian inner products, so we can define a continuous group homomorphism

$$\mathbf{U}(\varphi) \ : \ \mathbf{U}(V) \ \longrightarrow \ \mathbf{U}(W)$$

by conjugation with $\varphi_{\mathbb{C}}\colon V_{\mathbb{C}}\longrightarrow W_{\mathbb{C}}$ and the identity on the orthogonal complement of the image of $\varphi_{\mathbb{C}}$. The commutative multiplication of $\mathbf{U}$ is given by the direct sum of unitary automorphisms

$$\mathbf{U}(V)\times\mathbf{U}(W) \ \longrightarrow \ \mathbf{U}(V\oplus W) \ , \quad (A,B)\mapsto \ A\oplus B \ .$$

If $G$ is a compact Lie group, then the underlying $G$-space is the unitary group of a complete complex $G$-universe. Since unitary $G$-representations break up into isotypical summand, its $G$-fixed points decompose as a weak product, indexed by the isomorphism classes of irreducible unitary $G$-representations, of infinite unitary groups. We refer to [18, Example 2..37] for more details.

The global stable splitting morphism $s_k\colon (\mathbf{Gr}_k^{\mathbb{C}})^{\mathrm{ad}(k)} \longrightarrow \Sigma^\infty_+\mathbf{U}$ ultimately stems from a specific $U(k)$-equivariant stable splitting of the 'top cell' in $U(k)^{\mathrm{ad}}$, the unitary group $U(k)$ acting on itself by conjugation. We review this splitting now, following Crabb's exposition in [7, page 39]. In [18] and [19] we use different conventions on whether the suspension coordinate in an orthogonal suspension spectrum is written on the left of right of the argument. In this paper, we adopt the convention of [18], with the suspension coordinate written on the left; this entails some minor changes to some formulas in [19], moving some suspension coordinates to the other side.

**Construction 1.3** (Splitting the top cell off $U(k)^{\mathrm{ad}}$)**.** We write

$$\mathrm{ad}(k) \ = \ \{X \in M(k\times k;\mathbb{C})\colon X = -\bar{X}^t\}$$

for the $\mathbb{R}$-vector space of skew-hermitian complex $k\times k$ matrices. The unitary group $U(k)$ acts on $\mathrm{ad}(k)$ by conjugation, and this action witnesses $\mathrm{ad}(k)$ as the adjoint representation of $U(k)$, whence the name. The *Cayley transform* is the $U(k)$-equivariant open embedding

$$\mathrm{ad}(k) \ \longrightarrow \ U(k)^{\mathrm{ad}} \ , \quad X \ \longmapsto \ (X-1)(X+1)^{-1}$$

onto the subspace of $U(k)$ of those matrices that do not have $+1$ as an eigenvalue. The associated collapse map

$$U(k)^{\mathrm{ad}} \ \longrightarrow \ S^{\mathrm{ad}(k)}$$

admits a section in the stable homotopy category of genuine $U(k)$-spectra, as follows. We write

$$\mathrm{sa}(k) \ = \ \{Z \in M(k\times k;\mathbb{C})\colon Z = \bar{Z}^t\}$$

for the $\mathbb{R}$-vector space of hermitian complex $k\times k$ matrices, with $U(k)$-action by conjugation. A basic linear algebra fact, sometimes referred to as 'polar decomposition', is that the $U(k)$-equivariant map

$$\phi_k \ : \ \mathrm{sa}(k)\times U(k) \ \longrightarrow \ M(k\times k;\mathbb{C}) = \mathrm{sa}(k)\oplus\mathrm{ad}(k) \ , \quad (Z,A)\mapsto A\cdot\exp(-Z) \tag{1.4}$$

is an open embedding onto the general linear group $Gl_n(\mathbb{C})$; for a proof, see for example [19, Proposition B.17]. This open embedding has an associated $U(k)$-equivariant collapse map

$$t_k \ : \ S^{\mathrm{sa}(k)\oplus\mathrm{ad}(k)} \ \longrightarrow \ S^{\mathrm{sa}(k)}\wedge U(k)^{\mathrm{ad}}_+ \tag{1.5}$$

that is a stable section to the previous collapse map $U(k)^{\mathrm{ad}} \longrightarrow S^{\mathrm{ad}(k)}$, see the argument after the proof of Theorem 1.8 in [7, page 39], or the proof of [19, Theorem 4.7].

Next we recall how the $U(k)$-equivariant stable splitting (1.5) gives rise to a specific equivariant homotopy class $\langle t_k\rangle$ in $\pi^{U(k)}_{\mathrm{ad}(k)}(\Sigma^\infty_+\mathbf{U})$; this class in turn characterizes the global splitting morphism $s_k\colon \Sigma^\infty(\mathbf{Gr}_k^{\mathbb{C}})^{\mathrm{ad}(k)} \longrightarrow \Sigma^\infty_+\mathbf{U}$ by the relation (1.10).

**Construction 1.6** (The global splitting morphism)**.** We let $W$ be a unitary representation of a compact Lie group $G$. We write $uW$ for the underlying orthogonal $G$-representation, i.e., the underlying $\mathbb{R}$-vector space with the euclidean inner product $\langle v,w\rangle = \mathrm{Re}(v,w)$, the real part of the hermitian inner product. The map

$$\zeta^W \ : \ W \ \longrightarrow \ \mathbb{C}\otimes_{\mathbb{R}}(uW) = (uW)_{\mathbb{C}} \ , \quad w\longmapsto (1\otimes w - i\otimes iw)/\sqrt{2} \tag{1.7}$$

is a $G$-equivariant $\mathbb{C}$-linear isometric embedding. So conjugation by $\zeta^W$ and extension by the identity on the orthogonal complement of its image is a continuous group monomorphism

$$\zeta^W_* \ : \ U(W) \ \longrightarrow \ U((uW)_{\mathbb{C}}) \ = \ \mathbf{U}(uW) \ .$$

Since $\zeta^W$ is $G$-equivariant, the map $\zeta^W_*$ is $G$-equivariant for the conjugation action on the source and for the $G$-action on $\mathbf{U}(uW)$ through the functoriality of the orthogonal space $\mathbf{U}$. One should beware that $\zeta^W_*$ is *different* from the monomorphism that sense a unitary automorphism $\varphi\colon W\longrightarrow W$ to the unitary automorphism $(u\varphi)_{\mathbb{C}}$ of $(uW)_{\mathbb{C}}$.

We let $\nu_k$ denote the tautological unitary $U(k)$-representation on $\mathbb{C}^k$. In this special case we obtain a $U(k)$-equivariant monomorphism

$$\zeta_*^k = \zeta_*^{\nu_k} \ : \ U(k)^{\mathrm{ad}} = U(\nu_k) \ \longrightarrow \ \mathbf{U}(u(\nu_k)) \ . \tag{1.8}$$

The $U(k)$-equivariant collapse map $t_k$ was defined in (1.5). In [19, Construction 4.4], we define (for $\mathbb{K} = \mathbb{C}$, $m = 0$, and in slightly different notation) a class

$$\langle t_k \rangle \ \in \ \pi_{\mathrm{ad}(k)}^{U(k)}(\Sigma_+^\infty \mathbf{U})$$

as the one represented by the $U(k)$-map

$$\begin{aligned} S^{\nu_k \oplus \mathrm{sa}(k) \oplus \mathrm{ad}(k)} &\xrightarrow{S^{\nu_k} \wedge t_k} S^{\nu_k \oplus \mathrm{sa}(k)} \wedge U(k)_+^{\mathrm{ad}} \xrightarrow{S^{\nu_k \oplus \mathrm{sa}(k)} \wedge \zeta_*^k} S^{\nu_k \oplus \mathrm{sa}(k)} \wedge \mathbf{U}(u(\nu_k))_+ \\ &\xrightarrow{S^{\nu_k \oplus \mathrm{sa}(k)} \wedge \mathbf{U}(i_1)} S^{\nu_k \oplus \mathrm{sa}(k)} \wedge \mathbf{U}(u(\nu_k) \oplus \mathrm{sa}(k))_+ = (\Sigma_+^\infty \mathbf{U})(u(\nu_k) \oplus \mathrm{sa}(k)) \ . \end{aligned} \tag{1.9}$$

Here $i_1 \colon u(\nu_k) \longrightarrow u(\nu_k) \oplus \mathrm{sa}(k)$ is the embedding of the second summand. In [19], we consider $\langle t_k \rangle$ as an equivariant stable homotopy class of the $k$-th stage of the eigenspace filtration of $\mathbf{U}$, but now we work in the ambient orthogonal space $\mathbf{U}$.

The *tautological class*

$$e_{U(k),\mathrm{ad}(k)} \ \in \ \pi_{\mathrm{ad}(k)}^{U(k)}(\Sigma^\infty (\mathbf{Gr}_k^{\mathbb{C}})^{\mathrm{ad}(k)})$$

is defined in [19, (A.16)]. By [19, Theorem A.17 (i)], the pair $(\Sigma^\infty (\mathbf{Gr}_k^{\mathbb{C}})^{\mathrm{ad}(k)}, e_{U(k),\mathrm{ad}(k)})$ represents the functor $\pi_{\mathrm{ad}(k)}^{U(k)} \colon \mathcal{GH} \longrightarrow \mathcal{A}b$ on the global stable homotopy category. The global splitting morphism $s_k \colon \Sigma^\infty (\mathbf{Gr}_k^{\mathbb{C}})^{\mathrm{ad}(k)} \longrightarrow \Sigma_+^\infty \mathbf{U}$ is defined in [19, (4.6)] by the property that it takes the tautological class to $\langle t_k \rangle$, i.e., by the relation

$$(s_k)_*(e_{U(k),\mathrm{ad}(k)}) \ = \ \langle t_k \rangle \ . \tag{1.10}$$

In this paper, we shall mostly work with the reduced suspension spectrum $\Sigma^\infty \mathbf{U}$ (as opposed to the unreduced one), and with the 'reduced' version of the classes $\langle t_k \rangle$. We endow $\mathbf{U}$ with the intrinsic basepoint $1 \in \mathbf{U}$ consisting of the multiplicative units. We write $\mathbf{U}_+$ for $\mathbf{U}$ with an additional basepoint added. This comes with based maps $\mathbf{U}_+ \longrightarrow S^0$ and $\mathbf{U}_+ \longrightarrow \mathbf{U}$; the first of these maps $\mathbf{U}$ to the non-basepoint of $S^0$, and the second is the identity of $\mathbf{U}$ and maps the extra basepoint to the intrinsic basepoint 1. We write

$$\epsilon \ : \ \Sigma_+^\infty \mathbf{U} \ \longrightarrow \ \Sigma_+^\infty * = \mathbb{S} \qquad \text{and} \qquad q \ : \ \Sigma_+^\infty \mathbf{U} \ \longrightarrow \ \Sigma^\infty \mathbf{U}$$

for the morphisms induced on reduced suspension spectra. The combined morphism

$$(\epsilon, q) \ : \ \Sigma_+^\infty \mathbf{U} \ \xrightarrow{\sim} \ \mathbb{S} \times (\Sigma^\infty \mathbf{U}) \tag{1.11}$$

is then a global equivalence. We set

$$\sigma_k \ = \ q_* \langle t_k \rangle \ \in \pi_{\mathrm{ad}(k)}^{U(k)}(\Sigma^\infty \mathbf{U}) \ . \tag{1.12}$$

The following representability result is a fairly direct consequence of the stable splitting (1.1). We shall use it to construct global stable morphisms with source $\Sigma^\infty \mathbf{U}$, and to check commutativity of diagrams whose initial object is $\Sigma^\infty \mathbf{U}$. We let $[\![-,-]\!]$ denote the group of morphisms in the global stable homotopy category.

**Theorem 1.13.** *For every global spectrum $X$, the evaluation map*

$$[\![\Sigma^\infty \mathbf{U}, X]\!] \ \xrightarrow{\cong} \ \prod\nolimits_{k \geq 1} \pi_{\mathrm{ad}(k)}^{U(k)}(X) \ , \quad f \ \longmapsto \ (f_*(\sigma_k))_{k \geq 1}$$

*is an isomorphism.*

*Proof.* The splitting (1.1) proved in [19, Theorem 4.10] and the representability property of $(\Sigma^\infty(\mathbf{Gr}_k^{\mathbb{C}})^{\mathrm{ad}(k)}, e_{U(k),\mathrm{ad}(k)})$ together provide the natural isomorphism

$$\llbracket \Sigma^\infty_+\mathbf{U}, X\rrbracket \xrightarrow{\cong} \prod\nolimits_{k\geq 0} \pi^{U(k)}_{\mathrm{ad}(k)}(X)\ , \quad f \longmapsto (f_*\langle t_k\rangle)_{k\geq 0}\ . \tag{1.14}$$

The global equivalence (1.11) induces another isomorphism

$$\llbracket \mathbb{S}, X\rrbracket \times \llbracket \Sigma^\infty\mathbf{U}, X\rrbracket \xrightarrow{\cong} \llbracket \Sigma^\infty_+\mathbf{U}, X\rrbracket\ , \quad (a,b) \longmapsto a\circ\epsilon + b\circ q\ .$$

The morphism $\epsilon\colon \Sigma^\infty_+\mathbf{U}\longrightarrow \mathbb{S}$ sends the class $\langle t_0\rangle$ to $1\in\pi_0(\mathbb{S})$, so the composite

$$\llbracket \mathbb{S}, X\rrbracket \xrightarrow{\epsilon^*} \llbracket \Sigma^\infty_+\mathbf{U}, X\rrbracket \xrightarrow{f\mapsto f_*\langle t_0\rangle} \pi_0(X)$$

is an isomorphism. Moreover, $q_*\langle t_0\rangle = 0$ and $q_*\langle t_k\rangle = \sigma_k$, so the isomorphism (1.14) restricts an isomorphism as in the statement of the theorem, where now the factor indexed by $k=0$ is omitted. $\square$

**Construction 1.15.** The commutative multiplication on $\mathbf{U}$ induces the structure of commutative orthogonal ring spectrum on its unreduced suspension spectrum. The multiplication in turn provides graded-commutative ring structures on the equivariant homotopy groups, and external multiplication maps

$$\times\ :\ \pi_k^G(\Sigma^\infty_+\mathbf{U})\times\pi_l^K(\Sigma^\infty_+\mathbf{U}) \longrightarrow \pi_{k+l}^{G\times K}(\Sigma^\infty_+\mathbf{U})\ ,\quad x\times y\ =\ p_G^*(x)\cdot p_K^*(y)\ ;$$

here $p_G\colon G\times K\longrightarrow G$ and $p_K\colon G\times K\longrightarrow K$ are the projections. We let $T^k$ denote the diagonal maximal torus of $U(k)$. The following proposition will show that $\langle t_k\rangle$ and the class

$$\langle t_1\rangle\times\cdots\times\langle t_1\rangle\ \in\ \pi_k^{T^k}(\Sigma^\infty_+\mathbf{U})\ ,$$

the $k$-fold exterior product of copies of $\langle t_1\rangle\in\pi_1^T(\Sigma^\infty_+\mathbf{U})$, have the same $T^k$-geometric fixed points. The $T^k$-fixed points of the adjoint representation are the diagonal matrices inside $\mathrm{ad}(k)$, and we use the map

$$\mathbb{R}^k \longrightarrow \mathrm{ad}(k)\ ,\quad (x_1,\dots,x_k) \longmapsto \begin{pmatrix} ix_1 & \dots & 0\\ \vdots & \ddots & \vdots\\ 0 & \dots & ix_k\end{pmatrix}$$

to identify $\mathrm{ad}(k)^{T^k}$ with $\mathbb{R}^k$. The $T^k$-geometric fixed point map then becomes a homomorphism

$$\Phi^{T^k}\ :\ \pi^{U(k)}_{\mathrm{ad}(k)}(\Sigma^\infty_+\mathbf{U}) \longrightarrow \Phi_k^{T^k}(\Sigma^\infty_+\mathbf{U})\ .$$

**Theorem 1.16.** *For every $k\geq 2$, the relation*

$$\Phi^{T^k}(\langle t_1\rangle\times\cdots\times\langle t_1\rangle)\ =\ \Phi^{T^k}\langle t_k\rangle$$

*holds in the group $\Phi_k^{T^k}(\Sigma^\infty_+\mathbf{U})$.*

*Proof.* We write

$$\Delta\ :\ \mathbb{C}^k \longrightarrow M(k\times k;\mathbb{C})\ ,\quad (z_1,\dots,z_k) \mapsto \begin{pmatrix} z_1 & \dots & 0\\ \vdots & \ddots & \vdots\\ 0 & \dots & z_k\end{pmatrix}$$

for the embedding as diagonal matrices. This map embeds $U(1)^k = T^k$ diagonally into $U(k)$, it embeds $\mathrm{ad}(1)^k$ diagonally into $\mathrm{ad}(k)$, and it embeds $\mathrm{sa}(1)^k$ diagonally into $\mathrm{sa}(k)$. We denote all these restrictions by $\Delta$, too.

We will show that the following diagram of based continuous maps commutes:

(1.17)
$$\begin{tikzcd}
(S^{\mathrm{sa}(1)\oplus\mathrm{ad}(1)})^{\wedge k} \arrow[r, "(t_1)^{\wedge k}"] \arrow[d, "\mathrm{shuffle}"', "\cong"] & (S^{\mathrm{sa}(1)} \wedge U(1)_+)^{\wedge k} \arrow[r, "(S^{\mathrm{sa}(1)}\wedge\zeta^1_*)^{\wedge k}"] \arrow[d, "\cong"', "\mathrm{shuffle}"] & (S^{\mathrm{sa}(1)} \wedge \mathbf{U}(u(\nu_1))^T_+)^{\wedge k} \arrow[d, "\mathrm{shuffle}"] \\
S^{\mathrm{sa}(1)^k \oplus \mathrm{ad}(1)^k} \arrow[d, "\Delta\wedge\Delta"', "\cong"] & S^{\mathrm{sa}(1)^k} \wedge U(1)^k_+ \arrow[r, "S^{\mathrm{sa}(1)^k}\wedge(\zeta^1_*)^k_+"] \arrow[d, "\cong"', "\Delta\wedge\Delta"] & (S^{\mathrm{sa}(1)^k} \wedge \mathbf{U}(u(\nu_1))^k_+)^{T^k} \arrow[d, "(\Delta\wedge\mu^{(k)}_{u(\nu_1)})^{T^k}"] \\
(S^{\mathrm{sa}(k)\oplus\mathrm{ad}(k)})^{T^k} \arrow[r, "(t_k)^{T^k}"] & (S^{\mathrm{sa}(k)} \wedge U(k)^{\mathrm{ad}}_+)^{T^k} \arrow[r, "(S^{\mathrm{sa}(k)}\wedge\zeta^k_*)^{T^k}"'] \arrow[d, "(S^{\mathrm{sa}(k)}\wedge(\mathbf{U}(i_1)\circ\zeta^k_*))^{T^k}"'] & (S^{\mathrm{sa}(k)} \wedge \mathbf{U}(u(\nu_k))_+)^{T^k} \arrow[d, "(S^{\mathrm{ad}(k)}\wedge\mathbf{U}(i_1)_+)^{T^k}"] \\
& ((\Sigma^\infty_+\mathbf{U})(u(\nu_k)\oplus\mathrm{sa}(k)))^{T^k} \arrow[r, equal] & (S^{\mathrm{ad}(k)} \wedge \mathbf{U}(u(\nu_k)\oplus\mathrm{sa}(k))_+)^{T^k}
\end{tikzcd}$$

The clockwise composite represents the class $\Phi^{T^k}(\langle t_1\rangle \times \cdots \times \langle t_1\rangle)$; here we exploit that $(\nu_1)^T = 0$, so the smash factor $S^{\nu_1}$ in the representative (1.9) for $\langle t_1\rangle$ does not contribute to the representative for its $T$-geometric fixed points, and we omit it. The counter clockwise composite represents the class $\Phi^{T^k}\langle t_k\rangle$; here we exploit that $(\nu_k)^{T^k} = 0$, so the smash factor $S^{\nu_k}$ in the representative (1.9) for $\langle t_k\rangle$ does not contribute to the representative for its $T^k$-geometric fixed points. Granted its commutativity, the diagram (1.17) witnesses the desired relation.

Now we justify the commutativity of the diagram (1.17). The following diagram of open embeddings commutes:

$$\begin{tikzcd}
(\mathrm{sa}(1)\times U(1))^k \arrow[r, "(\phi_1)^k"] \arrow[d, "\mathrm{shuffle}"', "\cong"] & \mathbb{C}^k \arrow[r, equal] \arrow[dd, "\cong"', "\Delta"] & (\mathrm{sa}(1)\oplus\mathrm{ad}(1))^k \arrow[d, "\cong"', "\mathrm{shuffle}"] \\
\mathrm{sa}(1)^k \times U(1)^k \arrow[d, "\Delta\times\Delta"', "\cong"] & & \mathrm{sa}(1)^k \times \mathrm{ad}(1)^k \arrow[d, "\cong"', "\Delta\times\Delta"] \\
(\mathrm{sa}(k)\times U(k)^{\mathrm{ad}})^{T^k} \arrow[r, "(\phi_k)^{T^k}"'] & M(k\times k,\mathbb{C})^{T^k} \arrow[r, equal] & (\mathrm{sa}(k)\times\mathrm{ad}(k))^{T^k}
\end{tikzcd}$$

The vertical maps in this diagram are homeomorphisms. So associated diagram of collapse maps commutes, too, which is the left part of diagram (1.17). The iterated multiplication morphism

$$\mu^{(k)}_{u(\nu_1)} \ : \ \mathbf{U}(u(\nu_1)) \times \cdots \times \mathbf{U}(u(\nu_1)) \ \longrightarrow \ \mathbf{U}(u(\nu_1) \oplus \cdots \oplus u(\nu_1)) = \mathbf{U}(u(\nu_k))$$

of the ultra-commutative monoid $\mathbf{U}$ is given by orthogonal direct sum. So it participates in a commutative diagram:

$$\begin{tikzcd}
U(1)^k \arrow[r, "(\zeta^1_*)^k"] \arrow[d, "\Delta"'] & \mathbf{U}(u(\nu_1))^k \arrow[d, "\mu^{(k)}_{u(\nu_1)}"] \\
U(k)^{\mathrm{ad}} \arrow[r, "\zeta^k_*"'] & \mathbf{U}(u(\nu_k))
\end{tikzcd}$$

This implies the commutativity of the middle right part of diagram (1.17). □

**Construction 1.18.** We write

(1.19)
$$\mathfrak{c} \ : \ S^1 \ \xrightarrow{\cong} \ U(1) \ , \quad \mathfrak{c}(x) = (x+i)(x-i)^{-1}$$

for the Cayley transform. We write

$$\partial \ : \ \mathbb{C} \ \longrightarrow \ M(k\times k;\mathbb{C}) \ , \quad z \ \mapsto \ \begin{pmatrix} z & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & z \end{pmatrix}$$

for the embedding as constant diagonal matrices. This map identifies $U(1)$ with the center of $U(k)$, consisting of the constant diagonal matrices, which also equals the $U(k)$-fixed space under the conjugation action $(U(k)^{\mathrm{ad}})^{U(k)}$. We consider the composite

$$\delta_k \ : \ S^1 \ \xrightarrow[\cong]{\mathfrak{c}} \ U(1) \ \xrightarrow{\partial} \ U(k)^{\mathrm{ad}} \ \xrightarrow{\zeta^k_*} \ \mathbf{U}(u(\nu_k)) \ ,$$

where $\zeta^k_*$ was defined in (1.8). We define

$$d_k \ \in \ \pi_1^{U(k)}(\Sigma^\infty \mathbf{U}) \tag{1.20}$$

as the class of the map

$$S^{\nu_k}\wedge \delta_k \ : \ S^{\nu_k}\wedge S^1 \ \longrightarrow \ S^{\nu_k}\wedge \mathbf{U}(u(\nu_k)) = (\Sigma^\infty\mathbf{U})(u(\nu_k)) \ .$$

We let $H$ be a closed subgroup of $U(k)$ whose tautological action on $\mathbb{C}^k$ is irreducible. Equivalently, $H$ is not subconjugate to a block subgroup $U(j,k-j)$ for any $1\le j\le k-1$. Then scalar multiplications are the only $H$-equivariant automorphisms of $\mathbb{C}^k$, and thus $(U(k)^{\mathrm{ad}})^H$ coincides with the center of $U(k)$. Moreover, the $H$-fixed points of the adjoint representation are the diagonal matrices inside $\mathrm{ad}(k)$, and we use the map

$$\mathbb{R} \ \longrightarrow \ \mathrm{ad}(k) \ , \quad y \ \longmapsto \partial(iy)$$

to identify $\mathrm{ad}(k)^H$ with $\mathbb{R}$. The $H$-geometric fixed point map then becomes a homomorphism

$$\Phi^H \ : \ \pi^{U(k)}_{\mathrm{ad}(k)}(\Sigma^\infty\mathbf{U}) \ \longrightarrow \ \Phi^H_1(\Sigma^\infty\mathbf{U}) \ .$$

**Theorem 1.21.**
(i) *The relation $\sigma_1 = d_1$ holds in the group $\pi_1^T(\Sigma^\infty\mathbf{U})$.*
(ii) *For $k\ge 2$, let $H$ be a closed subgroup of $U(k)$ whose tautological action on $\mathbb{C}^k$ is irreducible. Then the relation*

$$\Phi^H(\sigma_k) \ = \ \Phi^H(d_k)$$

*holds in the group $\Phi^H_1(\Sigma^\infty\mathbf{U})$.*

*Proof.* We start with a preliminary study that we need in both parts of the proof. We have $\mathrm{sa}(1)=\mathbb{R}$ and $\mathrm{ad}(1)=i\cdot\mathbb{R}$, both with trivial action by $T=U(1)$. We recall from (1.4) the open embedding

$$\phi_1 \ : \ \mathbb{R}\times U(1) \ \longrightarrow \ \mathbb{C}=\mathbb{R}^2 \ , \quad (z,\lambda) \ \longmapsto \ \exp(-z)\cdot\lambda \ = \ (\exp(-z)\cdot\mathrm{Re}(\lambda), \exp(-z)\cdot\mathrm{Im}(\lambda))$$

whose image is $\mathbb{C}\setminus\{0\}$. The associated collapse map is $t_1\colon S^2\longrightarrow S^1\wedge U(1)_+$. We claim that the composite $(S^1\wedge q)\circ t_1\colon S^2\longrightarrow S^1\wedge U(1)$ is homotopic to the suspension of the Cayley transform (1.19). The Cayley transform a homeomorphism, so we may show that the composite

$$S^2 \ \xrightarrow{t_1} \ S^1\wedge U(1)_+ \ \xrightarrow{S^1\wedge q} \ S^1\wedge U(1) \ \xrightarrow[\cong]{S^1\wedge\mathfrak{c}^{-1}} \ S^2 \tag{1.22}$$

is homotopic to the identity. This composite collapses the contractible subset $[0,\infty]\times\{0\}$ of $S^2$ to the basepoint and factors through a homeomorphism

$$S^2/([0,\infty]\times\{0\}) \ \cong \ S^2 \ .$$

So the composite (1.22) is a homotopy equivalence. Expanding all formulas shows that the composite (1.22) is given on $\mathbb{R} \times \mathbb{R}_{>0}$ by the formula

$$F \ : \ \mathbb{R}^2 \ \longrightarrow \ \mathbb{R}^2 \ , \quad F(x,y) \ = \ \left( -\ln(\sqrt{x^2+y^2}), \ x/y + \sqrt{(x/y)^2+1} \right) \ .$$

So (1.22) fixes the point $(0,1)$, and it is smooth near $(0,1)$ with differential $D_{(0,1)}F = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. Since this differential has determinant 1, the composite (1.22) is indeed homotopic to the identity. This concludes the proof of the claim that $(S^1 \wedge q) \circ t_1 \sim S^1 \wedge \mathfrak{c}$.

(i) The class $\sigma_1 = q_*\langle t_1 \rangle$ is represented by the $U(1)$-map

$$S^{\nu_1 \oplus \mathbb{R}^2} \ \xrightarrow{S^{\nu_1} \wedge t_1} \ S^{\nu_1 \oplus \mathbb{R}} \wedge U(1)_+ \ \xrightarrow{S^{\nu_1 \oplus \mathbb{R}} \wedge (\mathbf{U}(i_1) \circ \zeta^1_* \circ q)} \ S^{\nu_1 \oplus \mathbb{R}} \wedge \mathbf{U}(u(\nu_1) \oplus \mathbb{R}) = (\Sigma^\infty \mathbf{U})(u(\nu_1) \oplus \mathbb{R}) \ .$$

The class $d_1$ is represented by the composite

$$S^{\nu_1} \wedge \delta_1 \ : \ S^{\nu_1 \oplus \mathbb{R}} \ \xrightarrow{S^{\nu_1} \wedge \mathfrak{c}} \ S^{\nu_1} \wedge U(1) \ \xrightarrow{S^{\nu_1} \wedge \zeta^1_*} \ S^{\nu_1} \wedge \mathbf{U}(u(\nu_1)) = (\Sigma^\infty \mathbf{U})(u(\nu_1)) \ .$$

Since $(S^1 \wedge q) \circ t_1$ is homotopic to $S^1 \wedge \mathfrak{c}$, this proves the first claim.

(ii) Since the class $\sigma_k$ arises from the collapse map $t_k$ defined in (1.5), we first study the $H$-fixed points of $t_k$. The vertical maps in the following commutative diagram are homeomorphisms:

$$\begin{tikzcd}
\mathbb{R} \times U(1) \arrow[r, "\phi_1"] \arrow[d, "\partial \times \partial"', "\cong"] & \mathbb{C} \arrow[r, equal] \arrow[d, "\cong"', "\partial"] & \mathbb{R}^2 \arrow[d, "\cong"', "\partial \times \partial(i \cdot -)"] \\
(\mathrm{sa}(k) \times U(k)^{\mathrm{ad}})^H \arrow[r, "(\phi_k)^H"'] & M(k \times k, \mathbb{C})^H \arrow[r, equal] & (\mathrm{sa}(k) \times \mathrm{ad}(k))^H
\end{tikzcd}$$

So the collapse maps for the horizontal open embeddings participate in a commutative diagram that is the upper left rectangle in the following diagram of based continuous maps:

$$\begin{tikzcd}
S^2 \arrow[r, "t_1"'] \arrow[rr, bend left, "S^1 \wedge (\delta_k)^H"] \arrow[d, "\partial \wedge \partial(i \cdot -)"', "\cong"] & S^1 \wedge U(1)_+ \arrow[r, "S^1 \wedge (\zeta^k_* \circ \partial \circ q)^H"'] \arrow[d, "\cong"', "\partial \wedge \partial"] & S^1 \wedge \mathbf{U}(u(\nu_k))^H \arrow[d, "\cong"', "\partial \wedge \mathbf{U}(u(\nu_k))^H"] \\
(S^{\mathrm{sa}(k) \oplus \mathrm{ad}(k)})^H \arrow[r, "(t_k)^H"'] \arrow[dr, bend right, "\text{represents } \Phi^H(\sigma_k)"'] & (S^{\mathrm{sa}(k)} \wedge (U(k)^{\mathrm{ad}})_+)^H \arrow[r, "(S^{\mathrm{sa}(k)} \wedge (\zeta^k_* \circ q))^H"'] & (S^{\mathrm{ad}(k)} \wedge \mathbf{U}(u(\nu_k)))^H \arrow[d, "(S^{\mathrm{sa}(k)} \wedge \mathbf{U}(i_1))^H"] \\
& ((\Sigma^\infty \mathbf{U})(u(\nu_k) \oplus \mathrm{sa}(k)))^H \arrow[r, equal] & (S^{\mathrm{sa}(k)} \wedge \mathbf{U}(u(\nu_k) \oplus \mathrm{sa}(k)))^H
\end{tikzcd}$$

The uppermost part commutes because $(S^1 \wedge q) \circ t_1 \sim S^1 \wedge \mathfrak{c}$ and $\delta_k = \zeta^k_* \circ \partial \circ \mathfrak{c}$. The class $\sigma_k$ is represented by the $U(k)$-map

$$\begin{aligned} S^{\nu_k \oplus \mathrm{sa}(k) \oplus \mathrm{ad}(k)} \ &\xrightarrow{S^{\nu_k} \wedge t_k} \ S^{\nu_k \oplus \mathrm{sa}(k)} \wedge U(k)^{\mathrm{ad}}_+ \\ &\xrightarrow{S^{\nu_k \oplus \mathrm{sa}(k)} \wedge (\mathbf{U}(i_1) \circ \zeta^k_* \circ q)} \ S^{\nu_k \oplus \mathrm{sa}(k)} \wedge \mathbf{U}(u(\nu_k) \oplus \mathrm{sa}(k)) = (\Sigma^\infty \mathbf{U})(u(\nu_k) \oplus \mathrm{sa}(k)) \ . \end{aligned}$$

The hypothesis on $H$ guarantees that $(\nu_k)^H = 0$. So the smash factor $S^{\nu_k}$ in the representative for $\sigma_k$ does not contribute to the representative for its $H$-geometric fixed points. The class $\Phi^H(\sigma_k)$ is thus represented by the lower left diagonal composite, as indicated in the diagram.

The class $d_k$ is represented by the map $S^{\nu_k} \wedge \delta_k$, by definition. Again because $(\nu_k)^H = 0$, the smash factor $S^{\nu_k}$ does not contribute to the representative for $\Phi^H(d_k)$, which is thus represented by the map $(\delta_k)^H$. The big diagram above thus witnesses the desired relation $\Phi^H(\sigma_k) = \Phi^H(d_k)$. $\square$

## 2. The eigenspace morphism

The connective global K-theory spectrum **ku** is defined in [18, Construction 3.6.9], generalizing a configuration space model of Segal [22, Section 1] to the global equivariant context. We recall the definition in Construction 2.2 below. The *eigenspace morphism* of based orthogonal spaces

$$\mathrm{eig} \ : \ \mathbf{U} \ \longrightarrow \ \Omega^\bullet(\mathrm{sh}\,\mathbf{ku}) \tag{2.1}$$

is defined in [18, (6.3.26)]. Here 'sh' denotes the shift of an orthogonal spectrum [18, Construction 3.1.21], which is globally equivalent to the suspension. And $\Omega^\bullet$ is the functor from orthogonal spectra to based orthogonal spaces that is right adjoint to the reduced suspension spectrum functor, see [18, Construction 4.1.6]. For every orthogonal spectrum $X$, the orthogonal space $\Omega^\bullet X$ models the underlying 'global infinite loop space' of the global spectrum $X$.

As the name suggest, this morphism assigns to a unitary automorphism the configuration of eigenvalues and eigenspaces; the shift coordinate in $\mathrm{sh}\,\mathbf{ku}$ is the place that stores the eigenvalues. The eigenspace morphism (2.1) is a $\mathcal{F}in$-global equivalence by [18, Theorem 6.3.27]. We will mostly work with the adjoint

$$\mathrm{eig}^\natural \ : \ \Sigma^\infty \mathbf{U} \ \longrightarrow \ \mathrm{sh}\,\mathbf{ku}$$

of the eigenspace morphism, and we recall the definition of $\mathrm{eig}^\natural$ in Construction 2.3 below. The purpose of this section is to show that for every compact Lie group $G$, the map of equivariant homotopy groups

$$(\mathrm{eig}^\natural \circ q)_* \ : \ \pi_*^G(\Sigma^\infty_+ \mathbf{U}) \ \longrightarrow \ \pi_*^G(\mathrm{sh}\,\mathbf{ku})$$

annihilates the square of the augmentation ideal, see Theorem 2.6.

The eigenspace morphism (2.1) is *not* a fully global equivalence, but the composite $\Omega^\bullet(\mathrm{sh}\,j)\circ\mathrm{eig}\colon \mathbf{U} \longrightarrow \Omega^\bullet(\mathrm{sh}\,\mathbf{KU})$ is, see [18, Theorem 6.4.21]. This is related to the fact that the periodization morphism $j\colon \mathbf{ku} \longrightarrow \mathbf{KU}$ is a $\mathcal{F}in$-global connective cover, but it is *not* a connective cover for compact Lie groups of positive dimensions, see for example [18, Remark 6.3.38].

**Construction 2.2** (The connective global K-theory spectrum)**.** We recall from [18, Construction 6.3.9] the definition of the connective global K-theory spectrum **ku**, an adaptation of Segal's configuration space model [22, Section 1] to the global equivariant context. We let $\mathcal{U}$ be a hermitian inner product space of countable dimension (finite or infinite). We recall the $\Gamma$-space $\mathscr{C}(\mathcal{U})$ of 'orthogonal subspaces in $\mathcal{U}$'. For a finite based set $A$ we let $\mathscr{C}(\mathcal{U}, A)$ be the space of tuples $(E_a)_{a\in A\setminus\{0\}}$, indexed by the non-basepoint elements of $A$, of finite-dimensional, pairwise orthogonal $\mathbb{C}$-subspaces of $\mathcal{U}$. The topology on $\mathscr{C}(\mathcal{U}, A)$ is that of a disjoint union of subspaces of a product of Grassmannians. The basepoint of $\mathscr{C}(\mathcal{U}, A)$ is the tuple where $E_a = \{0\}$ for all $a \in A\setminus\{0\}$. For a based map $\alpha\colon A \longrightarrow B$ the induced map $\mathscr{C}(\mathcal{U},\alpha)\colon \mathscr{C}(\mathcal{U}, A) \longrightarrow \mathscr{C}(\mathcal{U}, B)$ sends $(E_a)$ to $(F_b)$ where

$$F_b \ = \ \bigoplus\nolimits_{\alpha(a)=b} E_a .$$

Every $\Gamma$-space can be evaluated on a based space by a coend construction, see for example [18, (4.5.14)]. Categorically speaking, this coend realizes the enriched Kan extension along the inclusion of $\Gamma$ into the category of based spaces. We write $\mathscr{C}(\mathcal{U}, K) = \mathscr{C}(\mathcal{U})(K)$ for the value of the $\Gamma$-space $\mathscr{C}(\mathcal{U})$ on a based space $K$. Elements of $\mathscr{C}(\mathcal{U}, K)$ can be interpreted as 'labeled configurations': a point is represented by an unordered tuple

$$[E_1, \dots, E_n;\, k_1, \dots, k_n]$$

where $(E_1, \dots, E_n)$ is an $n$-tuple of finite-dimensional, pairwise orthogonal subspaces of $\mathcal{U}$, and $k_1, \dots, k_n$ are points of $K$, for some $n$. The topology is such that, informally speaking, the labels sum up whenever two points collide, and a label disappears whenever a point approaches the basepoint of $K$.

The value of the orthogonal spectrum **ku** on a euclidean inner product space $V$ is

$$\mathbf{ku}(V) \ = \ \mathscr{C}(\mathrm{Sym}(V_\mathbb{C}), S^V) \ ,$$

the value of the Γ-space $\mathscr{C}(\mathrm{Sym}(V_{\mathbb{C}}))$ on the sphere $S^V$; the inner product on the symmetric algebra is described in [18, Proposition 6.3.8]. The action of $O(V)$ on $V$ then extends to a unitary action on $\mathrm{Sym}(V_{\mathbb{C}})$. We let the orthogonal group $O(V)$ act diagonally, via the action on the sphere $S^V$ and the action on the Γ-space $\mathscr{C}(\mathrm{Sym}(V_{\mathbb{C}}))$. For the structure maps we refer to [18, Construction 6.3.9].

**Construction 2.3** (The adjoint eigenspace morphism)**.** We recall the adjoint

$$\mathrm{eig}^\natural \ : \ \Sigma^\infty \mathbf{U} \ \longrightarrow \ \mathrm{sh}\,\mathbf{ku} \ . \tag{2.4}$$

of the eigenspace morphism (2.1), the latter being defined in [18, (6.3.26)]. Its value

$$\mathrm{eig}^\natural(V) \ : \ S^V \wedge \mathbf{U}(V) = S^V \wedge U(V_{\mathbb{C}}) \ \longrightarrow \ \mathscr{C}(\mathrm{Sym}((V\oplus\mathbb{R})_{\mathbb{C}}), S^{V\oplus\mathbb{R}}) = \mathbf{ku}(V\oplus\mathbb{R}) = (\mathrm{sh}\,\mathbf{ku})(V)$$

at an inner product space $V$ is defined as follows. For a unitary endomorphism $A \in U(V_{\mathbb{C}})$, we let $\lambda_1,\dots,\lambda_n \in U(1)\setminus\{1\}$ be the set of its eigenvalues different from 1, and we let $E(\lambda_j)$ be the eigenspace of $A$ for the eigenvalue $\lambda_j$. We let

$$\mathfrak{c}^{-1} \ : \ U(1) \ \xrightarrow{\cong} \ S^1 \ , \quad \mathfrak{c}^{-1}(\lambda) = i\cdot(\lambda+1)(\lambda-1)^{-1} \tag{2.5}$$

be the inverse of the Cayley transform (1.19). Then the map $\mathrm{eig}(V)$ is defined by

$$\mathrm{eig}^\natural(V)(v\wedge A) \ = \ [E(\lambda_1),\dots,E(\lambda_n);\,(v,\mathfrak{c}^{-1}(\lambda_1)),\dots,(v,\mathfrak{c}^{-1}(\lambda_n))] \ .$$

In other words, $\mathrm{eig}^\natural(V)(v\wedge A)$ is the configuration of the points $(v,\mathfrak{c}^{-1}(\lambda_i)) \in S^{V\oplus\mathbb{R}}$ labeled by the eigenspace $E(\lambda_i)$ of $A$, whence the name. Strictly speaking, $E(\lambda_i)$ is a subspace of $V_{\mathbb{C}}$, which we embed into the linear summand of $\mathrm{Sym}((V\oplus\mathbb{R})_{\mathbb{C}})$.

The ultra-commutative multiplication on $\mathbf{U}$ induces an ultra-commutative ring spectrum structure on the unreduced suspension spectrum $\Sigma^\infty_+\mathbf{U}$. This, in turn, induces a product structure on the equivariant homotopy groups of $\Sigma^\infty_+\mathbf{U}$. The morphism $\epsilon\colon \Sigma^\infty_+\mathbf{U} \longrightarrow \mathbb{S}$ arising from the unique morphism of orthogonal spaces $\mathbf{U} \longrightarrow *$ is a morphism of ultra-commutative ring spectra, and thus induces a morphism of graded-commutative equivariant homotopy rings

$$\epsilon_* \ : \ \pi_*^G(\Sigma^\infty_+\mathbf{U}) \ \longrightarrow \ \pi_*^G(\mathbb{S}) \ .$$

Because the morphism $\mathbf{U} \longrightarrow *$ has a section given by the identity elements of the unitary groups, $\epsilon_*$ is surjective. The *augmentation ideal* of $\pi_*^G(\Sigma^\infty_+\mathbf{U})$ is the kernel of this homomorphism $\epsilon_*$.

**Theorem 2.6.** *For every compact Lie group $G$, the composite*

$$\pi_*^G(\Sigma^\infty_+\mathbf{U}) \ \xrightarrow{q_*} \ \pi_*^G(\Sigma^\infty\mathbf{U}) \ \xrightarrow{\mathrm{eig}^\natural_*} \ \pi_*^G(\mathrm{sh}\,\mathbf{ku})$$

*annihilates the square of the augmentation ideal.*

*Proof.* We define the exterior multiplication

$$\boxtimes \ : \ \pi_k^G(\Sigma^\infty_+\mathbf{U})\times\pi_l^G(\Sigma^\infty_+\mathbf{U}) \ \longrightarrow \ \pi_{k+l}^G(\Sigma^\infty_+(\mathbf{U}\boxtimes\mathbf{U}))$$

as the composite

$$\pi_k^G(\Sigma^\infty_+\mathbf{U})\times\pi_l^G(\Sigma^\infty_+\mathbf{U}) \ \xrightarrow{\cdot} \ \pi_{k+l}^G((\Sigma^\infty_+\mathbf{U})\wedge(\Sigma^\infty_+\mathbf{U})) \ \cong \ \pi_{k+l}^G(\Sigma^\infty_+(\mathbf{U}\boxtimes\mathbf{U})) \ ,$$

where the isomorphism is induced by the strong symmetric monoidal structure on the unreduced suspension spectrum functor, see [18, (4.1.17)]. Then

$$(\Sigma^\infty_+\mu)_*(x\boxtimes y) \ = \ x\cdot y \ ,$$

for all equivariant homotopy classes $x,y\in\pi_*^G(\Sigma^\infty_+\mathbf{U})$, where $\mu\colon \mathbf{U}\boxtimes\mathbf{U}\longrightarrow\mathbf{U}$ denotes the multiplication morphism.

We let $\rho_1, \rho_2 \colon \mathbf{U} \boxtimes \mathbf{U} \longrightarrow \mathbf{U}$ denote the projections to the two factors. The morphism $\Sigma^\infty_+ \rho_1 \colon \Sigma^\infty_+(\mathbf{U} \boxtimes \mathbf{U}) \longrightarrow \Sigma^\infty_+ \mathbf{U}$ factors as the composite

$$\Sigma^\infty_+(\mathbf{U} \boxtimes \mathbf{U}) \ \cong (\Sigma^\infty_+ \mathbf{U}) \wedge (\Sigma^\infty_+ \mathbf{U}) \ \xrightarrow{\mathrm{Id} \wedge \epsilon} \ (\Sigma^\infty_+ \mathbf{U}) \wedge \mathbb{S} \ \cong \ \Sigma^\infty_+ \mathbf{U} \ ,$$

where the final isomorphism is the unit isomorphism of the smash product, and similarly for $\Sigma^\infty_+ \rho_2$. So

$$(2.7) \qquad (\Sigma^\infty_+ \rho_1)_*(x \boxtimes y) \ = \ x \cdot \epsilon_*(y) \qquad \text{and} \qquad (\Sigma^\infty_+ \rho_2)_*(x \boxtimes y) \ = \ \epsilon_*(x) \cdot y \ .$$

Below we introduce an orthogonal spectrum $\mathbf{ku}^{[2]}$ together with a morphism $\mathrm{eig}^{[2]} \colon \Sigma^\infty(\mathbf{U} \boxtimes \mathbf{U}) \longrightarrow \mathrm{sh}\, \mathbf{ku}^{[2]}$. The spectrum $\mathbf{ku}^{[2]}$ is globally equivalent to a product of two copies of $\mathbf{ku}$. The upshot of the construction will be a commutative diagram of orthogonal spectra:

$$(2.8) \qquad \begin{tikzcd}
(\Sigma^\infty_+ \mathbf{U}) \times (\Sigma^\infty_+ \mathbf{U}) \arrow[r, "q \times q"] & (\Sigma^\infty \mathbf{U}) \times (\Sigma^\infty \mathbf{U}) \arrow[r, "\mathrm{eig}^\natural \times \mathrm{eig}^\natural"] & (\mathrm{sh}\, \mathbf{ku}) \times (\mathrm{sh}\, \mathbf{ku}) \\
\Sigma^\infty_+(\mathbf{U} \boxtimes \mathbf{U}) \arrow[u, "(\Sigma^\infty_+ \rho_1, \Sigma^\infty_+ \rho_2)"] \arrow[r, "q^{[2]}"'] \arrow[d, "\Sigma^\infty_+ \mu"'] & \Sigma^\infty(\mathbf{U} \boxtimes \mathbf{U}) \arrow[u, "(\Sigma^\infty \rho_1, \Sigma^\infty \rho_2)"] \arrow[r, "\mathrm{eig}^{[2]}"'] \arrow[d, "\Sigma^\infty \mu"'] & \mathrm{sh}\, \mathbf{ku}^{[2]} \arrow[u, "\sim", "(\mathrm{sh}\, p_1, \mathrm{sh}\, p_2)"'] \arrow[d, "\mathrm{sh}\, \nabla"] \\
\Sigma^\infty_+ \mathbf{U} \arrow[r, "q"'] & \Sigma^\infty \mathbf{U} \arrow[r, "\mathrm{eig}^\natural"'] & \mathrm{sh}\, \mathbf{ku}
\end{tikzcd}$$

The upward pointing morphisms $p_1, p_2 \colon \mathbf{ku}^{[2]} \longrightarrow \mathbf{ku}$ are incarnations of projections to the factors. The morphism $\nabla \colon \mathbf{ku}^{[2]} \longrightarrow \mathbf{ku}$ is a model for the fold map.

The value of the orthogonal spectrum $\mathbf{ku}^{[2]}$ on a euclidean inner product space $V$ is the configuration space

$$\mathbf{ku}^{[2]}(V) \ = \ \mathscr{C}(\mathrm{Sym}(V_\mathbb{C}), S^V \vee S^V) \ ,$$

the value of the $\Gamma$-space $\mathscr{C}(\mathrm{Sym}(V_\mathbb{C}))$ on the wedge of two copies of $S^V$. The action of $O(V)$ and the structure maps of $\mathbf{ku}^{[2]}$ are defined in much the same way as for $\mathbf{ku}$. The projection and fold maps

$$p_1,\, p_2,\, \nabla \ : \ S^V \vee S^V \ \longrightarrow \ S^V$$

induce continuous, based and $O(V)$-equivariant maps of configuration spaces

$$p_1(V),\, p_2(V),\, \nabla(V) \ : \ \mathbf{ku}^{[2]}(V) \ = \ \mathscr{C}(\mathrm{Sym}(V_\mathbb{C}), S^V \vee S^V) \ \longrightarrow \ \mathscr{C}(\mathrm{Sym}(V_\mathbb{C}), S^V) \ = \ \mathbf{ku}(V) \ .$$

For varying inner product spaces $V$, these maps assemble into morphisms of orthogonal spectra

$$p_1,\, p_2,\, \nabla \ : \ \mathbf{ku}^{[2]} \ \longrightarrow \ \mathbf{ku} \ .$$

We claim that the morphism

$$(p_1, p_2) \ : \ \mathbf{ku}^{[2]} \ \longrightarrow \ \mathbf{ku} \times \mathbf{ku}$$

is a global equivalence of orthogonal spectra. We call an orthogonal $G$-representation $V$ *ample* if the complex symmetric algebra $\mathrm{Sym}(V_\mathbb{C})$ is a complete complex $G$-universe. If $V$ is ample, then the $G$-$\Gamma$-space $\mathscr{C}(\mathrm{Sym}(V_\mathbb{C}), -)$ is special by [18, Theorem 6.3.19 (i)], and it is $G$-cofibrant by [18, Example 6.3.16]. So the $G$-$\Gamma$-space $\mathscr{C}(\mathrm{Sym}(V_\mathbb{C}), S^1 \wedge -)$ is very special and cofibrant, and thus takes wedges of finite based $G$-CW-complexes to products, up to $G$-weak equivalence, by [18, Theorem B.61 (i)]. In particular, if $V$ is ample and $V^G \neq 0$, then the map

$$\begin{aligned}(p_1(V), p_2(V)) \ : \ \mathbf{ku}^{[2]}(V) \ = \ \mathscr{C}(\mathrm{Sym}(V_\mathbb{C}), S^V \vee S^V) \ \longrightarrow \\ \mathscr{C}(\mathrm{Sym}(V_\mathbb{C}), S^V) \times \mathscr{C}(\mathrm{Sym}(V_\mathbb{C}), S^V) \ = \ \mathbf{ku}(V) \times \mathbf{ku}(V)\end{aligned}$$

is a $G$-weak equivalence. The ample $G$-representations with nonzero $G$-fixed points are cofinal in all orthogonal $G$-representations, so this proves the claim that the morphism $(p_1, p_2)$ is a global equivalence.

Since shifting orthogonal spectra preserves global equivalences and products, also the upper right vertical morphism $(\mathrm{sh}\, p_1, \mathrm{sh}\, p_2)$ in (2.8) is a global equivalence.

The morphism $\mathrm{eig}^{[2]}\colon \Sigma^\infty(\mathbf{U}\boxtimes\mathbf{U}) \longrightarrow \mathrm{sh}\,\mathbf{ku}^{[2]}$ is a variation of the adjoint eigenspace morphism (2.4), but with two (instead of one) unitary parameters. Its value at an inner product space $V$ is the map

$$\mathrm{eig}^{[2]}(V) \ :\ S^V \wedge (\mathbf{U}\boxtimes\mathbf{U})(V) \ \longrightarrow\ \mathscr{C}(\mathrm{Sym}((V\oplus\mathbb{R})_{\mathbb{C}}), S^{V\oplus\mathbb{R}}\vee S^{V\oplus\mathbb{R}}) = \mathbf{ku}^{[2]}(V\oplus\mathbb{R}) = (\mathrm{sh}\,\mathbf{ku}^{[2]})(V)$$

defined as follows. Elements of $(\mathbf{U}\boxtimes\mathbf{U})(V)$ are pairs $(A,B)$ of unitary endomorphisms $A,B\in U(V)$ that are *transverse* in the following sense: there exists an orthogonal direct sum decomposition $V = V'\oplus V''$ such that $A$ is the identity on $V'$, and $B$ is the identity on $V''$. The transversality hypothesis in particular means that $A$ and $B$ commute (but it is stronger than that), so $A$ and $B$ are simultaneously diagonalizable. We let $\lambda_1,\dots,\lambda_n \in U(1)\setminus\{1\}$ be the set of all eigenvalues of $A$ and $B$ different from 1, we let $E(\lambda_j)$ be the eigenspace of $A$ for eigenvalue $\lambda_j$, and we let $F(\lambda_j)$ be the eigenspace of $B$ for eigenvalue $\lambda_j$. By the transversality hypothesis, all these eigenspaces $E(\lambda_i)$ and $F(\lambda_j)$ are pairwise orthogonal. We can then define the map $\mathrm{eig}^{[2]}(V)$ by

$$\mathrm{eig}^{[2]}(V)(v\wedge(A,B)) \ =\ \{E(\lambda_j),(v,\mathfrak{c}^{-1}(\lambda_j))_1\}_{1\le j\le n} \ \cup\ \{F(\lambda_j),(v,\mathfrak{c}^{-1}(\lambda_j))_2\}_{1\le j\le n}\ .$$

As before, $\mathfrak{c}^{-1}$ is the inverse Cayley transform (2.5). The subscripts '1' and '2' indicate in which of the two wedge summands of $S^{V\oplus\mathbb{R}}\vee S^{V\oplus\mathbb{R}}$ the respective point lies. In other words, the eigenspace $E(\lambda_j)$ of $A$ is attached to the point $(v,\mathfrak{c}^{-1}(\lambda_j))$ in the first wedge summand, and the eigenspace $F(\lambda_j)$ of $B$ is attached to the point $(v,\mathfrak{c}^{-1}(\lambda_j))$ in the second wedge summand.

The relation

$$(\mathrm{sh}\, p_1)\circ\mathrm{eig}^{[2]} \ =\ \mathrm{eig}^\natural\circ(\Sigma^\infty\rho_1) \ :\ \Sigma^\infty(\mathbf{U}\boxtimes\mathbf{U}) \ \longrightarrow\ \mathrm{sh}\,\mathbf{ku}$$

holds because for every pair $(A,B)$ of transverse unitary matrices, both composites forget $B$ and record the eigenvalues and eigenspaces of $A$. And similarly for the projections to the second factor. So the upper part of the diagram (2.8) commutes. The lower part of the diagram (2.8) commutes by the fact that for every pair $(A,B)$ of transverse unitary matrices and all $\lambda\in U(1)\setminus\{1\}$, the $\lambda$-eigenspace of the commuting product $A\cdot B$ is the orthogonal direct sum of the $\lambda$-eigenspaces of $A$ and $B$.

Now we can proceed to prove the theorem. We let $x,y\in\pi_*^G(\Sigma_+^\infty\mathbf{U})$ be classes in the augmentation ideal. The commutativity of the upper part of (2.8) now provides the relation

$$\begin{aligned}((\mathrm{sh}\, p_1)\circ\mathrm{eig}^{[2]}\circ q^{[2]})_*(x\boxtimes y) \ &=\ (\mathrm{eig}^\natural\circ q\circ(\Sigma_+^\infty\rho_1))_*(x\boxtimes y)\\ \underset{(2.7)}{=}\ &\ (\mathrm{eig}^\natural\circ q)_*(x\cdot\epsilon_*(y)) \ =\ 0\ .\end{aligned}$$

And similarly, $((\mathrm{sh}\, p_2)\circ\mathrm{eig}^{[2]}\circ q^{[2]})_*(x\boxtimes y) = 0$. Because the upper right vertical morphism $(\mathrm{sh}\, p_1,\mathrm{sh}\, p_2)$ in (2.8) is a global equivalence, this proves that

$$(\mathrm{eig}^{[2]}\circ q^{[2]})_*(x\boxtimes y) \ =\ 0\ .$$

The commutativity of the lower part of (2.8) then provides the desired relation

$$\begin{aligned}(\mathrm{eig}^\natural\circ q)_*(x\cdot y) \ &=\ (\mathrm{eig}^\natural\circ q_*\circ(\Sigma_+^\infty\mu))_*(x\boxtimes y)\\ &=\ ((\mathrm{sh}\,\nabla)\circ\mathrm{eig}^{[2]}\circ q^{[2]})_*(x\boxtimes y) \ =\ 0\ .\end{aligned}$$

□

A more careful analysis in the spirit of the previous proof shows the relation

$$\mathrm{eig}_*^\natural(q_*(x\cdot y)) \ =\ \mathrm{eig}_*^\natural(q_*(x)\cdot\epsilon_*(y)) \ +\ \mathrm{eig}_*^\natural(\epsilon_*(x)\cdot q_*(y))\ .$$

for all equivariant homotopy classes $x,y\in\pi_*^G(\Sigma_+^\infty\mathbf{U})$, not necessarily in the augmentation ideal.

## 3. The interplay of the global splitting and the eigenspace morphism

In this section we establish a subtle connection between two a priori unrelated features of the ultra-commutative monoid $\mathbf{U}$, namely its global stable splitting and its preferred infinite delooping. As we show in Theorem 3.10, the adjoint $\Sigma^\infty\mathbf{U} \longrightarrow \mathrm{sh}\,\mathbf{KU}$ of the preferred infinite delooping $\mathbf{U} \sim \Omega^\bullet(\mathrm{sh}\,\mathbf{KU})$ from [18, Theorem 6.4.21] annihilates all the higher terms of the stable global splitting (1.1). This fact is fundamental for all other results in this paper.

**Construction 3.1** (The global ultra-commutative monoid $\mathbf{P}$)**.** We recall the orthogonal space $\mathbf{P}$ made from complex projective spaces, compare [18, (2.3.20)], a specific ultra-commutative model for the global classifying space of the circle group $T = U(1)$. In [18], we use the notation $\mathbf{P}^{\mathbb{C}}$ to distinguish this version made from complex projective spaces from the real version. In this paper, the real version plays no role, so we simplify notation and drop the superscript '$\mathbb{C}$'. The value of $\mathbf{P}$ at the inner product space $V$ is

$$\mathbf{P}(V) \ = \ P(\mathrm{Sym}(V_{\mathbb{C}})) \ ,$$

the complex projective space of the symmetric algebra of the complexification. The structure map $\mathbf{P}(\varphi)\colon \mathbf{P}(V) \longrightarrow \mathbf{P}(W)$ induced by a linear isometric embedding $\varphi\colon V \longrightarrow W$ takes a complex line to its image under $\mathrm{Sym}(\varphi_{\mathbb{C}})\colon \mathrm{Sym}(V_{\mathbb{C}}) \longrightarrow \mathrm{Sym}(W_{\mathbb{C}})$. The orthogonal space $\mathbf{P}$ also has a commutative multiplication by tensor product of line bundles, see [18, Example 2.3.8]. Since this multiplication plays no particular role for the present paper, we do not go into more details.

The inclusions $V_{\mathbb{C}} \longrightarrow \mathrm{Sym}(V_{\mathbb{C}})$ as the linear summands induce maps of projective spaces

$$\ell(V) \ : \ \mathbf{Gr}_1^{\mathbb{C}}(V) \ = \ P(V_{\mathbb{C}}) \ \longrightarrow \ P(\mathrm{Sym}(V_{\mathbb{C}})) \ = \ \mathbf{P}(V) \ .$$

As $V$ varies, these maps form a morphism of orthogonal spaces

$$\ell \ : \ \mathbf{Gr}_1^{\mathbb{C}} \ \xrightarrow{\ \sim\ } \ \mathbf{P} \tag{3.2}$$

that is a global equivalence by [18, Theorem 1.1.10]. The reason for using the 'bigger' model $\mathbf{P}$ in the first place is that tensor product of line bundles makes $\mathbf{P}$ into an ultra-commutative monoid, and hence $\Sigma^\infty_+\mathbf{P}$ into an ultra-commutative ring spectra; the ultra-commutative multiplication of $\mathbf{P}$ does *not* restrict to a multiplication on $\mathbf{Gr}_1^{\mathbb{C}}$.

**Construction 3.3** (Tautological classes)**.** The underlying global spaces of $\mathbf{Gr}_1^{\mathbb{C}}$ and $\mathbf{P}$ are global classifying spaces, in the sense of [18, Definition 1.1.27], for the circle group $T = U(1)$. In particular, the associated Rep-functors $\underline{\pi}_0(\mathbf{Gr}_1^{\mathbb{C}})$ and $\underline{\pi}_0(\mathbf{P})$ are represented by $T$, see [18, Proposition 1.5.12]. We will later need to refer to the universal elements, the *tautological classes*, so we recall them here. The line

$$\mathbb{L} \ = \ \mathbb{C}\cdot(1\otimes 1 - i\otimes i) \ = \ \mathrm{im}(\zeta^{\nu_1}\colon \nu_1 \longrightarrow u(\nu_1)_{\mathbb{C}})$$

is a $T$-fixed point of $P(u(\nu_1)_{\mathbb{C}}) = \mathbf{Gr}_1^{\mathbb{C}}(u(\nu_1))$; the *unstable tautological class* in $\pi_0^T(\mathbf{Gr}_1^{\mathbb{C}})$ is its homotopy class $[\mathbb{L}]$. The *stable tautological class*

$$\tilde{e}_T \ \in \ \pi_0^T(\Sigma^\infty_+\mathbf{Gr}_1^{\mathbb{C}}) \tag{3.4}$$

is the class represented by the $T$-equivariant map

$$S^{\nu_1} \ \xrightarrow{-\wedge\mathbb{L}} \ S^{\nu_1}\wedge P(u(\nu_1)_{\mathbb{C}})_+ \ = \ (\Sigma^\infty_+\mathbf{Gr}_1^{\mathbb{C}})(u(\nu_1)) \ .$$

We also set

$$u_T \ = \ \ell_*[\mathbb{L}] \ \in \ \pi_0^T(\mathbf{P}) \qquad \text{and} \qquad e_T \ = \ (\Sigma^\infty_+\ell)_*(\tilde{e}_T) \ \in \ \pi_0^T(\Sigma^\infty_+\mathbf{P}) \tag{3.5}$$

for the images of the tautological classes under the global equivalence induced by (3.2). Then both the pairs $(\mathbf{Gr}_1^{\mathbb{C}}, [\mathbb{L}])$ and $(\mathbf{P}, u_T)$ represent the functor $\pi_0^T$ on the unstable global homotopy category. And both the pairs $(\Sigma^\infty_+\mathbf{Gr}_1^{\mathbb{C}}, \tilde{e}_T)$ and $(\Sigma^\infty_+\mathbf{P}, e_T)$ represents the functor $\pi_0^T$ on the global stable homotopy category.

**Construction 3.6** (The morphism $\eta\colon \Sigma^\infty_+\mathbf{P} \longrightarrow \mathbf{KU}$)**.** The morphism of non-equivariant spectra from $\Sigma^\infty_+\mathbb{C}P^\infty$ to $KU$ that classifies the tautological complex line bundle over $\mathbb{C}P^\infty$ has a particularly nice and prominent global-equivariant refinement, a morphism of ultra-commutative ring spectra

$$\eta \ : \ \Sigma^\infty_+\mathbf{P} \ \longrightarrow \ \mathbf{KU} \ .$$

The global K-theory spectrum $\mathbf{KU}$ was introduced by Joachim [12], see also [18, Construction 6.4.9]; for every compact Lie group $G$, the underlying genuine $G$-spectrum of $\mathbf{KU}$ represents $G$-equivariant complex K-theory, see [12, Theorem 4.4] or [18, Corollary 6.4.23]. In particular, the equivariant homotopy group $\pi_0^G(\mathbf{KU})$ is isomorphic to the complex representation ring of the compact Lie group $G$; a specific natural isomorphism $R(G) \cong \pi_0^G(\mathbf{KU})$ is defined in [18, Theorem 6.4.24].

The morphism $\eta$ is defined as a composite of two morphisms of ultra-commutative ring spectra

$$\Sigma^\infty_+\mathbf{P} \ \xrightarrow{\ \mu\ } \ \mathbf{ku} \ \xrightarrow{\ j\ } \ \mathbf{KU} \ .$$

The first morphism $\mu$ is the inclusion of the 'rank 1' part in the rank filtration, compare [18, Construction 6.3.40]; its value at an inner product space $V$ is the map

$$\mu(V) \ : \ (\Sigma^\infty_+\mathbf{P})(V) = S^V \wedge P(\mathrm{Sym}(V_\mathbb{C}))_+ \ \longrightarrow \ \mathscr{C}(\mathrm{Sym}(V_\mathbb{C}), S^V) = \mathbf{ku}(V) \ , \quad v \wedge L \longmapsto [L; v] \ .$$

The second morphism $j$ is a 'periodization morphism' $j\colon \mathbf{ku} \longrightarrow \mathbf{KU}$ defined in [18, Construction 6.4.13]. For finite groups of equivariance, the degree zero equivariant cohomology theory represented by $\mathbf{ku}$ is equivariant K-theory, see [18, Theorem 6.3.31], and the effect of $j$ on homotopy groups is inversion of the Bott class; for non-finite compact Lie groups, the situation is somewhat more subtle, see for example [18, Remark 6.3.38]. We will not need to know what the orthogonal spectrum $\mathbf{KU}$ or the morphism $j\colon \mathbf{ku} \longrightarrow \mathbf{KU}$ look like explicitly.

The underlying morphism of global spectra of $\eta$ classifies the tautological $T$-representation $\nu_1$, in the following sense. As mentioned earlier, the pair $(\Sigma^\infty_+\mathbf{P}, e_T)$ represents the functor $\pi_0^T$, where $e_T$ is the stable tautological class (3.5). Under the preferred identification [18, Theorem 6.4.24] of $\pi_0^T(\mathbf{KU})$ with the complex representation ring $R(T)$, the element $e_T$ maps to the class of the tautological $T$-representation, i.e.,

$$\eta_*(e_T) \ = \ [\nu_1] \quad \text{in } \pi_0^T(\mathbf{KU}) \cong R(T).$$

The morphism $\eta$ is extremely highly structured, and has a range of marvelous properties. Because $\eta$ is a morphism of ultra-commutative ring spectra, its effect on equivariant homotopy groups is not only compatible with restriction, inflations and transfers, but also with multiplicative power operations and norms. Moreover, in [21], the author establishes a global generalization of Snaith's celebrated theorem [25, 26], saying that $KU$ can be obtained from $\Sigma^\infty_+\mathbb{C}P^\infty$ by 'inverting the Bott class'.

**Proposition 3.7.** *Let $V$ be an orthogonal representation of the unitary group $U(k)$. Let $T^k$ be the diagonal maximal torus of $U(k)$. Then the geometric fixed point homomorphism*

$$\Phi^{T^k} \circ \mathrm{res}^{U(k)}_{T^k} \ : \ \pi_V^{U(k)}(\mathbf{KU}) \ \longrightarrow \ \Phi_d^{T^k}(\mathbf{KU})$$

*is injective, where $d = \dim(V^{T^k})$.*

*Proof.* For every orthogonal representation $V$ of a compact Lie group $G$, the group $\pi_V^G(\mathbf{KU})$ is isomorphic to the reduced equivariant $K$-group $\tilde{\mathbf{K}}_G(S^V)$, in a way compatible with restriction to subgroups. The restriction homomorphism

$$\mathrm{res}^{U(k)}_{T^k} \ : \ \tilde{\mathbf{K}}_{U(k)}(S^V) \ \longrightarrow \ \tilde{\mathbf{K}}_{T^k}(S^V)$$

is split injective, see for example [3, Proposition (4.9)]. So also the restriction homomorphism

$$\mathrm{res}^{U(k)}_{T^k} \ : \ \pi_V^{U(k)}(\mathbf{KU}) \ \longrightarrow \ \pi_V^{T^k}(\mathbf{KU}) \tag{3.8}$$

is split injective.

We let $W = V - V^{T^k}$ denote the orthogonal complement of the $T^k$-fixed subrepresentation. Then $W$ is an orthogonal $T^k$-representation with trivial fixed points, and thus admits a unitary structure. We choose a linear isomorphism $V^{T^k} \cong \mathbb{R}^d$ and a unitary structure on $W$. Then Bott periodicity for any unitary structure on $W$ provides an isomorphism

$$\pi_V^{T^k}(\mathbf{KU}) \;\cong\; \pi_{\mathbb{R}^d\oplus W}^{T^k}(\mathbf{KU}) \;\cong\; \pi_{d+\dim(W)}^{T^k}(\mathbf{KU}) \;\cong\; \pi_{\dim(V)}^{T^k}(\mathbf{KU}) \ .$$

Since the coefficient ring $\pi_*^{T^k}(\mathbf{KU})$ is 2-periodic and concentrated in even degrees, we conclude that $\pi_V^{T^k}(\mathbf{KU})$ is free of rank 1 over $\pi_0^{T^k}(\mathbf{KU}) \cong R(T^k)$ whenever the dimension of $V$ is even, and $\pi_V^{T^k}(\mathbf{KU}) = 0$ whenever the dimension of $V$ is odd. Because $\pi_V^{U(k)}(\mathbf{KU})$ injects into $\pi_V^{T^k}(\mathbf{KU})$, this group, too, vanishes whenever the dimension of $V$ is odd.

For the rest of the argument we assume that the dimension of $V$ is even. As we just argued, the orthogonal $T^k$-representation underlying $V$ then admits a unitary structure. A choice of unitary structure provides a Bott class $\beta_V \in \tilde{\mathbf{K}}_{T^k}(S^V) \cong \pi_V^{T^k}(\mathbf{KU})$ such that $\pi_V^{T^k}(\mathbf{KU})$ is free of rank one over the representation ring $R(T^k) \cong \pi_0^{T^k}(\mathbf{KU})$.

Since $\mathbf{KU}$ is equivariantly complex oriented by the Bott classes, the geometric fixed point homomorphism $\Phi^{T^k} : \pi_0^{T^k}(\mathbf{KU}) \longrightarrow \Phi_0^{T^k}(\mathbf{KU})$ presents the target as the localization of the source by inverting the Euler classes of all non-trivial irreducible unitary $T^k$-representations. Since $T^k$ is abelian, all irreducible unitary representations are 1-dimensional, and thus given by characters $\lambda\colon T^k \longrightarrow T$. Under the isomorphism $\pi_0^{T^k}(\mathbf{KU}) \cong R(T^k)$, the Euler class of $\lambda$ corresponds to $1-\lambda \in R(T^k)$. So the composite ring homomorphism

$$R(T) \;\cong\; \pi_0^{T^k}(\mathbf{KU}) \;\xrightarrow{\Phi^{T^k}}\; \Phi_0^{T^k}(\mathbf{KU})$$

extends uniquely to an isomorphism

$$R(T)[(1-\lambda)^{-1}] \;\cong\; \Phi_0^{T^k}(\mathbf{KU})$$

where the localization inverts $1-\lambda$ for all non-trivial $T^k$-characters $\lambda$. In the commutative diagram

$$\begin{tikzcd}
R(T^k) \arrow[r, "\cong"] \arrow[d] & \pi_0^{T^k}(\mathbf{KU}) \arrow[r, "\beta_V\cdot -", "\cong"'] \arrow[d, "\Phi^{T^k}"'] & \pi_V^{T^k}(\mathbf{KU}) \arrow[d, "\Phi^{T^k}"] \\
R(T^k)[(1-\lambda^{-1})] \arrow[r, "\cong"'] & \Phi_0^{T^k}(\mathbf{KU}) \arrow[r, "\cong", "\Phi^{T^k}(\beta_V)\cdot -"'] & \Phi_d^{T^k}(\mathbf{KU})
\end{tikzcd}$$

all horizontal maps are isomorphisms. Since the representation ring $R(T^k)$ is a domain, the left vertical localization homomorphism is injective. So the right vertical geometric fixed point map is injective. Since the restriction homomorphism (3.8) is injective, too, this proves the claim when the dimension of $V$ is even. $\square$

Theorem 1.13 lets us define global morphisms from $\Sigma^\infty\mathbf{U}$ by specifying their values on the classes $\sigma_k \in \pi_{\mathrm{ad}(k)}^{U(k)}(\Sigma^\infty\mathbf{U})$ defined in (1.12). So we let

$$a \;:\; \Sigma^\infty\mathbf{U} \;\longrightarrow\; \Sigma_+^\infty\mathbf{P}\wedge S^1 \tag{3.9}$$

denote the unique morphism in the global stable homotopy category such that

$$a_*(\sigma_k) \;=\; \begin{cases} e_T \wedge S^1 & \text{for } k=1, \text{ and} \\ 0 & \text{for } k\geq 2. \end{cases}$$

In the case $k=1$ we have implicitly identified $\mathbb{R}\cong \mathrm{ad}(1)$ by sending $x$ to $i\cdot x$.

**Theorem 3.10.** *The following diagram commutes in the global stable homotopy category:*

$$\begin{tikzcd}
\Sigma^\infty \mathbf{U} \arrow[r, "a", "(3.9)"'] \arrow[d, "\mathrm{eig}^\natural"', "(2.4)"] & \Sigma^\infty_+ \mathbf{P} \wedge S^1 \arrow[r, "\eta\wedge S^1"] & \mathbf{KU}\wedge S^1 \arrow[d, "\sim"', "\lambda_{\mathbf{KU}}"] \\
\mathrm{sh}\,\mathbf{ku} \arrow[rr, "\mathrm{sh}\, j"'] & & \mathrm{sh}\,\mathbf{KU}
\end{tikzcd}$$

*Proof.* By Theorem 1.13, it suffices to show that both composites agree on the classes $\sigma_k$ for all $k \geq 1$. We start with the case $k = 1$. We define a morphism of based orthogonal spaces

$$b \ : \ (\mathbf{Gr}_1^{\mathbb{C}})_+ \wedge S^1 \ \longrightarrow \ \mathbf{U}$$

at an inner product space $V$ as the map

$$b(V) \ : \ ((\mathbf{Gr}_1^{\mathbb{C}})_+ \wedge S^1)(V) = P(V_{\mathbb{C}})_+ \wedge S^1 \ \longrightarrow \ U(V_{\mathbb{C}}) = \mathbf{U}(V)$$

that sends $L \wedge x$ to the unitary automorphism $b(V)(L \wedge x)$ of $V_{\mathbb{C}}$ that is multiplication by $\mathfrak{c}(x) \in U(1)$ on the complex line $L$, and the identity on the orthogonal complement of $L$. By inspection of definitions, the following diagram commutes:

$$\begin{tikzcd}
S^1 \arrow[rr, "\mathfrak{c}", "\cong"'] \arrow[d, "\mathbb{L}\wedge -"'] & & U(1) \arrow[d, "\zeta^1_*"] \\
P(u(\nu_1)_{\mathbb{C}})_+ \wedge S^1 \arrow[r, equal] & ((\mathbf{Gr}_1^{\mathbb{C}})_+ \wedge S^1)(u(\nu_1)) \arrow[r, "b(u(\nu_1))"'] & \mathbf{U}(u(\nu_1))
\end{tikzcd}$$

Here $\mathbb{L} = \mathbb{C} \cdot (1 \otimes 1 - i \otimes i)$ is the $T$-invariant line that defines the tautological class (3.4). Smashing with $S^{\nu_1}$ and passing to homotopy classes proves the relation

$$(\Sigma^\infty b)_*(\tilde{e}_T \wedge S^1) \ = \ d_1$$

in $\pi_1^T(\Sigma^\infty \mathbf{U})$, where the class $d_1$ was defined in (1.20).

The global equivalence $\ell \colon \mathbf{Gr}_1^{\mathbb{C}} \xrightarrow{\sim} \mathbf{P}$ was defined in (3.2). The right square in the following diagram commutes by naturality of the $\lambda$-morphisms:

$$\begin{tikzcd}
\Sigma^\infty_+ \mathbf{Gr}_1^{\mathbb{C}} \wedge S^1 \arrow[r, "\Sigma^\infty_+ \ell\wedge S^1", "\sim"'] \arrow[d, "\Sigma^\infty b"'] & \Sigma^\infty_+ \mathbf{P} \wedge S^1 \arrow[r, "\mu\wedge S^1"'] \arrow[rr, bend left, "\eta\wedge S^1"] & \mathbf{ku}\wedge S^1 \arrow[r, "j\wedge S^1"'] \arrow[d, "\sim"', "\lambda_{\mathbf{ku}}"] & \mathbf{KU}\wedge S^1 \arrow[d, "\sim"', "\lambda_{\mathbf{KU}}"] \\
\Sigma^\infty \mathbf{U} \arrow[rr, "\mathrm{eig}^\natural"'] & & \mathrm{sh}\,\mathbf{ku} \arrow[r, "\mathrm{sh}\, j"'] & \mathrm{sh}\,\mathbf{KU}
\end{tikzcd}$$

We claim that the left part also commutes. Indeed, expanding definitions shows that both composites send an element

$$v \wedge L \wedge x \ \in \ S^V \wedge P(V_{\mathbb{C}})_+ \wedge S^1 = (\Sigma^\infty_+ \mathbf{Gr}_1^{\mathbb{C}} \wedge S^1)(V)$$

to the one-element configuration $[L, (v,x)]$ in $\mathscr{C}(\mathrm{Sym}((V \oplus \mathbb{R})_{\mathbb{C}}), S^{V\oplus\mathbb{R}}) = (\mathrm{sh}\,\mathbf{ku})(V)$ of the point $(v,x) \in S^{V\oplus\mathbb{R}}$ labeled by the line $L$, embedded via $V_{\mathbb{C}} \longrightarrow (V \oplus \mathbb{R})_{\mathbb{C}} \longrightarrow \mathrm{Sym}((V \oplus \mathbb{R})_{\mathbb{C}})$. Given the commutativity of the previous diagram, we obtain:

$$\begin{aligned}
((\mathrm{sh}\, j) \circ \mathrm{eig}^\natural)_*(\sigma_1) \ &= \ ((\mathrm{sh}\, j) \circ \mathrm{eig}^\natural)_*(d_1) \\
&= \ ((\mathrm{sh}\, j) \circ \mathrm{eig}^\natural \circ (\Sigma^\infty b))_*(\tilde{e}_T \wedge S^1) \\
&= \ (\lambda_{\mathbf{KU}} \circ (\eta \wedge S^1) \circ (\Sigma^\infty_+ \ell \wedge S^1))_*(\tilde{e}_T \wedge S^1) \\
\overset{(3.5)}{=} \ & \ (\lambda_{\mathbf{KU}} \circ (\eta \wedge S^1))_*(e_T \wedge S^1) \ = \ (\lambda_{\mathbf{KU}} \circ (\eta \wedge S^1) \circ a)_*(\sigma_1)
\end{aligned}$$

The first equation is Theorem 1.21 (i). The final equation is part of the definition (3.9) of the morphism $a$.

For $k \geq 2$ we will show that both composites in the diagram of the theorem annihilate the class $\sigma_k$. On the one hand, $a_*(\sigma_k) = 0$ by the definition (3.9) of the morphism $a$. On the other hand, we claim that the class $\langle t_1 \rangle \in \pi_1^T(\Sigma^\infty_+ \mathbf{U})$ belongs to the augmentation ideal, i.e., $\epsilon_* \langle t_1 \rangle = 0$. Indeed the composite

$$S^{\mathrm{sa}(1)\oplus \mathrm{ad}(1)} \xrightarrow{\ t_1\ } S^{\mathrm{sa}(1)} \wedge U(1)_+ \xrightarrow{\ S^{\mathrm{ad}(1)}\wedge \epsilon\ } S^{\mathrm{sa}(1)}$$

is a map from a 2-sphere to a 1-sphere, hence null-homotopic. The group $U(1)$ acts trivially on source and target, so this composite is $U(1)$-equivariantly null-homotopy, and thus $\epsilon_*\langle t_1 \rangle = 0$, as claimed. Theorem 2.6 thus shows that the map

$$(\mathrm{eig}^\natural \circ q)_* \ : \ \pi_k^{T^k}(\Sigma^\infty_+ \mathbf{U}) \longrightarrow \pi_k^{T^k}(\mathrm{sh}\, \mathbf{ku})$$

annihilates the class $\langle t_1 \rangle \times \dots \times \langle t_1 \rangle$. Hence

$$\begin{aligned} \Phi^{T^k}((\mathrm{eig}^\natural)_*(\sigma_k)) \ {}_{(1.12)}\!\!= \ & (\mathrm{eig}^\natural \circ q)_*(\Phi^{T^k}\langle t_k \rangle) \\ = \ & (\mathrm{eig}^\natural \circ q)_*(\Phi^{T^k}(\langle t_1 \rangle \times \dots \times \langle t_1 \rangle)) \\ = \ & \Phi^{T^k}((\mathrm{eig}^\natural \circ q)_*(\langle t_1 \rangle \times \dots \times \langle t_1 \rangle)) \ = \ 0 \end{aligned}$$

in the geometric fixed point group $\Phi_k^{T^k}(\mathrm{sh}\, \mathbf{ku})$; the second equation is Theorem 1.16. From this we deduce

$$\Phi^{T^k}(((\mathrm{sh}\, j) \circ \mathrm{eig}^\natural)_*(\sigma_k)) \ = \ (\mathrm{sh}\, j)_*(\Phi^{T^k}(\mathrm{eig}^\natural)_*(\sigma_k)) \ = \ 0$$

in $\Phi_k^{T^k}(\mathrm{sh}\, \mathbf{KU})$. The geometric fixed point homomorphism $\Phi^{T^k} : \pi^{U(k)}_{\mathrm{ad}(k)}(\mathrm{sh}\, \mathbf{KU}) \longrightarrow \Phi_k^{T^k}(\mathrm{sh}\, \mathbf{KU})$ is compatible with the shift isomorphisms, and hence isomorphic to $\Phi^{T^k} : \pi^{U(k)}_{\widetilde{\mathrm{ad}}(k)}(\mathbf{KU}) \longrightarrow \Phi_{k-1}^{T^k}(\mathbf{KU})$. This homomorphism is injective by Proposition 3.7. Hence $((\mathrm{sh}\, j) \circ \mathrm{eig}^\natural)_*(\sigma_k) = 0$. This concludes the proof. $\square$

**Remark 3.11.** The proof of Theorem 3.10, and hence of all subsequent results in this paper, hinges on the fact that for $k \geq 2$, the composite morphism of global spectra

$$\Sigma^\infty \mathbf{U} \xrightarrow[(2.4)]{\mathrm{eig}^\natural} \mathrm{sh}\, \mathbf{ku} \xrightarrow{\mathrm{sh}\, j} \mathrm{sh}\, \mathbf{KU}$$

annihilates the classes $\sigma_k$ that encodes the $k$-th summand in the global splitting (1.1).

We alert the reader that the eigenspace morphism $\mathrm{eig}^\natural : \Sigma^\infty \mathbf{U} \longrightarrow \mathrm{sh}\, \mathbf{ku}$ itself does *not* annihilate $\sigma_k$ for any $k \geq 1$. So $\mathrm{eig}^\natural$ does not factor through $a : \Sigma^\infty \mathbf{U} \longrightarrow \Sigma^\infty_+ \mathbf{P} \wedge S^1$, and the composite

$$\mathbf{U} \xrightarrow{\ d\ } \Omega^\bullet(\Sigma^\infty_+ \mathbf{P} \wedge S^1) \xrightarrow{\ \Omega^\bullet(\mu \wedge S^1)\ } \Omega^\bullet(\mathbf{ku} \wedge S^1) \xrightarrow{\ \Omega^\bullet(\lambda_{\mathbf{ku}})\ } \Omega^\bullet(\mathrm{sh}\, \mathbf{ku})$$

is *different* from $\mathrm{eig} : \mathbf{U} \longrightarrow \Omega^\bullet(\mathrm{sh}\, \mathbf{ku})$. At this point one might want to recall that the name 'connective global K-theory' has to be taken with a grain of salt, in that the morphism $j : \mathbf{ku} \longrightarrow \mathbf{KU}$ is an equivariant connective cover for *finite* groups by Theorems 6.3.27 and 6.4.21 of [18], but not generally for compact Lie groups of positive dimension.

We show that even the composite

$$\Sigma^\infty \mathbf{U} \xrightarrow{\mathrm{eig}^\natural} \mathrm{sh}\, \mathbf{ku} \xrightarrow{\mathrm{sh}\dim} \mathrm{sh}(Sp^\infty)$$

does not annihilate $\sigma_k$ for any $k \geq 1$. Here $\dim : \mathbf{ku} \longrightarrow Sp^\infty$ is dimension homomorphism to the infinite symmetric product spectrum, defined in [18, Example 6.3.36]. The spectrum $Sp^\infty$ is $\mathcal{F}in$-globally equivalent to the Eilenberg–MacLane spectrum for the constant global functor $\underline{\mathbb{Z}}$, see Propositions 5.3.9 and 5.3.12 of [18]. However, for compact Lie groups $G$ of positive dimension, the group $\pi_*^G(Sp^\infty)$ are typically not concentrated in dimension 0, and the ring $\pi_0^G(Sp^\infty)$ need not be isomorphic to $\mathbb{Z}$. For example, $\pi_1^T(Sp^\infty) \cong \mathbb{Q}$, see [18, Theorem 5.3.16], and the abelian group $\pi_0^{SU(2)}(Sp^\infty)$ has rank 2, see [17, Example 4.16].

We detect the classes $((\mathrm{sh}\dim)\circ\mathrm{eig}^\natural)_*(\sigma_k)$ in geometric fixed points. For an orthogonal representation $V$ of a *connected* compact Lie group $G$, the map

$$Sp^\infty(S^{V^G}) \longrightarrow (Sp^\infty(S^V))^G$$

induced by the fixed point inclusion $V^G \longrightarrow V$ is a homeomorphism by [18, Proposition B.42]. So the $G$-geometric fixed point spectrum of $Sp^\infty$ is an Eilenberg–MacLane spectrum for $\mathbb{Z}$. In particular, the ring $\Phi_0^G(Sp^\infty)$ is isomorphic to $\mathbb{Z}$ whenever $G$ is connected.

The class $d_k = [S^{\nu_k}\wedge\delta_k]$ in $\pi_1^{U(k)}(\Sigma^\infty\mathbf{U})$ was defined in (1.20). Because $\nu_k$ has trivial $U(k)$-fixed points, $(S^{\nu_k})^{U(k)} = S^0$, and the class $\Phi^{U(k)}(d_k)$ is thus represented by the map

$$\delta_k = \zeta_*^k\circ\partial\circ\mathfrak{c} \ : \ S^1 \longrightarrow \mathbf{U}(u(\nu_k))^{U(k)} = ((\Sigma^\infty\mathbf{U})(u(\nu_k)))^{U(k)}$$

that sends $x\in S^1$ to the unitary automorphism of $u(\nu_k)_{\mathbb{C}}$ that is multiplication by $\mathfrak{c}(x)$ on the image of the $\mathbb{C}$-linear monomorphism $\zeta^k\colon \nu_k \longrightarrow u(\nu_k)_{\mathbb{C}}$, and the identity on its orthogonal complement. The morphism $\mathrm{eig}^\natural\colon \Sigma^\infty\mathbf{U}\longrightarrow \mathrm{sh}\,\mathbf{ku}$ extracts eigenvalues and eigenspaces; and as the name suggests, the morphism $\dim\colon \mathbf{ku}\longrightarrow Sp^\infty$ takes a configuration of vector spaces to the configuration of the dimensions. So the class $((\mathrm{sh}\dim)\circ\mathrm{eig}^\natural)_*(\Phi^{U(k)}(d_k))$ is represented by the map

$$S^1 \longrightarrow (Sp^\infty(S^{\nu_k\oplus\mathbb{R}}))^{U(k)} = (\mathrm{sh}(Sp^\infty)(u(\nu_k)))^{U(k)} , \quad x \longmapsto k\cdot(0,x) ,$$

the point $(0,x)\in S^{\nu_k\oplus\mathbb{R}}$ with multiplicity $k$. This map represents $\mathrm{sh}(k\cdot 1)$, the $k$-fold multiple of the shifted multiplicative unit in $\Phi_1^{U(k)}(\mathrm{sh}(Sp^\infty))\cong\mathbb{Z}$. Consequently,

$$\begin{aligned}\Phi^{U(k)}(((\mathrm{sh}\dim)\circ\mathrm{eig}^\natural)_*(\sigma_k)) \ &= \ ((\mathrm{sh}\dim)\circ\mathrm{eig}^\natural)_*(\Phi^{U(k)}(\sigma_k))\\ &= \ ((\mathrm{sh}\dim)\circ\mathrm{eig}^\natural)_*(\Phi^{U(k)}(d_k)) \ = \ \mathrm{sh}(k\cdot 1) \ \neq \ 0 .\end{aligned}$$

The second equation is Theorem 1.21 (ii). This proves that $((\mathrm{sh}\dim)\circ\mathrm{eig}^\natural_*(\sigma_k))\neq 0$.

## 4. The global Segal–Becker splitting

In this section we construct the global Segal–Becker splitting $c\colon \mathbf{BUP}\longrightarrow\Omega^\bullet(\Sigma^\infty_+\mathbf{P})$, see (4.7). This morphism comes into existence as a global loop map, the delooping being the morphism $d\colon \mathbf{U}\longrightarrow\Omega^\bullet(\Sigma^\infty_+\mathbf{P}\wedge S^1)$ adjoint to the morphism $a\colon\Sigma^\infty\mathbf{U}\longrightarrow\Sigma^\infty_+\mathbf{P}\wedge S^1$ defined in (3.9). The fact that the morphisms $d$ and $c$ are indeed sections to $\Omega^\bullet(\eta\wedge S^1)\colon\Omega^\bullet(\Sigma^\infty_+\mathbf{P}\wedge S^1)\longrightarrow\Omega^\bullet(\mathbf{KU}\wedge S^1)$ and to $\Omega^\bullet(\eta)\colon\Omega^\bullet(\Sigma^\infty_+\mathbf{P})\longrightarrow\Omega^\bullet(\mathbf{KU})$, respectively, are proved in Theorem 4.2 and Corollary 4.8.

**Construction 4.1** (The deloop of the global Segal–Becker splitting)**.** The morphism $a\colon\Sigma^\infty\mathbf{U}\longrightarrow\Sigma^\infty_+\mathbf{P}\wedge S^1$ was defined in (3.9), essentially as the projection of the global stable splitting (1.1) onto the summand indexed by $k=1$. As a Quillen adjoint functor for the global model structures, the pair $(\Sigma^\infty,\Omega^\bullet)$ derives to an adjoint functor pair at the level of global homotopy categories. The stable morphism $a$ from (3.9) is thus adjoint to an unstable morphism in the homotopy category of based global spaces

$$d \ : \ \mathbf{U} \longrightarrow \Omega^\bullet(\Sigma^\infty_+\mathbf{P}\wedge S^1) .$$

This morphism is our deloop of the global Segal–Becker splitting.

We can now prove Theorem A from the introduction:

**Theorem 4.2.** *The composite*

$$\mathbf{U} \xrightarrow{\ d\ } \Omega^\bullet(\Sigma^\infty_+\mathbf{P}\wedge S^1) \xrightarrow{\Omega^\bullet(\eta\wedge S^1)} \Omega^\bullet(\mathbf{KU}\wedge S^1)$$

*is a global equivalence.*

*Proof.* We take the commutative diagram established in Theorem 3.10 and pass to adjoints for the adjunction $(\Sigma^\infty, \Omega^\bullet)$. We obtain a commutative diagram in the unstable global homotopy category:

$$\begin{tikzcd} \mathbf{U} \arrow[r, "d"] \arrow[d, "\mathrm{eig}"'] & \Omega^\bullet(\Sigma^\infty_+ \mathbf{P} \wedge S^1) \arrow[r, "\Omega^\bullet(\eta \wedge S^1)"] & \Omega^\bullet(\mathbf{KU} \wedge S^1) \arrow[d, "\sim"', "\Omega^\bullet(\lambda_{\mathbf{KU}})"] \\ \Omega^\bullet(\mathrm{sh}\,\mathbf{ku}) \arrow[rr, "\Omega^\bullet(\mathrm{sh}\, j)"'] & & \Omega^\bullet(\mathrm{sh}\,\mathbf{KU}) \end{tikzcd}$$

The composite $\Omega^\bullet(\mathrm{sh}\, j) \circ \mathrm{eig}\colon \mathbf{U} \longrightarrow \Omega^\bullet(\mathrm{sh}\,\mathbf{KU})$ is a global equivalence by [18, Theorem 6.4.21]. The morphism $\lambda_{\mathbf{KU}}\colon \mathbf{KU} \wedge S^1 \longrightarrow \mathrm{sh}\,\mathbf{KU}$ is a global equivalence of orthogonal spectra by [18, Proposition 4.1.4 (i)]; hence the right vertical morphism $\Omega^\bullet(\lambda_{\mathbf{KU}})$ is a global equivalence of orthogonal spaces. Thus $\Omega^\bullet(\eta \wedge S^1) \circ d\colon \mathbf{U} \longrightarrow \Omega^\bullet(\mathbf{KU} \wedge S^1)$ is a global equivalence, as claimed. $\square$

Now we construct the actual global Segal–Becker splitting $c\colon \mathbf{BUP} \longrightarrow \Omega^\bullet(\Sigma^\infty_+ \mathbf{P})$, essentially by looping the morphism $d\colon \mathbf{U} \longrightarrow \Omega^\bullet(\Sigma^\infty_+ \mathbf{P} \wedge S^1)$.

**Construction 4.3** (The global Segal–Becker splitting)**.** The orthogonal space

$$\mathbf{Gr}^{\mathbb{C}} \ = \ \coprod\nolimits_{k\geq 0} \mathbf{Gr}^{\mathbb{C}}_k$$

made from the complex Grassmannians of varying dimensions is an ultra-commutative monoid under direct sum, compare [18, Example 2.3.16]. The ultra-commutative monoid $\mathbf{BUP}$ is the complex analog of the ultra-commutative monoid $\mathbf{BOP}$ introduced in [18, Example 2.4.1]. So the values of $\mathbf{BUP}$ are

$$\mathbf{BUP}(V) \ = \ \coprod\nolimits_{n\geq 0} Gr^{\mathbb{C}}_n(V^2_{\mathbb{C}}) \ ,$$

the full Grassmannian of complex subspace of $V^2_{\mathbb{C}}$. The structure map $\mathbf{BUP}(\varphi)\colon \mathbf{BUP}(V) \longrightarrow \mathbf{BUP}(W)$ associated with a linear isometric embedding $\varphi\colon V \longrightarrow W$ is given by

$$\mathbf{BUP}(\varphi)(L) \ = \ \varphi^2_{\mathbb{C}}(L) \ + \ ((W - \varphi(V))_{\mathbb{C}} \oplus 0) \ .$$

The ultra-commutative monoids $\mathbf{BUP}$ and $\Omega\mathbf{U}$ are two global group completions of $\mathbf{Gr}^{\mathbb{C}}$, see Theorems 2.5.33 and 2.5.40 of [18]; this fact is witnessed by specific morphisms of ultra-commutative monoids $i\colon \mathbf{Gr}^{\mathbb{C}} \longrightarrow \mathbf{BUP}$ and $\beta\colon \mathbf{Gr}^{\mathbb{C}} \longrightarrow \Omega\mathbf{U}$ defined in (2.4.3) and (2.5.38) of [18], respectively. The former is given by

$$i(V) \ : \ \mathbf{Gr}^{\mathbb{C}}(V) = Gr^{\mathbb{C}}(V_{\mathbb{C}}) \ \longrightarrow \ Gr^{\mathbb{C}}(V^2_{\mathbb{C}}) = \mathbf{BUP}(V) \ , \quad i(V)(L) \ = \ V_{\mathbb{C}} \oplus L \ . \tag{4.4}$$

The value of the latter at an inner product space $V$

$$\beta(V) \ : \ \mathbf{Gr}^{\mathbb{C}}(V) = Gr^{\mathbb{C}}_n(V_{\mathbb{C}}) \ \longrightarrow \ \Omega(U(V_{\mathbb{C}})) = (\Omega\mathbf{U})(V) \tag{4.5}$$

sends a complex subspace $L \subset V_{\mathbb{C}}$ to the loop $\beta(V)(L)\colon S^1 \longrightarrow U(V_{\mathbb{C}})$ such that $\beta(V)(L)(x)$ is multiplication by $c(x) \in U(1)$ on $L$, and the identity on the orthogonal complement of $L$.

Since $i\colon \mathbf{Gr}^{\mathbb{C}} \longrightarrow \mathbf{BUP}$ and $\beta\colon \mathbf{Gr}^{\mathbb{C}} \longrightarrow \Omega\mathbf{U}$ are both global group completions, there is a unique morphism in the homotopy category of ultra-commutative monoids

$$\gamma \ : \ \mathbf{BUP} \ \xrightarrow{\ \sim\ } \ \Omega\mathbf{U} \tag{4.6}$$

such that $\gamma \circ i \sim \beta\colon \mathbf{Gr}^{\mathbb{C}} \longrightarrow \Omega\mathbf{U}$. Moreover, the morphism $\gamma$ is a global equivalence, and it witnesses a global form of equivariant Bott periodicity. In [18, Theorem 2.5.41], the morphism $\gamma$ is realized by a zizag of two global equivalences in the model.

Another ingredient is a natural global equivalence of orthogonal spaces

$$\xi_X \ : \ \Omega^\bullet X \ \xrightarrow{\ \sim\ } \ \Omega(\Omega^\bullet(X \wedge S^1)) \ ,$$

where $X$ is an orthogonal spectrum. Its value at an inner product space $V$ is adjoint to the assembly map

$$(\Omega^\bullet X)(V) \wedge S^1 = \mathrm{map}_*(S^V, X(V)) \wedge S^1 \longrightarrow \mathrm{map}_*(S^V, X(V) \wedge S^1) = (\Omega^\bullet(X \wedge S^1))(V) .$$

We can then define the *global Segal–Becker splitting*

$$c \ : \ \mathbf{BUP} \longrightarrow \Omega^\bullet(\Sigma^\infty_+ \mathbf{P}) \tag{4.7}$$

as the unique morphism in the unstable global homotopy category that makes the following diagram commute:

$$\begin{tikzcd} \mathbf{BUP} \arrow[r, "c"] \arrow[d, "\gamma"', "\sim"] & \Omega^\bullet(\Sigma^\infty_+ \mathbf{P}) \arrow[d, "\sim"', "\xi_{\Sigma^\infty_+ \mathbf{P}}"] \\ \Omega \mathbf{U} \arrow[r, "\Omega d"'] & \Omega(\Omega^\bullet(\Sigma^\infty_+ \mathbf{P} \wedge S^1)) \end{tikzcd}$$

The next corollary verifies the fact that global Segal–Becker splitting is indeed a splitting of the morphism $\Omega^\bullet(\eta) \colon \Omega^\bullet(\Sigma^\infty_+ \mathbf{P}) \longrightarrow \Omega^\bullet(\mathbf{KU})$. In Theorem 5.15 will we show that the global Segal–Becker realizes the 'classical' equivariant Segal–Becker splittings at the level of equivariant cohomology theories, thereby justifying its name.

**Corollary 4.8.** *The composite*

$$\mathbf{BUP} \xrightarrow{c} \Omega^\bullet(\Sigma^\infty_+ \mathbf{P}) \xrightarrow{\Omega^\bullet(\eta)} \Omega^\bullet(\mathbf{KU})$$

*is a global equivalence.*

*Proof.* We consider the commutative diagram in the homotopy category of based global spaces:

$$\begin{tikzcd} \mathbf{BUP} \arrow[r, "c"] \arrow[d, "\gamma"', "\sim"] & \Omega^\bullet(\Sigma^\infty_+ \mathbf{P}) \arrow[r, "\Omega^\bullet(\eta)"] \arrow[d, "\xi_{\Sigma^\infty_+ \mathbf{P}}"', "\sim"] & \Omega^\bullet(\mathbf{KU}) \arrow[d, "\sim"', "\xi_{\mathbf{KU}}"] \\ \Omega \mathbf{U} \arrow[r, "\Omega d"'] & \Omega(\Omega^\bullet(\Sigma^\infty_+ \mathbf{P} \wedge S^1)) \arrow[r, "\Omega(\Omega^\bullet(\eta \wedge S^1))"'] & \Omega(\Omega^\bullet(\mathbf{KU} \wedge S^1)) \end{tikzcd}$$

The lower horizontal composite is an equivalence by Theorem 4.2. Since the left and right vertical morphisms are equivalences, this proves that the upper horizontal composite is an equivalence. □

**Remark 4.9** (No section deloops twice)**.** Since $\Omega^\bullet(\eta) \colon \Omega^\bullet(\Sigma^\infty_+ \mathbf{P}) \longrightarrow \Omega^\bullet(\mathbf{KU})$ is a global infinite loop map with an unstable section, one can wonder how often one can deloop an unstable section. Our construction of the section $c \colon \mathbf{BUP} \longrightarrow \Omega^\bullet(\Sigma^\infty_+ \mathbf{P})$ presents it as a global loop map, the deloop being the morphism $d \colon \mathbf{U} \longrightarrow \Omega^\bullet(\Sigma^\infty_+ \mathbf{P} \wedge S^1)$. However, one cannot do better than this, not even non-equivariantly, as we now recall.

The loop structure on the infinite unitary group $U$ coming from Bott periodicity coincides with the group structure. Under the Pontryagin product, the integral homology $H_*(U;\mathbb{Z})$ is an exterior $\mathbb{F}_2$-algebra on classes $a_i \in H_{2i+1}(U;\mathbb{F}_2)$ for $i \geq 0$. In contrast, $H_*(\Omega^\infty(\Sigma^\infty_+ \mathbb{C}P^\infty \wedge S^1);\mathbb{F}_2)$ is a polynomial $\mathbb{F}_2$-algebra on the iterated Kudo–Araki operations on a basis of $\tilde{H}_*(\mathbb{C}P^\infty_+ \wedge S^1;\mathbb{F}_2)$, see for example [9, Theorem 5.1]. So the epimorphism of graded-commutative rings

$$(\Omega^\infty(\eta \wedge S^1))_* \ : \ H_*(\Omega^\infty(\Sigma^\infty_+ \mathbb{C}P^\infty \wedge S^1);\mathbb{F}_2) \longrightarrow H_*(\Omega^\infty(KU \wedge S^1);\mathbb{F}_2) \ \cong \ H_1(U;\mathbb{F}_2)$$

does not admit a multiplicative section. Hence the map $\Omega^\infty(\eta \wedge S^1) \colon \Omega^\infty(\Sigma^\infty_+ \mathbb{C}P^\infty \wedge S^1) \longrightarrow \Omega^\bullet(KU \wedge S^1)$ does not have a section that is an H-map, much less a loop map.

We conclude this section with some calculations that will ultimately allow us to determine the effect of the global Segal–Becker splitting and the global Adams operations on equivariant K-theory. We assign to a unitary representation $W$ of a compact Lie group $G$ a class in $\pi_0^G(\mathbf{BUP})$ as follows. The image of the $\mathbb{C}$-linear and $G$-equivariant isometric embedding $\zeta^W\colon W \longrightarrow (uW)_{\mathbb{C}}$ from (1.7) is a $G$-invariant $\mathbb{C}$-linear subspace of $(uW)_{\mathbb{C}}$. So the subspace $(uW)_{\mathbb{C}} \oplus \mathrm{im}(\zeta^W)$ of $(uW)_{\mathbb{C}}^2$ is a $G$-fixed point of $\mathbf{BUP}(uW) = \coprod_{n\geq 0} Gr_n^{\mathbb{C}}((uW)_{\mathbb{C}}^2)$. We write

$$\{W\} \ = \ [(uW)_{\mathbb{C}} \oplus \mathrm{im}(\zeta^W)] \ \in \ \pi_0^G(\mathbf{BUP}) \tag{4.10}$$

for the homotopy class represented by this element of $(\mathbf{BUP}(uW))^G$. The construction is additive in the sense that $\{V \oplus W\} = \{V\} + \{W\}$.

The next theorem relates the image of the class $\{\nu_k\}$ under the map

$$c_* \ : \ \pi_0^{U(k)}(\mathbf{BUP}) \ \xrightarrow{\ c_*\ } \ \pi_0^{U(k)}(\Omega^\bullet(\Sigma^\infty_+\mathbf{P})) = \pi_0^{U(k)}(\Sigma^\infty_+\mathbf{P})$$

to the class $d_k \in \pi_1^{U(k)}(\Sigma^\infty\mathbf{U})$ defined in (1.20) and to the stable tautological class $e_T \in \pi_0^T(\Sigma^\infty_+\mathbf{P})$ defined in (3.5).

**Theorem 4.11.** *Let $k \geq 1$.*

(i) *The relation*
$$c_*\{\nu_k\} \wedge S^1 \ = \ a_*(d_k)$$
*holds in $\pi_1^{U(k)}(\Sigma^\infty_+\mathbf{P} \wedge S^1)$.*

(ii) *The relation*
$$c_*\{\nu_k\} \ = \ \mathrm{tr}_{U(1,k-1)}^{U(k)}(q^*(e_T))$$
*holds in $\pi_0^{U(k)}(\Sigma^\infty_+\mathbf{P})$, where $q\colon U(1,k-1) \longrightarrow U(1) = T$ is the projection to the first block.*

*Proof.* (i) The following diagram commutes by the definition of the morphism $c\colon \mathbf{BUP} \longrightarrow \Omega^\bullet(\Sigma^\infty_+\mathbf{P})$ from the morphism $d\colon \mathbf{U} \longrightarrow \Omega^\bullet(\Sigma^\infty_+\mathbf{P} \wedge S^1)$, which in turn was defined as the adjoint of $a\colon \Sigma^\infty\mathbf{U} \longrightarrow \Sigma^\infty_+\mathbf{P} \wedge S^1$:

$$\begin{tikzcd}
\pi_0^G(\mathbf{Gr}^{\mathbb{C}}) \arrow[r, "i_*"] \arrow[dr, bend right, "\beta_*"'] & \pi_0^G(\mathbf{BUP}) \arrow[rrr, "c_*"] \arrow[d, "\gamma_*", "\cong"'] & & & \pi_0^G(\Sigma^\infty_+\mathbf{P}) \arrow[d, "-\wedge S^1", "\cong"'] \\
& \pi_0^G(\Omega\mathbf{U}) \arrow[r, "\cong"'] & \pi_1^G(\mathbf{U}) \arrow[r, "\sigma^G"] \arrow[rr, bend right, "d_*"'] & \pi_1^G(\Sigma^\infty\mathbf{U}) \arrow[r, "a_*"] & \pi_1^G(\Sigma^\infty_+\mathbf{P} \wedge S^1)
\end{tikzcd}$$

The map $\sigma^G\colon \pi_1^G(\mathbf{U}) \longrightarrow \pi_1^G(\Sigma^\infty\mathbf{U})$ is the stabilization map [18, (3.3.12)]. The image of $\zeta^k\colon \nu_k \longrightarrow u(\nu_k)_{\mathbb{C}}$ is a $U(k)$-fixed point of $\mathbf{Gr}^{\mathbb{C}}(u(\nu_k))$, and we write $\langle \nu_k \rangle$ for its homotopy class in $\pi_0^{U(k)}(\mathbf{Gr}^{\mathbb{C}})$. By inspection of definitions, the map $\beta_*\colon \pi_0^{U(k)}(\mathbf{Gr}) \longrightarrow \pi_0^{U(k)}(\Omega\mathbf{U})$ sends the class $\langle \nu_k \rangle$ to the homotopy class of the map

$$S^1 \ \xrightarrow{\ \partial\circ\mathfrak{c}\ } \ U(k) \ \xrightarrow{\ \zeta^k_*\ } \ \mathbf{U}(u(\nu_k)) \ .$$

Comparing with (1.20) shows that

$$d_k \ = \ \sigma^{U(k)}(\beta_*\langle \nu_k \rangle) \ .$$

Since the map $i_*\colon \pi_0^{U(k)}(\mathbf{Gr}) \longrightarrow \pi_0^{U(k)}(\mathbf{BUP})$ sends $\langle \nu_k \rangle$ to $\{\nu_k\}$, the commutativity of the above diagram thus shows the desired relation.

(ii) We argue by induction on $k$. For $k = 1$, we have

$$c_*\{\nu_1\} \wedge S^1 \ = \ a_*(d_1) \ = \ a_*(\sigma_1) \ = \ e_T \wedge S^1 \ .$$

by part (i), Theorem 1.21 (i) and the definition of the morphism $a$. Since the suspension isomorphism is bijective, this proves that $c_*\{\nu_1\} = e_T$. Now we suppose that $k \geq 2$. The geometric fixed point homomorphisms $\Phi^H\colon \pi_0^{U(k)}(\Sigma^\infty_+\mathbf{P}) \longrightarrow \Phi_0^H(\Sigma^\infty_+\mathbf{P})$ are jointly injective as $H$ ranges over all closed subgroups of $U(k)$, see [18, Theorem 3.3.15 (ii)]. In other words: it suffices to show the desired relation after taking $H$-geometric fixed points for all closed subgroups $H$ of $U(k)$. To this end we distinguish two cases.

Case 1: The group $H$ is subconjugate to $U(i,k-i)$ for some $1\le i\le k-1$. We disambiguate our notation by writing $q_k\colon U(1,k-1)\longrightarrow U(1)$ for the projection to the first block, i.e., we also record the size of the ambient group. Then

$$\begin{aligned}
\mathrm{res}^{U(k)}_{U(i,k-i)}(\mathrm{tr}^{U(k)}_{U(1,k-1)}(q_k^*(e_T))) \ &= \ \mathrm{tr}^{U(i,k-i)}_{U(1,i-1,k-i)}(\mathrm{res}^{U(1,k-1)}_{U(1,i-1,k-i)}(q_k^*(e_T)))\\
&\quad + \ \mathrm{tr}^{U(i,k-i)}_{U(i,1,k-i-1)}((i+1,i,\dots,2,1)_\star(\mathrm{res}^{U(1,k-1)}_{U(1,i,k-i-1)}(q_k^*(e_T))))\\
&= \ \mathrm{tr}^{U(i,k-i)}_{U(1,i-1,k-i)}(\mathrm{pr}_1^*(e_T)) \ + \mathrm{tr}^{U(i,k-i)}_{U(i,1,k-i-1)}(\mathrm{pr}_2^*(e_T))\\
&= \ p_1^*(\mathrm{tr}^{U(i)}_{U(1,i-1)}(q_i^*(e_T))) \ + \ p_2^*(\mathrm{tr}^{U(k-i)}_{U(1,k-i-1)}(q_{k-i}^*(e_T))) \ .
\end{aligned}$$

Here $\mathrm{pr}_1\colon U(1,i-1,k-i)\longrightarrow U(1)$ and $\mathrm{pr}_2\colon U(i,1,k-i-1)\longrightarrow U(1)$ are the projections to the first and second block, respectively. And $p_1\colon U(i,k-i)\longrightarrow U(i)$ and $p_2\colon U(i,k-i)\longrightarrow U(k-i)$ are the projections to the first and second block, respectively. The first equation is the double coset formula for $\mathrm{res}^{U(k)}_{U(i,k-i)}\circ\mathrm{tr}^{U(k)}_{U(1,k-1)}$, see [20, Proposition 1.3]. The third equation is the compatibility of transfers with inflation, see [18, Proposition 3.2.32]. We use the inductive hypothesis to obtain

$$\begin{aligned}
\mathrm{res}^{U(k)}_{U(i,k-i)}(\mathrm{tr}^{U(k)}_{U(1,k-1)}(q_k^*(e_T))) \ &= \ p_1^*(\mathrm{tr}^{U(i)}_{U(1,i-1)}(q_i^*(e_T))) \ + \ p_2^*(\mathrm{tr}^{U(k-i)}_{U(1,k-i-1)}(q_{k-i}^*(e)))\\
&= \ p_1^*(c_*\{\nu_i\}) \ + \ p_2^*(c_*\{\nu_{k-i}\}) \ = \ c_*(p_1^*\{\nu_i\}+p_2^*\{\nu_{k-i}\})\\
&= \ c_*\{p_1^*(\nu_i)\oplus p_2^*(\nu_{k-i})\}) \ = \ c_*(\mathrm{res}^{U(k)}_{U(i,k-i)}\{\nu_k\}) \ = \ \mathrm{res}^{U(k)}_{U(i,k-i)}(c_*\{\nu_k\}) \ .
\end{aligned}$$

We have exploited that the map $c_*$ is additive because $c$ is a global loop map. Since the classes $c_*\{\nu_k\}$ and $\mathrm{tr}^{U(k)}_{U(1,k-1)}(q_k^*(e_T))$ have the same restrictions to $U(i,k-i)$, they also have the same geometric fixed points for all closed subgroups of $U(i,k-i)$, and hence for all those that are subconjugate to $U(i,k-i)$.

Case 2: The group $H$ is not subconjugate to $U(i,k-i)$ for any $1\le i\le k-1$. We will show that for such groups, the $H$-geometric fixed points of both sides of the equation vanish. On the one hand, since $H$ is not subconjugate to $U(1,k-1)$, we have $\Phi^H\circ\mathrm{tr}^{U(k)}_{U(1,k-1)}=0$, see for example [18, Proposition 3.1.11 (i)]. In particular, $\Phi^H(\mathrm{tr}^{U(k)}_{U(1,k-1)}(q^*(e_T)))=0$. On the other hand,

$$\begin{aligned}
\Phi^H(c_*\{\nu_k\})\wedge S^1 \ &= \ \Phi^H(c_*\{\nu_k\}\wedge S^1) \ = \ \Phi^H(a_*(d_k))\\
&= \ a_*(\Phi^H(d_k)) \ = \ a_*(\Phi^H(\sigma_k)) \ = \ \Phi^H(a_*(\sigma_k)) \ = \ 0 \ .
\end{aligned}$$

The second equation is part (i). The fourth equation is Theorem 1.21 (ii). The final equation is a defining property (3.9) of $a$. Since the suspension isomorphism is bijective, this proves that $\Phi^H(c_*\{\nu_k\})=0$. This completes the inductive step, and hence the proof of the theorem. $\square$

**Construction 4.12.** The unstable tautological class $u_T\in\pi_0^T(\mathbf{P})$ was defined in (3.5). Since the pair $(\mathbf{P},u_T)$ represents the functor $\pi_0^T$, we can define a morphism $h\colon\mathbf{P}\longrightarrow\mathbf{BUP}$ in the unstable global homotopy category by the requirement that

$$h_*(u_T) \ = \ \{\nu_1\}$$

in $\pi_0^T(\mathbf{BUP})$. The following diagram commutes on the unstable tautological class in $\pi_0^T(\mathbf{Gr}_1^{\mathbb{C}})$:

$$\begin{tikzcd} \mathbf{Gr}_1^{\mathbb{C}} \arrow[r, "\ell", "\sim"'] \arrow[d, "\mathrm{incl}"'] & \mathbf{P} \arrow[d, "h"] \\ \mathbf{Gr}^{\mathbb{C}} \arrow[r, "i"'] & \mathbf{BUP} \end{tikzcd}$$

Hence the diagram commutes in the unstable global homotopy category, which shows that the morphism $h$ represents the inclusion of line bundles into virtual vector bundles.

Theorem 4.11 (ii) shows that

$$c_*(h_*(u_T)) \;=\; c_*\{\nu_1\} \;=\; e_T \;.$$

The adjunction unit $\mathbf{P} \longrightarrow \Omega^\bullet(\Sigma^\infty_+\mathbf{P})$ also sends the unstable tautological class $u_T \in \pi_0^T(\mathbf{P})$ to the stable tautological class $e_T$. Since the pair $(\mathbf{P}, u_T)$ represents the functor $\pi_0^T$ on the unstable homotopy category, this proves:

**Corollary 4.13.** *The composite*

$$\mathbf{P} \;\xrightarrow{\ h\ }\; \mathbf{BUP} \;\xrightarrow{\ c\ }\; \Omega^\bullet(\Sigma^\infty_+\mathbf{P})$$

*is the unit of the adjunction* $(\Sigma^\infty_+, \Omega^\bullet)$.

## 5. Explicit Brauer induction and $G$-equivariant Segal–Becker splitting

In this section we show that our global Segal–Becker splitting $c\colon \mathbf{BUP} \longrightarrow \Omega^\bullet(\Sigma^\infty_+\mathbf{P})$ induces the Boltje–Symonds 'explicit Brauer induction' on equivariant homotopy groups, and that it rigidifies and globalizes the 'classical' equivariant Segal–Becker splittings at the level of equivariant cohomology theories. The first fact is Theorem C of the introduction, and Theorem 5.7 below; the second fact is Theorem D of the introduction, and Theorem 5.15 below.

**Remark 5.1** (Explicit Brauer induction)**.** Brauer showed in [6, Theorem I] that the complex representation ring of a finite group is generated, as an abelian group, by representations that are induced from 1-dimensional representations of subgroups. Segal generalized this result to compact Lie groups in [23, Proposition 3.11 (ii)], where 'induction' refers to the smooth induction. We write $\mathbf{A}(T, G)$ for the free abelian group with a basis the symbols $[H, \chi]$, where $H$ runs over all conjugacy classes of closed subgroup of $G$ with finite Weyl group, and $\chi\colon H \longrightarrow T = U(1)$ runs over all characters of $H$. The Brauer–Segal theorem can then be paraphrased as the fact that the maps

$$\mathbf{A}(T, G) \;\longrightarrow\; R(G) \tag{5.2}$$

that sends $[H, \chi]$ to $\mathrm{tr}_H^G(\chi^*[\nu_1])$ are surjective. Informally speaking, an 'explicit Brauer induction' is a collection of sections to the maps (5.2) that are specified by a direct recipe, for example an explicit formula, and with naturality properties as the group $G$ varies. So such maps give an 'explicit and natural' way to write virtual representations as sums of induced representations of 1-dimensional representations. What qualifies as 'explicit' is, of course, in the eye of the beholder.

The first explicit Brauer induction was Snaith's formula [27, Theorem 2.16]; however, Snaith's maps are not additive and not compatible with restriction to subgroups. Boltje [5] specified a different explicit Brauer induction formula for finite groups by purely algebraic means; Symonds [28] gave a topological interpretation of the same section in the context of compact Lie groups. Symonds' construction [28, §4] of the sections

$$b_G \;:\; R(G) \;\longrightarrow\; \mathbf{A}(T, G) \tag{5.3}$$

is designed so that

$$b_{U(k)}[\nu_k] \;=\; [U(1, k-1), q] \qquad \text{in} \quad \mathbf{A}(T, U(k)) \tag{5.4}$$

for all $k \geq 1$, where $q\colon U(1, k-1) \longrightarrow U(1) = T$ is the projection to the first block. The Boltje–Symonds maps are additive and natural for restriction along continuous group homomorphisms; and the value of $b_G$ at a 1-dimensional representation with character $\chi\colon G \longrightarrow T$ is given by

$$b_G[\chi] \ = \ [G, \chi] \ \in \ \mathbf{A}(T, G) \ .$$

The Boltje–Symonds maps (5.3) are not (and in fact cannot be) in general compatible with transfers.

The next theorem shows that our global Segal–Becker splitting (4.7) induces the Boltje–Symonds explicit Brauer induction on equivariant homotopy groups. To make sense of that statement, we use a specific isomorphism

$$\{-\} \ : \ R(G) \ \cong \ \pi_0^G(\mathbf{BUP}) \ . \tag{5.5}$$

In construction (4.10) we assigned to a unitary representation $W$ of a compact Lie group $G$ a class $\{W\}$ in $\pi_0^G(\mathbf{BUP})$. This class only depends on the isomorphism class of $W$ and satisfies $\{V\} + \{W\} = \{V \oplus W\}$. Since the abelian monoid $\pi_0^G(\mathbf{BUP})$ is a group, there is a unique group homomorphism (5.5) that sends the class of a unitary $G$-representation $W$ to $\{W\}$. The homomorphism (5.5) is an isomorphism by the unitary analog of [18, Theorem 2.4.13].

Since the orthogonal space $\mathbf{P}$ is a global classifying space for the circle group $T$, the global functor $\underline{\pi}_0(\Sigma^\infty_+ \mathbf{P})$ is represented by $T$, see [18, Proposition 4.2.5]. In more down-to-earth terms, this means that the abelian group $\pi_0^G(\Sigma^\infty_+ \mathbf{P})$ is free, with a basis given by the classes $\mathrm{tr}_H^G(\chi^*(e_T))$, for $(H, \chi)$ ranging over all conjugacy classes of closed subgroups $H$ of $G$ with finite Weyl group, and all continuous homomorphisms $\chi\colon H \longrightarrow T$; see [18, Corollary 4.1.13]. The group $\mathbf{A}(T, G)$ was defined as a free abelian group with a corresponding basis, so the map

$$\mathbf{A}(T, G) \ \longrightarrow \ \pi_0^G(\Sigma^\infty_+ \mathbf{P}) \ , \quad [H, \chi] \ \longmapsto \ \mathrm{tr}_H^G(\chi^*(e_T)) \tag{5.6}$$

is an isomorphism of abelian groups.

**Theorem 5.7.** *For every compact Lie group $G$, the following square commutes:*

$$\begin{tikzcd} R(G) \arrow[r, "b_G"] \arrow[d, "(5.5)"', "\cong"] & \mathbf{A}(T, G) \arrow[d, "\cong"', "(5.6)"] \\ \pi_0^G(\mathbf{BUP}) \arrow[r, "c_*"'] & \pi_0^G(\Sigma^\infty_+ \mathbf{P}) \end{tikzcd}$$

*Proof.* We start with the tautological representation of the group $U(k)$. By (5.4), the composite through the upper right corner of the square takes $[\nu_k]$ to $\mathrm{tr}_{U(1,k-1)}^{U(k)}(q^*(e_T))$. Theorem 4.11 (ii) provides the relation

$$c_*\{\nu_k\} \ = \ \mathrm{tr}_{U(1,k-1)}^{U(k)}(q^*(e_T)) \ .$$

So the square commutes for $G = U(k)$ on the class of $\nu_k$. Every unitary representation of a compact Lie group $G$ is isomorphic to $\rho^*(\nu_k)$ for some $k \geq 0$ and some continuous homomorphism $\rho\colon G \longrightarrow U(k)$. So naturality for $\rho$ proves the case $(G, [\rho^*(\nu_k)])$. The claim follows for virtual representations by additivity. □

Now we proceed to prove Theorem D of the introduction, saying that our global Segal–Becker splitting induces the classical equivariant Segal–Becker at the level of equivariant cohomology theories.

**Construction 5.8** (The equivariant Segal–Becker splitting)**.** We let $G$ be a compact Lie group, and we let $A$ be a finite $G$-CW-complex. We write $\mathbf{K}_G(A)$ for the $G$-equivariant K-group of $A$, i.e., the group completion of the abelian monoid, under Whitney sum, of isomorphism classes of $G$-vector bundles over $A$. We recall the equivariant Segal–Becker splitting

$$\vartheta_{G,A} \ : \ \mathbf{K}_G(A) \ \longrightarrow \ [A, \Omega^\bullet(\Sigma^\infty_+ \mathbf{P})]^G \tag{5.9}$$

via equivariant transfers due to Iriye–Kono [11, §3], following Crabb's presentation [8]. Crabb denotes the circle group $T = U(1)$ by $\mathbb{T}$, and he writes $B_G\mathbb{T}$ for any classifying space for $G$-equivariant principal $T$-bundles; we may thus take $B_G\mathbb{T} = \mathbf{P}(\mathcal{U}_G)$. In Crabb's exposition, the target of the morphism $\vartheta_{G,A}$ is presented slightly differently, namely as $\omega^0_G\{A_+, (B_G\mathbb{T})_+\}$, the group of morphisms from $\Sigma^\infty_+ A$ to $\Sigma^\infty_+ B_G\mathbb{T}$ in the $G$-equivariant stable homotopy category. Since $A$ is a finite $G$-CW-complex, that group is isomorphic to

$$\mathrm{colim}_{V\in s(\mathcal{U}_G)}[A, \Omega^V(S^V \wedge \mathbf{P}(V)_+)]^G \ = \ [A, \Omega^\bullet(\Sigma^\infty_+\mathbf{P})]^G \ .$$

We let $\xi\colon E \longrightarrow A$ be a $G$-vector bundle. We denote by

$$P\xi \ : \ PE \ \longrightarrow \ A \tag{5.10}$$

the projectivized bundle; its fiber over $a \in A$ is the projective space of the complex vector space $E_a = \xi^{-1}(\{a\})$. The projection (5.10) has an associated transfer, a morphism in the homotopy category of genuine $G$-spectra from $\Sigma^\infty_+ A$ to $\Sigma^\infty_+ PE$. Since $A$ is a finite $G$-CW-complex, this transfer is represented by some continuous based $G$-map

$$\tau(P\xi) \ : \ S^V \wedge A_+ \ \longrightarrow \ S^V \wedge (PE)_+ \ ,$$

for some $G$-representation $V$. The tautological $G$-equivariant line bundle over $PE$ is classified by a continuous $G$-map

$$k \ : \ PE \ \longrightarrow \ \mathbf{P}(W)$$

for some sufficiently large $G$-subrepresentation $W$ of $\mathcal{U}_G$. By enlarging, if necessary, we may assume that $W = V$. We write

$$\vartheta(\xi) \ \in \ [A, \Omega^\bullet(\Sigma^\infty_+\mathbf{P})]^G$$

for the class of the adjoint to the composite

$$S^V \wedge A_+ \ \xrightarrow{\tau(P\xi)} \ S^V \wedge (PE)_+ \ \xrightarrow{S^V\wedge k_+} \ S^V \wedge \mathbf{P}(V)_+ \ .$$

In the classical sources one finds a verification that the class $\vartheta(\xi)$ only depends on the isomorphism class of the $G$-vector bundle $\xi$. Slightly less obvious is that the resulting map

$$\vartheta \ : \ \mathrm{Vect}_G(A) \ \longrightarrow \ [A, \Omega^\bullet(\Sigma^\infty_+\mathbf{P})]^G$$

is additive for the Whitney sum of vector bundles, see [8, Lemma 2.6]. The map thus extends uniquely to an additive map (5.9) on the group completion $\mathbf{K}_G(A)$ of $\mathrm{Vect}_G(A)$. The equivariant Segal–Becker splittings (5.9) are natural for continuous $G$-maps in $A$, and for restriction along continuous group homomorphisms between compact Lie groups. Moreover, they satisfy a normalization property on line bundles.

The Boltje–Symonds map (5.3) is a special case of the equivariant Segal–Becker splitting (5.9), namely when $A = *$ is a one-point $G$-space, in the sense that the composite

$$R(G) = \mathbf{K}_G(*) \ \xrightarrow{\vartheta_{G,*}} \ [*, \Omega^\bullet(\Sigma^\infty_+\mathbf{P})]^G \ = \ \pi_0^G(\Sigma^\infty_+\mathbf{P}) \ \cong_{(5.6)} \ \mathbf{A}(T, G)$$

agrees with $b_G\colon R(G) \longrightarrow \mathbf{A}(T, G)$. Indeed, we will argue in Example 5.11 below that the two coincide on the class of the tautological $U(k)$-representation on $\nu_k$. Since both are additive and natural in continuous group homomorphisms, they coincide on all virtual unitary representations of compact Lie groups. Theorem D is thus a special case of Theorem C; but we deduce Theorem D from Theorem C by global functoriality.

**Example 5.11.** We calculate the $U(k)$-equivariant Segal–Becker splitting on class of the tautological $U(k)$-representation $\nu_k$, considered as a $U(k)$-equivariant vector bundle over a point. The construction of the class $\vartheta_{U(k),*}[\nu_k]$ involves the projective space $P(\nu_k)$ of the tautological representation. This projective space is a homogeneous space: the group $U(k)$ acts transitively on $P(\nu_k)$, and the complex line

$$l \ = \ \mathbb{C} \cdot (1, 0, \ldots, 0)$$

spanned by the first basis vector has stabilizer group $U(1,k-1)$. So the equivariant transfer

$$\tau(P(\nu_k)) \;:\; \mathbb{S} \;\longrightarrow\; \Sigma^\infty_+ P(\nu_k)$$

associated with the unique $U(k)$-map $P(\nu_k) \longrightarrow *$ sends $1 \in \pi_0^{U(k)}(\mathbb{S})$ to the class

$$\mathrm{tr}_{U(1,k-1)}^{U(k)}(\sigma^{U(1,k-1)}[l]) \;\in \pi_0^{U(k)}(\Sigma^\infty_+ P(\nu_k)) \;,$$

where $[l] \in \pi_0(P(\nu_k)^{U(1,k-1)})$ is the class represented by the $U(1,k-1)$-fixed point $l$, and

$$\sigma^{U(1,k-1)} \;:\; \pi_0(P(\nu_k)^{U(1,k-1)}) \;\longrightarrow\; \pi_0^{U(1,k-1)}(\Sigma^\infty_+ P(\nu_k))$$

is the stabilization map [18, (3.3.12)].

The group $U(1,k-1)$ acts on the invariant line $l$ through the homomorphism $q\colon U(1,k-1) \longrightarrow T$, so the classifying $U(k)$-map

$$k \;:\; P(\nu_k) \;\longrightarrow\; \mathbf{P}(\mathcal{U}_{U(k)})$$

for the tautological line bundle satisfies

$$k_*[l] \;=\; q^*(u_T) \quad \text{in} \;\; \pi_0^{U(1,k-1)}(\mathbf{P}) \;,$$

where $u_T \in \pi_0^T(\mathbf{P})$ is the unstable tautological class (3.5). Thus

$$(\Sigma^\infty_+ k)_*(\sigma^{U(1,k-1)}[l]) \;=\; \sigma^{U(1,k-1)}(k_*[l]) \;=\; \sigma^{U(1,k-1)}(q^*(u_T)) \;=\; q^*(\sigma^T(u_T)) \;=\; q^*(e_T) \;,$$

where $e_T = \sigma^T(u_T) \in \pi_0^T(\Sigma^\infty_+\mathbf{P})$ is the stable tautological class (3.5). Combining these observations yields

$$\begin{aligned} \vartheta_{U(k),*}[\nu_k] \;&=\; ((\Sigma^\infty_+ k)\circ\tau(P(\nu_k)))_*(1) \;=\; (\Sigma^\infty_+ k)_*(\mathrm{tr}_{U(1,k-1)}^{U(k)}(\sigma^{U(1,k-1)}[l])) \\ &=\; \mathrm{tr}_{U(1,k-1)}^{U(k)}((\Sigma^\infty_+ k)_*(\sigma^{U(1,k-1)}[l])) \;=\; \mathrm{tr}_{U(1,k-1)}^{U(k)}(q^*(e_T)) \end{aligned} \tag{5.12}$$

in $\pi_0^{U(k)}(\Sigma^\infty_+\mathbf{P})$.

**Construction 5.13.** To give rigorous meaning to the claim that our morphism $c\colon \mathbf{BUP} \longrightarrow \Omega^\bullet(\Sigma^\infty_+\mathbf{P})$ realizes the classical equivariant Segal–Becker splittings, we recall the natural isomorphism

$$\{-\} \;:\; \mathbf{K}_G(A) \;\cong\; [A, \mathbf{BUP}]^G \tag{5.14}$$

that is inverse to the complex analog of the isomorphism specified in [18, Theorem 2.4.10]. Here $G$ is a compact Lie group, and $A$ is a finite $G$-CW-complex.

Given a complex $G$-vector bundle $\xi\colon E \longrightarrow A$, we can choose a classifying $G$-map

$$f \;:\; A \;\longrightarrow\; \mathbf{Gr}^{\mathbb{C}}(V) \;=\; \coprod\nolimits_{n\geq 0} Gr_n^{\mathbb{C}}(V_{\mathbb{C}})$$

for some sufficiently large orthogonal $G$-representation $V$, i.e., such that $\xi$ is isomorphic to the pullback along $f$ of the tautological vector bundle (of non-constant rank) over $\mathbf{Gr}^{\mathbb{C}}(V)$. We write

$$\langle\xi\rangle \;\in\; [A, \mathbf{Gr}^{\mathbb{C}}]^G$$

for the class represented by the classifying map $f$. This class only depends on the isomorphism class $\xi$, and the construction satisfies $\langle\xi\rangle + \langle\zeta\rangle = \langle\xi\oplus\zeta\rangle$. Since the abelian monoid $[A,\mathbf{BUP}]^G$ is a group, we can define (5.14) by additive extension to group completions, i.e., as the unique homomorphism that sends the class of $\xi$ to $i_*\langle\xi\rangle$, with $i\colon \mathbf{Gr}^{\mathbb{C}} \longrightarrow \mathbf{BUP}$ being the group completion morphism (4.4). The homomorphism (5.14) is an isomorphism by the complex analog of [18, Theorem 2.4.10]. When $A$ is a single point, then $\mathbf{K}_G(*) = R(G)$, $[*,\mathbf{BUP}]^G = \pi_0^G(\mathbf{BUP})$, and the isomorphism (5.14) specializes to the isomorphism (5.5).

**Theorem 5.15.** *Let $G$ be a compact Lie group and $A$ a finite $G$-CW-complex. Then the composite*

$$\mathbf{K}_G(A) \;\cong_{(5.14)}\;\; [A,\mathbf{BUP}]^G \;\xrightarrow{[A,c]^G}\; [A, \Omega^\bullet(\Sigma^\infty_+\mathbf{P})]^G$$

*coincides with the $G$-equivariant Segal–Becker splitting $\vartheta_{G,A}$ defined in* (5.9).

*Proof.* We precompose the maps $[A, c]^G \circ (5.14)$ and the Segal–Becker splitting $\vartheta_{G,A}$ with the map

$$\mathrm{Vect}^k_G(A) \longrightarrow \mathbf{K}_G(A)$$

that sends a rank $k$ vector bundle to its K-theory class. Letting $G$ and $A$ vary yields two global transformations from $\mathrm{Vect}^k$ to $\Omega^\bullet(\Sigma^\infty_+ \mathbf{P})$ in the sense of Definition A.8. Theorem 5.7 shows that

$$c_*\{\nu_k\} = ((5.6) \circ b_{U(k)})[\nu_k] =_{(5.4)} \mathrm{tr}^{U(k)}_{U(1,k-1)}(q^*(e_T)) =_{(5.12)} \vartheta_{U(k),*}[\nu_k] .$$

So the two global transformations coincide on the class of the tautological $U(k)$-representation $\nu_k$. Thus the global transformations coincide for all equivariant rank $k$ vector bundles, over all compact Lie groups and all finite equivariant CW-complexes, by Corollary A.9.

This shows that maps $[A, c]^G \circ (5.14)$ and $\vartheta_{G,A}$ coincide on all classes in $\mathbf{K}_G(A)$ that are represented by an equivariant vector bundle of constant rank. Since the morphism $c$ is a loop map, the induced map $[A, c]^G$ is additive. The map $\vartheta_{G,A}$ is additive by [8, Lemma 2.6]. Since vector bundles of constant rank generate $\mathbf{K}_G(A)$ as an abelian group, this proves the theorem. □

## 6. Global Adams operations

In this section we give an application of the global Segal–Becker splitting: we construct global equivariant rigidifications (6.4) of the unstable Adams operations in equivariant K-theory. By design, the Adams operations will arise as global loop maps.

**Construction 6.1** (Adams operations in equivariant $K$-theory)**.** We let $G$ be a compact Lie group. We recall the construction of the $\lambda$-operations and Adams operations on the Grothendieck ring of $G$-equivariant vector bundles over a compact $G$-space. For all $G$-vector bundles $\xi$ and $\zeta$ over the same base, the $G$-vector bundle $\Lambda^n(\xi \oplus \zeta)$ is isomorphic to $\bigoplus_{i=0}^n \Lambda^i(\xi) \otimes \Lambda^{n-i}(\zeta)$. So the map

$$\Lambda \ : \ \mathrm{Vect}_G(A) \longrightarrow \mathbf{K}_G(A)[\![t]\!] \ , \quad [\xi] \longmapsto \sum\nolimits_{n \geq 0} [\Lambda^n(\xi)] \cdot t^n$$

takes addition in the abelian monoid of isomorphism classes of $G$-vector bundles to multiplication in the power series ring $\mathbf{K}_G(A)[\![t]\!]$. All these power series moreover have constant term $\Lambda^0(\xi) = 1$, and are thus invertible. So $\Lambda$ defines a monoid homomorphism from $\mathrm{Vect}_G(A)$ to the multiplicative group of the ring $\mathbf{K}_G(A)[\![t]\!]$. The universal property of the Grothendieck construction thus yields an extension to a monoid homomorphism

$$\Lambda \ : \ \mathbf{K}_G(A) \longrightarrow (\mathbf{K}_G(A)[\![t]\!])^\times .$$

The $\lambda$-operations $\lambda^i \colon \mathbf{K}_G(A) \longrightarrow \mathbf{K}_G(A)$ are then defined by

$$\Lambda(x) = \sum\nolimits_{i \geq 0} \lambda^i(x) \cdot t^i .$$

By design, these operations extend the exterior powers on classes of actual $G$-vector bundles. The $\lambda$-operations make the ring $\mathbf{K}_G(A)$ into a special $\lambda$-ring, see [1, Theorem 1.5 (i)]. Every special $\lambda$-ring supports *Adams operations*, i.e., ring homomorphisms $\psi^n \colon R \longrightarrow R$ for $n \geq 1$ that satisfy $\psi^n \circ \psi^m = \psi^{nm}$ for all $n, m \geq 1$, as well as the congruence $\psi^p(x) \equiv x^p$ modulo $(p)$ for every prime $p$, see [1, §5]. We are particularly interested in these Adams operation

$$\psi^n \ : \ \mathbf{KU}_G(A) \longrightarrow \mathbf{KU}_G(A)$$

in the case of equivariant $K$-theory. One key property of these operations is that on the class of a line bundle $\xi$, the Adams operation is given by

$$\psi^n[\xi] = [\xi^{\otimes n}] .$$

Taking exterior power of vector bundles in natural both for $G$-maps in $A$, and for restriction along continuous homomorphisms in the group $G$. Hence the $\lambda$-operations and the Adams operations inherit both kinds of naturalities.

We shall now use the power endomorphisms of $\mathbf{P}$ to define the global Adams operations on $\mathbf{BUP}$, by employing our splitting to 'retract' them off the induced endomorphisms of $\Sigma^\infty_+\mathbf{P}$.

**Construction 6.2** (Global Adams operations)**.** We let $n \geq 1$ be a positive natural number. We write

$$\mu_n \ : \ T \ \longrightarrow \ T \ , \quad \mu_n(\lambda) = \lambda^n$$

for the $n$-th power homomorphism. Since the pair $(\mathbf{P}, u_T)$ represents the functor $\pi_0^T$, we can define a morphism $\phi^n \colon \mathbf{P} \longrightarrow \mathbf{P}$ in the unstable global homotopy category by the requirement that

$$\phi^n_*(u_T) \ = \ \mu_n^*(u_T)$$

in $\pi_0^T(\mathbf{P})$. The morphism $\phi^n$ then represents raising a line bundle to its $n$-th power. We define

$$\kappa^n \ : \ \mathbf{U} \ \longrightarrow \ \mathbf{U} \tag{6.3}$$

as the unique morphism in the unstable global homotopy category making the following diagram commute:

$$\begin{tikzcd}
\mathbf{U} \arrow[rr, "\kappa^n"] \arrow[d, "d"'] & & \mathbf{U} \arrow[d, "\sim"', "\Omega^\bullet(\eta\wedge S^1)\circ d"] \\
\Omega^\bullet(\Sigma^\infty_+\mathbf{P}\wedge S^1) \arrow[r, "\Omega^\bullet(\Sigma^\infty_+\phi^n\wedge S^1)"'] & \Omega^\bullet(\Sigma^\infty_+\mathbf{P}\wedge S^1) \arrow[r, "\Omega^\bullet(\eta\wedge S^1)"'] & \Omega^\bullet(\mathbf{KU}\wedge S^1)
\end{tikzcd}$$

The global equivalence $\gamma \colon \mathbf{BUP} \xrightarrow{\sim} \Omega\mathbf{U}$ was defined in (4.6). We define the *$n$-th global Adams operation*

$$\psi^n \ : \ \mathbf{BUP} \ \longrightarrow \ \mathbf{BUP} \tag{6.4}$$

as the unique morphism in the unstable global homotopy category making the following diagram commute:

$$\begin{tikzcd}
\mathbf{BUP} \arrow[r, "\psi^n"] \arrow[d, "\gamma"', "\sim"] & \mathbf{BUP} \arrow[d, "\sim"', "\gamma"] \\
\Omega\mathbf{U} \arrow[r, "\Omega(\kappa^n)"'] & \Omega\mathbf{U}
\end{tikzcd}$$

Clearly, the morphism $\phi^1$ is the identity of $\mathbf{P}$, $\kappa^1$ is the identity of $\mathbf{U}$, and thus $\psi^1$ is the identity of $\mathbf{BUP}$.

The next proposition verifies a globally-coherent version of the design criterion for Adams operations, namely that on line bundles, $\psi^n$ is the $n$-th tensor power. The morphism of global spaces $h \colon \mathbf{P} \longrightarrow \mathbf{BUP}$ was defined in Construction 4.12 by the property $h_*(u_T) = \{\nu_1\}$ in $\pi_0^T(\mathbf{BUP})$; it represents the inclusion of line bundles into virtual vector bundles.

**Proposition 6.5.** *For every $n \geq 1$, the following diagram commutes in the unstable global homotopy category:*

$$\begin{tikzcd}
\mathbf{P} \arrow[r, "\phi^n"] \arrow[d, "h"'] & \mathbf{P} \arrow[d, "h"] \\
\mathbf{BUP} \arrow[r, "\psi^n"'] & \mathbf{BUP}
\end{tikzcd}$$

*Proof.* The following diagram commutes by Corollary 4.13 and naturality of the adjunction unit:

$$\begin{tikzcd}
 & \mathbf{P} \arrow[r, "\phi^n"] \arrow[dl, "h"'] \arrow[dd, "\text{unit}"] & \mathbf{P} \arrow[dd, "\text{unit}"'] \arrow[dr, "h"] & \\
\mathbf{BUP} \arrow[dr, "c"'] & & & \mathbf{BUP} \arrow[dl, "c"'] \arrow[d, "\Omega^\bullet(\eta)\circ c", "\sim"'] \\
 & \Omega^\bullet(\Sigma^\infty_+ \mathbf{P}) \arrow[r, "\Omega^\bullet(\Sigma^\infty_+ \phi^n)"'] & \Omega^\bullet(\Sigma^\infty_+ \mathbf{P}) \arrow[r, "\Omega^\bullet(\eta)"'] & \Omega^\bullet(\mathbf{KU})
\end{tikzcd}$$

Expanding the definition of $\psi^n$ and using that the morphism $d\colon \mathbf{U} \longrightarrow \Omega^\bullet(\Sigma^\infty_+ \mathbf{P} \wedge S^1)$ deloops $c\colon \mathbf{BUP} \longrightarrow \Omega^\bullet(\Sigma^\infty_+ \mathbf{P})$ proves the claim. $\square$

**Example 6.6.** The second Adams operation $\psi^2\colon \mathbf{K}_G(A) \longrightarrow \mathbf{K}_G(A)$ is given on the class of a $G$-vector bundle $\xi\colon E \longrightarrow A$ by the formula

$$\psi^2[\xi] \;=\; [\mathrm{Sym}^2(\xi)] \;-\; [\Lambda^2(\xi)]\ ,$$

where $\mathrm{Sym}^2(\xi)$ and $\Lambda^2(\xi)$ are, respectively, the second symmetric and exterior power of $\xi$. Indeed, this formula has the correct behavior on line bundles, is additive for Whitney sum in $\xi$, and natural in $(G, A)$. So the various naturality properties force $\psi^2$ to be given by this formula. In general, $\psi^n$ can be described on vector bundles by certain alternating sums of certain polynomial functors, but the general formula is not as simple. The formula for $\psi^2$ shows that the Adams operations do not generally send vector bundles (other than line bundles) to vector bundles, but rather to virtual vector bundles. So the global Adams operations $\psi^n\colon \mathbf{BUP} \longrightarrow \mathbf{BUP}$ do not restrict to endomorphisms of $\mathbf{Gr}^{\mathbb{C}}_k$ for $k \geq 2$.

The next theorem justifies the name 'global Adams operation' for the morphism $\psi^n\colon \mathbf{BUP} \longrightarrow \mathbf{BUP}$.

**Theorem 6.7.** *For every compact Lie group $G$, every finite $G$-CW-complex $A$ and every $n \geq 1$, the following square commutes:*

$$\begin{tikzcd}
\mathbf{K}_G(A) \arrow[r, "\psi^n"] \arrow[d, "(5.14)"', "\cong"] & \mathbf{K}_G(A) \arrow[d, "\cong"', "(5.14)"] \\
{[A, \mathbf{BUP}]^G} \arrow[r, "\psi^n_*"'] & {[A, \mathbf{BUP}]^G}
\end{tikzcd}$$

*Proof.* We observe that

$$\begin{aligned}
\psi^n_*\{\nu_1\} = \; \psi^n_*(h_*(u_T)) \;&=\; h_*(\phi^n_*(u_T)) \;=\; h_*(\mu_n^*(u_T)) \\
&=\; \mu_n^*(h_*(u_T)) \;=\; \mu_n^*\{\nu_1\} \;=\; \{\mu_n^*(\nu_1)\} \;=\; \{\psi^n[\nu_1]\}\ .
\end{aligned}$$

The second equation is Proposition 6.5.

Next we consider the $T^k$-representation $p_1^*(\nu_1) \oplus \cdots \oplus p_k^*(\nu_1)$, where $p_i\colon T^k \longrightarrow T$ denotes the projection to the $i$-th factor. Then

$$\begin{aligned}
\psi^n_*\{p_1^*(\nu_1) \oplus \cdots \oplus p_k^*(\nu_1)\} \;&=\; \sum_{i=1,\dots,k} p_i^*(\psi^n_*\{\nu_1\}) \\
&=\; \sum_{i=1,\dots,k} p_i^*\{\psi^n[\nu_1]\} \;=\; \{\psi^n[p_1^*(\nu_1) \oplus \cdots \oplus p_k^*(\nu_1)]\}\ .
\end{aligned}$$

The first equation uses that the maps $\psi^n_*$ and $\{-\}$ are additive and compatible with restriction along continuous homomorphisms, because $\psi^n\colon \mathbf{BUP} \longrightarrow \mathbf{BUP}$ is a global loop map. The third equation use that the Adams operations $\psi^n$ are additive and compatible with restriction along continuous homomorphisms.

For the tautological $U(k)$-representation $\nu_k$, we then obtain

$$\mathrm{res}^{U(k)}_{T^k}(\psi^n_*\{\nu_k\}) \ = \ \psi^n_*\{\mathrm{res}^{U(k)}_{T^k}(\nu_k)\} \ = \ \{\psi^n(\mathrm{res}^{U(k)}_{T^k}[\nu_k])\} \ = \ \mathrm{res}^{U(k)}_{T^k}\{\psi^n[\nu_k]\} \ .$$

The second equation uses the previous case and the fact that $\mathrm{res}^{U(k)}_{T^k}(\nu_k) = p_1^*(\nu_1) \oplus \cdots \oplus p_k^*(\nu_1)$. Because the restriction homomorphism $\mathrm{res}^{U(k)}_{T^k} \colon R(U(k)) \longrightarrow R(T^k)$ is injective, this proves the relation $\psi^n_*\{\nu_k\} = \{\psi^n[\nu_k]\}$.

Now we prove the theorem. We precompose the two composites in the statement of the theorem with the map

$$j \ : \ \mathrm{Vect}^k_G(A) \ \longrightarrow \ \mathbf{K}_G(A)$$

that sends a rank $k$ equivariant vector bundle to its K-theory class. Letting $G$ and $A$ vary yields two global transformations from $\mathrm{Vect}^k$ to $\mathbf{BUP}$ in the sense of Definition A.8. We just showed that these two global transformations coincide on the class of the tautological $U(k)$-representation $\nu_k$. So the two global transformations coincide for all equivariant rank $k$ vector bundles, for all compact Lie groups and over all finite equivariant CW-complexes, by Corollary A.9. Since the morphism $\psi^n$ is a loop map, the induced map $\psi^n_*$ is additive. So both composites in the diagram are additive. Since vector bundles of constant rank generate $\mathbf{K}_G(A)$ as an abelian group, this proves the theorem. □

**Remark 6.8.** The Adams operations in equivariant K-theory satisfy the relation $\psi^m \circ \psi^n = \psi^{mn}$ for all $m, n \geq 1$. So by Theorem 6.7, the morphisms of global spaces

$$\psi^m \circ \psi^n, \ \psi^{mn} \ : \ \mathbf{BUP} \ \longrightarrow \ \mathbf{BUP}$$

induce the same map on $[A, \mathbf{BUP}]^G$ for all finite $G$-CW-complexes $A$. We do not know if in fact $\psi^m \circ \psi^n = \psi^{mn}$ as endomorphisms of $\mathbf{BUP}$ in the unstable global homotopy category. Or even better, if the deloopings (6.3) of the global Adams operations satisfy $\kappa^m \circ \kappa^n = \kappa^{mn} \colon \mathbf{U} \longrightarrow \mathbf{U}$.

## Appendix A. Global transformations of equivariant homotopy sets

In the body of this paper, we verify that the global Segal–Becker splitting $c \colon \mathbf{BUP} \longrightarrow \Omega^\bullet(\Sigma^\infty_+ \mathbf{P})$ induces the classical equivariant Segal–Becker splittings on equivariant cohomology theories; and we verify that global Adams operation $\psi^n \colon \mathbf{BUP} \longrightarrow \mathbf{BUP}$ induces the classical Adams operation on equivariant K-groups. In both cases we are dealing with 'global' natural transformations from the functor of isomorphism classes of complex vector bundles of some fixed rank. In this appendix we develop a general tool to characterize such global natural transformations by their effect on certain universal classes, see Corollary A.9.

Equivariant complex vector bundles of rank $k$ arise from equivariant $U(k)$-principal bundles, and are thus represented by the global classifying space $B_{\mathrm{gl}}U(k)$, in the sense of [18, Proposition 1.1.30]. This fact makes the representability result for vector bundles a special case of a representability result for global transformations between the equivariant homotopy sets of orthogonal spaces, see Theorem A.6.

**Construction A.1** (Equivariant homotopy sets)**.** We recall the equivariant homotopy sets defined by orthogonal spaces, discussed in more detail in [18, Section 1.5]. We let $E$ be an orthogonal space, we let $G$ be a compact Lie group, and we let $A$ be a $G$-space. We set

$$[A, E]^G \ = \ \mathrm{colim}_{V \in s(\mathcal{U}_G)} [A, E(V)]^G \ .$$

On the right, $[-,-]^G$ denotes the set of equivariant homotopy classes of continuous $G$-maps, and the colimit is taken over the poset of finite-dimensional $G$-subrepresentations of the complete $G$-universe $\mathcal{U}_G$. If the $G$-space $A$ is compact and the orthogonal space $E$ is closed in the sense of [18, Definition 1.1.16], i.e., all structure maps are closed embeddings, then the canonical map

$$[A, E]^G \ \longrightarrow \ [A, E(\mathcal{U}_G)]^G$$

is bijective, see [18, Proposition 1.5.3], for $E(\mathcal{U}_G) = \operatorname{colim}_{V \in s(\mathcal{U}_G)} E(V)$.

The sets $[A, E]^G$ are contravariantly functorial for continuous $G$-maps in $A$ by precomposition, and they are contravariantly functorial for continuous homomorphisms in $G$: a continuous homomorphism $\alpha\colon K \longrightarrow G$ of compact Lie groups induces a restriction map

$$\alpha^* \ : \ [A, E]^G \ \longrightarrow \ [\alpha^*(A), E]^K$$

by restriction of actions along $\alpha$, much like in [18, (1.5.9)]. When $A$ is a finite $G$-CW-complex, then the assignment $E \mapsto [A, E]^G$ sends global equivalences of orthogonal spaces to bijections, see [18, Proposition 1.5.3 (iii)].

**Construction A.2** (Induction isomorphisms)**.** We let $E$ be an orthogonal space, and we let $H$ be a closed subgroup of a compact Lie group $G$. For an $H$-space $B$, we write

$$[1, -] \ : \ B \ \longrightarrow \ G \times_H B \ , \quad y \ \longmapsto \ [1, y]$$

for the unit of the adjunction $(G \times_H -, \operatorname{res}^G_H)$, an $H$-equivariant continuous map. The adjunction bijections

$$[G \times_H B, E(V)]^G \ \cong \ [B, E(V)]^H$$

and the fact that the underlying $H$-universe of $\mathcal{U}_G$ is a complete $H$-universe provide an *induction isomorphism*: the composite

$$[G \times_H B, E]^G \ \xrightarrow{\operatorname{res}^G_H} \ [G \times_H B, E]^H \ \xrightarrow{[1,-]^*} \ [B, E]^H \tag{A.3}$$

is bijective.

If $H$ is a closed subgroup of a compact Lie group $G$ of smaller dimension, then the underlying $H$-space of a $G$-CW-complex need not admit an $H$-CW-structure; an example is given by Illman in [10, Section 2]. Nevertheless, the underlying $H$-space of a finite $G$-CW-complex is always $H$-homotopy equivalent to a finite $H$-CW-complex, see [10, Corollary B]. Consequently, for every continuous homomorphism $\alpha\colon K \longrightarrow G$ of compact Lie groups, the restriction functor $\alpha^*$ takes $G$-spaces of the $G$-homotopy type of a finite $G$-CW-complex to $K$-spaces of the $K$-homotopy type of a finite $K$-CW-complex.

**Definition A.4** (Global transformations)**.** We let $E$ and $F$ be orthogonal spaces. A *global transformation* $\tau$ from $F$ to $E$ consists of maps

$$\tau_{G,A} \ : \ [A, F]^G \ \longrightarrow \ [A, E]^G$$

for all compact Lie groups $G$ and all $G$-spaces $A$ of the $G$-homotopy type of a finite $G$-CW-complex that are natural for restriction along $G$-maps in $A$, and natural for restriction along continuous homomorphisms in $G$.

**Construction A.5.** We let $K$ be a compact Lie group, and we let $B$ be a $K$-space. We choose a faithful $K$-representation $V$ and define the *global quotient* orthogonal space as

$$K \backslash\!\backslash B \ = \ \mathbf{L}(V, -) \times_K B \ .$$

In [18, Example 1.1.24] we use the more precise notation $\mathbf{L}_{K,V} B$ for $K \backslash\!\backslash B$. However, the orthogonal space $K \backslash\!\backslash B$ is independent of the choice of faithful representation up to a preferred zigzag of global equivalences, see the remark immediately after [18, Definition 1.1.27], which justifies the simplified notation that does not record the representation $V$. The continuous map

$$B \ \xrightarrow{[\mathrm{Id}_V, -]} \ \mathbf{L}(V, V) \times_K B \ = \ (K \backslash\!\backslash B)(V)$$

is $K$-equivariant, so it represents a class

$$u_{K,B} \ \in \ [B, K \backslash\!\backslash B]^K \ ,$$

the *tautological class*.

**Theorem A.6.** *Let $E$ be an orthogonal space, let $K$ be a compact Lie group, and let $B$ be a finite $K$-CW-complex. For every class $y$ in $[B,E]^K$, there is a unique global transformation $\tau$ from $K\backslash\!\!\backslash B$ to $E$ such that the map*

$$\tau_{K,B} \ : \ [B, K\backslash\!\!\backslash B]^K \ \longrightarrow \ [B,E]^K$$

*sends the tautological class $u_{K,B}$ to $y$.*

*Proof.* We start with uniqueness, and we let $\tau$ be any global transformation from $K\backslash\!\!\backslash B$ to $E$. We let $V$ be the faithful $K$-representation that is implicit in the definition of $K\backslash\!\!\backslash B$. We let $W$ be an inner product space and $\varphi\colon V \longrightarrow W$ a linear isometric embedding. The group $O(W)$ acts transitively on $\mathbf{L}(V,W)/K$, and the stabilizer of $\varphi K$ is the subgroup

$$O[\varphi] \ = \ \{A \in O(W)\colon \text{there is } k \in K \text{ such that } A\varphi = \varphi \circ l_k\} \ .$$

We write $\alpha\colon O[\varphi] \longrightarrow K$ for the continuous epimorphism that sends $A \in O[\varphi]$ to the unique element $\alpha(A) \in K$ such that $A\varphi = \varphi \circ l_{\alpha(A)}$.

The identity of $(K\backslash\!\!\backslash B)(W)$ represents a class $[\mathrm{Id}_{(K\backslash\!\!\backslash B)(W)}]$ in $[(K\backslash\!\!\backslash B)(W), K\backslash\!\!\backslash B]^{O(W)}$. By design, the map $\varphi\colon \alpha^*(V) \longrightarrow K$ is $O[\varphi]$-equivariant. So

$$\begin{aligned}[\varphi,-]^*(\mathrm{res}^{O(W)}_{O[\varphi]}[\mathrm{Id}_{(K\backslash\!\!\backslash B)(W)}]) \ &= \ [\varphi\colon \alpha^*(V) \longrightarrow W, -] \\ &= [(K\backslash\!\!\backslash B)(\varphi)[\mathrm{Id}_{\alpha^*(V)}, -]] \ = [\mathrm{Id}_{\alpha^*(V)}, -] \ = \ \alpha^*(u_{K,B})\end{aligned}$$

in $[\alpha^*(B), K\backslash\!\!\backslash B]^{O[\varphi]}$. The naturality properties of the global transformation $\tau$ yield the relation

$$\begin{aligned}[\varphi,-]^*(\mathrm{res}^{O(W)}_{O[\varphi]}(\tau_{O(W),(K\backslash\!\!\backslash B)(W)}[\mathrm{Id}_{(K\backslash\!\!\backslash B)(W)}])) \ &= \ \tau_{O[\varphi],\alpha^*(B)}([\varphi,-]^*(\mathrm{res}^{O(W)}_{O[\varphi]}[\mathrm{Id}_{(K\backslash\!\!\backslash B)(W)}])) \\ &= \ \tau_{O[\varphi],\alpha^*(B)}(\alpha^*(u_{K,B})) \ = \ \alpha^*(\tau_{K,B}(u_{K,B}))\end{aligned} \tag{A.7}$$

in $[\alpha^*(B), E]^{O[\varphi]}$. The map

$$O(W) \times_{O[\varphi]} \alpha^*(B) \ \longrightarrow \ \mathbf{L}(V,W) \times_K B = (K\backslash\!\!\backslash B)(W) \ , \quad [A,b] \ \longmapsto \ [A\varphi, b]$$

is an $O(W)$-equivariant homeomorphism, so the induction isomorphism (A.3) shows that the composite

$$[(K\backslash\!\!\backslash B)(W), E]^{O(W)} \ \xrightarrow{\mathrm{res}^{O(W)}_{O[\varphi]}} \ [(K\backslash\!\!\backslash B)(W), E]^{O[\varphi]} \ \xrightarrow{[\varphi,-]^*} \ [\alpha^*(B), E]^{O[\varphi]}$$

is bijective. Since $[\varphi,-]^* \circ \mathrm{res}^{O(W)}_{O[\varphi]}$ is bijective, the relation (A.7) shows that, and how, the class $\tau_{O(W),(K\backslash\!\!\backslash B)(W)}[\mathrm{Id}_{(K\backslash\!\!\backslash B)(W)}]$ in $[(K\backslash\!\!\backslash B)(W), E]^{O(W)}$ is determined by the class $\tau_{K,B}(u_{K,B})$.

Now we let $G$ be any compact Lie group, and we let $x \in [A, K\backslash\!\!\backslash B]^G$ be a homotopy class. We represent $x$ by a continuous $G$-map $f\colon A \longrightarrow (K\backslash\!\!\backslash B)(W)$ for some $G$-representation $W$. If $A = \emptyset$, then $[A,E]^G$ has only one element, and there is nothing to show. If $A$ is nonempty, then also $(K\backslash\!\!\backslash B)(W)$ is nonempty, and there exists a linear isometric embedding $\varphi\colon V \longrightarrow W$. If $\rho\colon G \longrightarrow O(W)$ parameterizes the $G$-action on $W$, then

$$f^*(\rho^*[\mathrm{Id}_{(K\backslash\!\!\backslash B)(W)}]) \ = \ [f] \ = \ x \ .$$

The naturality properties of the global transformation $\tau$ then yield the relation

$$f^*(\rho^*(\tau_{O(W),(K\backslash\!\!\backslash B)(W)}[\mathrm{Id}_{(K\backslash\!\!\backslash B)(W)}])) \ = \ \tau_{G,A}(f^*(\rho^*[\mathrm{Id}_{(K\backslash\!\!\backslash B)(W)}])) \ = \ \tau_{G,A}(x)$$

in $[A,E]^G$. Since $\tau_{O(W),(K\backslash\!\!\backslash B)(W)}[\mathrm{Id}_{(K\backslash\!\!\backslash B)(W)}]$ is determined by $\tau_{K,B}(u_{K,B})$ by (A.7), this relation shows that the class $\tau_{G,A}(x)$ is also determined by $\tau_{K,B}(u_{K,B})$. This completes the proof of injectivity.

For surjectivity we represent the class $y \in [B,E]^K$ by a continuous $K$-map $g\colon B \longrightarrow E(U)$, for some $K$-representation $U$. The class $E(i^1) \circ g\colon B \longrightarrow E(U \oplus V)$ then also represents $y$. The Yoneda lemma provides a unique morphism of orthogonal spaces $g^\flat\colon \mathbf{L}(U \oplus V, -) \times_K B \longrightarrow E$ such that the composite

$$B \ \xrightarrow{[\mathrm{Id}_{U\oplus V}, -]} \ \mathbf{L}(U \oplus V, U \oplus V) \times_K B \ \xrightarrow{g^\flat(U\oplus V)} \ E(U \oplus V)$$

equals $E(i^1)\circ g$. The associated global transformation from $\mathbf{L}(U\oplus V,-)\times_K B$ to $E$ then sends the class $[\mathrm{Id}_{U\oplus V},-]$ in $[B,\mathbf{L}(U\oplus V,-)\times_K B]^K$ to the class $y$. Restriction of linear isometric embeddings along $i^2\colon V\longrightarrow U\oplus V$ is a global equivalence of orthogonal spaces

$$\rho \ : \ \mathbf{L}(U\oplus V,-)\times_K B \ \xrightarrow{\ \sim\ } \ \mathbf{L}(V,-)\times_K B = K\backslash\!\!\backslash B \ .$$

Moreover,

$$\rho_*[\mathrm{Id}_{U\oplus V},-] \ = \ [\mathrm{Id}_V,-] \ = \ u_{K,B}$$

in $[B,K\backslash\!\!\backslash B]^K$. As a global equivalence, the induced global transformation from $\mathbf{L}(U\oplus V,-)\times_K B$ to $K\backslash\!\!\backslash B$ is bijective. So the composite global transformation from $K\backslash\!\!\backslash B$ through $\mathbf{L}(U\oplus V,-)\times_K B$ to $E$ sends the tautological class $u_{K,B}\in[B,K\backslash\!\!\backslash B]^K$ to the class $y$. □

We alert the reader that Theorem A.6 is reminiscent of, but different from, another representability result for $[B,E]^K$. For a fixed finite $K$-CW-complex $B$, the functor $[B,-]^K$ takes global equivalences of orthogonal spaces to bijections, by [18, Proposition 1.5.3 (ii)]. So we can consider $[B,-]^K$ as a functor on the unstable global homotopy category. As such, it is represented by the pair $(K\backslash\!\!\backslash B,u_{K,B})$. In other words: for every orthogonal space $E$ and every class $y$ in $[B,E]^K$, there is a unique morphism $\psi\colon K\backslash\!\!\backslash B\longrightarrow E$ in the unstable global homotopy category such that $\psi_*(u_{K,B})=y$. The morphism $\psi$ induces a global transformation that takes $u_{K,B}$ to $y$, which is the surjectivity part of Theorem A.6. The additional power of Theorem A.6 is the injectivity part, and that it is about global transformations, as opposed to morphisms in the unstable global homotopy category. This is relevant for our applications because the equivariant Segal–Becker splitting (5.9) and the Adams operations in equivariant K-theory come to us are precisely as global transformations.

Now we use Theorem A.6 to deduce a representability statement for vector bundles, by exploiting that rank $k$ complex vector bundles are represented by the global classifying space

$$B_{\mathrm{gl}}U(k) \ = \ U(k)\backslash\!\!\backslash * \ = \ \mathbf{L}(u(\nu_k),-)/U(k) \ .$$

**Definition A.8.** We let $E$ be an orthogonal space and $k\geq 1$. A *global transformation* $\tau$ from $\mathrm{Vect}^k$ to $E$ consists of maps

$$\tau_{G,A} \ : \ \mathrm{Vect}^k_G(A) \ \longrightarrow \ [A,E]^G$$

for all compact Lie groups $G$ and all $G$-spaces $A$ of the $G$-homotopy type of a finite $G$-CW-complex that are natural for restriction along $G$-maps in $A$, and natural for restriction along continuous homomorphisms in $G$.

For every compact Lie group $G$ and every paracompact $G$-space $A$, the associated vector bundle construction passes to a natural bijection

$$\mathrm{Prin}^G_{U(k)}(A) \ \xrightarrow{\ \cong\ } \ \mathrm{Vect}^k_G(A)$$

between isomorphism classes of $G$-equivariant $U(k)$-principal bundles and $G$-equivariant rank $k$ complex vector bundles. If moreover $A$ is a finite $G$-CW-complex, then [18, Proposition 1.1.30] provides a natural bijection

$$[A,B_{\mathrm{gl}}U(k)]^G \ \cong \ \mathrm{Prin}^G_{U(k)}(A) \ .$$

For varying $(G,A)$, the composite natural bijections

$$\mathrm{Vect}^k_G(A) \ \cong \ [A,B_{\mathrm{gl}}U(k)]^G$$

are moreover natural for restriction along continuous homomorphisms in $G$. So they form a bijective global transformation from $\mathrm{Vect}^k$ to $B_{\mathrm{gl}}U(k)$. Unraveling all definitions shows that for $G=U(k)$ and $A=*$, the bijection $\mathrm{Vect}^k_{U(k)}(*)\cong[*,B_{\mathrm{gl}}U(k)]^G$ taked the tautological $U(k)$-representation, considered as a $U(k)$-vector bundle over a point, to the tautological class

$$u_{U(k),*} \ \in \ [*,U(k)\backslash\!\!\backslash *]^{U(k)} = \pi_0^{U(k)}(B_{\mathrm{gl}}U(k)) \ .$$

We can thus specialize Theorem A.6 to $K = U(k)$ and $B = *$, and deduce the following corollary:

**Corollary A.9.** *Let $E$ be an orthogonal space, and $k \geq 0$. For every class $y$ in $\pi_0^{U(k)}(E)$, there is a unique global transformation $\tau$ from $\mathrm{Vect}^k$ to $E$ such that the map*

$$\tau_{U(k),*} \ : \ \mathrm{Vect}^k_{U(k)}(*) \ \longrightarrow \ \pi_0^{U(k)}(E)$$

*sends the class $[\nu_k]$ to $y$.*

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