What is the strength of two cofinal well founded branches?

In these notes we address the above question. In §1 - §4 we show that if $\mathcal{I} = \langle\langle M_i\rangle, \langle\nu_i\rangle, \langle\pi_{i,j}\rangle, T\rangle$ is a Σ^*-iteration of a premouse M with distinct cofinal well founded branches b_0, b_1, and

$$N = J_\delta^E =_{\text{Df}} \bigcup_{i<lh(\mathcal{I})} J_{\varkappa_i}^{E^{M_i}},$$

then δ is Woodin in $J_\gamma(N) =_{\text{Df}} J_{\delta+\gamma}^{E^N}$, where γ is least s.t. $\gamma > 0$ and $J_\gamma(N)$ is admissible. (This holds even if $M_{b_0} = M_{b_1} = N$.) We also show that the first projectum of $J_\gamma(N)$ is δ. In §1 - §3 these facts are proven under the additional assumption that $\mathcal{I}$ is a truncation-free iteration of a passive mouse. In §4.1 we then show how to dispense with this assumption.

(Though we work with λ-mice, the results hold for Steel mice as well. Dispensing with the above assumption is easier in the case of Steel mice.)

In §4.2 we then show that if M is actually a mouse, then δ is not only Woodin but <u>E-Woodin</u> in $J_\gamma(N)$ (meaning that the Woodinness is witnessed by extenders lying on the sequence).

In §5 we then prove a converse, showing that the above results are best possible. Call a premouse N <u>royal</u> iff $\delta = On_N$ is E-Woodin in $M = J_\gamma(N)$ and $\rho^1_M = \delta$, where γ is the least $\gamma > 0$ s.t. $J_\gamma(N)$ is admissible. M is then called the <u>crown</u> of N.

Our main theorem says that, if N is royal and its crown is itself a mouse, then there is an iteration of N' (in fact an alternating chain) with distinct cofinal branches b_0, b_1 s.t. $N_{b_0} = N_{b_1} = \bigcup_{i<\omega} J^{E^{N_i}}_{\kappa_i}$.

(The basic method is to use the "one step lemma" of Martin and Steel to produce an iteration of the crown M whose restriction to N has the desired property. However, this seems to demand an infinite descending chain of ordinals in M, so we instead iterate an ill founded end extension of M.)

We then show that, if N is royal, then any non-truncating iterate of N will also be royal. If there is no inner model with a Woodin cardinal, we can iterate any royal mouse N against K^c. The final iterate N' will be absorbed on the K^c side, but then so will its crown. Thus N has an iterate N' whose crown is also a mouse. (Similarly, if N is a 1-small royal mouse and $M_1^\#$ exists, we get the same result by iterating

N against $M_1^\#$). Call N a <u>normed</u> <u>royal</u> <u>mouse</u> iff its crown is also a mouse.

Assume that there is no inner model with a Woodin or that $M_1^\#$ exists. It follows easily that there is a <u>minimal</u> normed royal mouse which is minimal w.r.t. all other normed mice w.r.t. height and coiterability. By minimality we then have $\omega\rho_M^2 = \omega$, where M is its crown. This minimal N has the property that there is a royal N′ and an alternating chain on N′ with distinct cofinal branches b_0, b_1 s.t.

$$N'_{b_0} = N'_{b_1} = N = \bigcup_{i<\omega} J_{\eta_i}^{E^{N_i'}}.$$

But $\delta = \mathrm{On}_N$ is <u>not</u> Woodin in $J_{\gamma+1}(N)$, where $M = J_\gamma(N)$, since M projects to ω. Hence our initial theorem is best possible.

Bibliography

[MO] Jensen "Measures of Order 0" */

[NFS] Jensen "A New Fine Structure for Higher Core Models" */

[SPSC] Jensen "Subproper and Subcomplete Forcing" */

[PD] Martin and Steel "A Proof of Projective Determinacy" Journal of the AMS Vol 3 No. 1, January 1989

*/ These handwritten notes are available on my website:
www.mathematik.hu-berlin.de/~raesch/org/jensen/html
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