§5 Inflations

After we had finished the first version of §1–§4, we received a remarkable set of notes by Farmer Schlutzenberg. In §2 we proved that, if M is uniquely iterable, then then any normal iteration I of M is uniquely "inserable" – i.e., if $\mathfrak{S}$ is an inseration of I of limit length η, then there is exactly one extension of $\mathfrak{S}$ to an inseration of length $\eta+1$. In §4 we make use of the consequence that I is finitely uniquely smoothly inserable. This can be defined as follows:

<u>Def</u> By a <u>finite smooth inseration</u> of I we mean a sequence $\langle \mathfrak{S}_0, \ldots, \mathfrak{S}_{n-1} \rangle$ s.t.

(a) $\mathfrak{S}_i = \langle \langle I_i^j \rangle, \langle \nu_i^j \rangle, \langle e_i^{jk} \rangle, T^i \rangle$ is an inseration

(b) $I_0^0 = I$

(c) If $i < n-1$, then $I_{i+1}^0 = I_{\eta_i}^i$, where $\eta_i + 1 = lh(\mathfrak{S}_i)$

It is then obvious that if I is a normal iterate of M and $\langle \mathfrak{S}_0, \ldots, \mathfrak{S}_{n-1} \rangle$ is a finite smooth iteration in which $\mathcal{S}_{n-1}$ has limit length η, then I^0_{n-1} is a normal iteration of M and, hence, $\mathcal{S}_{n-1}$ has a unique extension to an iteration of length $\eta+1$. This is what unique smooth iterability says. Essentially, Schlutzenberg showed that we can drop the requirement of finiteness, showing that I is uniquely smoothly iterable, even for smooth iterations which are infinite. However, the precise definitions of "smooth iteration" and "smooth iterability" are not immediately obvious in the infinite case. It is clear what it means for $\langle \mathcal{S}_i \mid i<\omega \rangle$ to be a smooth iteration of length ω ($\langle \mathcal{S}_i \mid i<n \rangle$ is smooth for $n<\omega$). $\omega+1$ - iterability would then require that this can be extended to an iteration of length $\omega+1$. What does this mean? We can define a commutative sequence of partial insertions $e^{ij}: I_i^0 \longrightarrow I_j^0$ by: $\tilde{e}^{i,i+1} = \tilde{e}_i^{0,\eta_i}$; $\tilde{e}^{ij} \cdot \tilde{e}^{hi} = \tilde{e}^{hj}$, The iterability requirement is then that there be a good limit:

I^0_ω, $\langle e^{i,\omega} \mid i<\omega \rangle$ of

$\langle I^0_i \mid i<\omega \rangle$, $\langle e^{ij} \mid i \leq j < \omega \rangle$.

However, the maps $e^{ii'}$ are partial. In order that the concept of good limit make sense, we need that for some $i_0 < \omega$, $e^{ii'}$ is total on I^0_i for $i_0 \leq i \leq i'$. This will be the case if at most finitely many $\mathcal{S}_i$ ($i < \omega$) have a truncation on the main branch. Hence we make it a requirement of $\omega+1$ - ~~in~~serability that only finitely many $\mathcal{S}_i$ have such a truncation. A smooth iteration of length $\omega+1$ is then a sequence $\langle \mathcal{S}_i \mid i \leq \omega \rangle$ s.t. I^0_ω, $\langle e^{i,\omega} \mid i < \omega \rangle$ be given as above. $\omega+2$ iserability then requires that, if $\mathcal{S}_\omega$ is of limit length η, ~~then~~ it can be extended to an iteration of length $\eta+1$. (We already know this, since I^0_ω is a normal iteration of M.) Proceeding in this way we arrive at the general definition of "smooth iteration" and "smooth iterability".

Before proceeding to the precise definition, however, we give an introduction to Schlutzenberg's methods, Let $\mathcal{S} = \langle\langle I^i\rangle, \langle \nu^i\rangle, \langle e^{ij}\rangle, \hat{T}\rangle$ be an interation, where I^0 is a normal iteration of M. Schlutzenberg calls I^i an "inflation" of I^0, since it was obtained by successively introducing new extenders into the original sequence. He observes that there is a unique record of the changes made in going from I^0 to I^i. We shall call that record the history of I^i and denote it by $\mathrm{hist}(I^0, I^i)$.

Let $\alpha \leq \eta_i$ (where $lh(I^j) = \eta_j + 1$ for all $j < lh(\mathcal{S})$).

Set: $l(\alpha) = l^i(\alpha) =:$ the least l s.t.

$$I^i | \alpha+1 = I^l | \alpha+1.$$

Defining s_j, t_j ($j < lh(\mathcal{S})$) as we did in §2, we have:

<u>Lemma 1</u> (Let $\hat{s}_l =: \underset{j<l}{\mathrm{lub}}\, s_j$)

(a) $l(\alpha) =$ that $l \leq i$ s.t. $\hat{s}_l \leq \alpha$ and either $\alpha \leq s_l$ or $l = i$.

(b) $I^j | \alpha+1 = I^l | \alpha+1$ for $l \leq j$.

proof.

(a) $\underset{j<l}{\mathrm{lub}}\, s_j \leq \alpha$, since otherwise $s_j + 1 > \alpha$ for a $j < l$. Hence $I^l | s_j + 1 = I^i | s_j + 1$, where $\alpha + 1 \leq s_j + 1$. Hence $j \geq l$. Contr.!

But if $l \neq i$, then $\alpha \leq s_l$, since otherwise $s_l + 1 < \alpha$
and $I^{l+1} | s_l + 2 \neq I^l | s_l + 2$,
since $\nu^{l+1}_{s_l} < \nu^l_{s_l}$. But $s_l + 2 \leq \alpha + 1$,
hence $I^i | s_{l+1} + 1 = I^{l+1} | s_{l+1} + 1$)
where $s_{l+1} \geq s_l + 2$, Hence
$I^i | \alpha + 1 \neq I^l | \alpha + 1$. Contr! QED(a)

(b) This is trivial for $i = l$. Otherwise
$\alpha \leq s_l$ and $I^l | s_l + 1 = I^j | s_l + 1$
for $l \leq j \leq lh(S)$. QED (Lemma 1)

Clearly, then: $\hat{s}_i \leq \alpha \rightarrow l^i(\alpha) = i$.

Lemma 2 If $h \le i$ and $I^h|\alpha+1 = I^i|\alpha+1$, then $\nu_\alpha^i \le \nu_\alpha^h$.

proof By induction on i.

Case 1 $i = 0$ (trivial)

Case 2 $i = h+1$
Since $I^i|s_h+1 = I^h|s_h+1$ and $\nu_{s_h}^i < \nu_{s_h}^h$, it holds for $\alpha \le s_h$ by the induction hypothesis. But $l(\alpha) = i$ for $\alpha > s_h$.
QED (Case 2)

Case 3 i is a limit
Set $\tilde{s} = \underset{j<i}{\mathrm{lub}}\, s_j$. Then $I^i|s_j+1 = I^j|s_j+1$ for $j < i$. Hence it holds by the induction hypothesis for $\alpha < \tilde{s}$. But $l(\alpha) = i$ for $\tilde{s} \le \alpha$.
QED (Lemma 2)

Lemma 3 Set $a =: \{\gamma < \eta_0 \mid \tilde{e}^{0,l}(\gamma) < \alpha\}$. There is a unique e inserting $I^0|a+1$ into $I^l|\alpha+1 = I^i|\alpha+1$ s.t. $\tilde{e} \restriction a = \tilde{e}^{0,l} \restriction a$, and $\tilde{e}(\alpha) = \alpha$.

proof By induction on i.

Case 1 $i = 0$. Set: $a = \alpha$, $e = \mathrm{id} \restriction \alpha+1$.

Case 3 $i = h+1$

If $\alpha \le s_h$, then $I^i|\alpha+1 = I^h|\alpha+1$; hence $l = l^h(\alpha)$ and the result holds by the induction hypothesis

If $\alpha > s_h$, then $\alpha = s_h + 1 + j = \tilde{e}^{\mu, h+1}(\bar{\alpha})$, where $\bar{\alpha} = t_h + j$ and $\mu = \tilde{T}(h+1)$. But then $\mu = l(t_h) =$ = the least μ s.t. $I^i | t_h + 1 = I^\mu | t_h + 1$. Hence $\mu = l^\mu(t_h) = l^\mu(\bar{\alpha})$, since $\bar{\alpha} \geq t_h$. Since $\tilde{e}^{0 l} = \tilde{e}^{\mu l} \tilde{e}^{0 \mu}$, we have:
$a = \{\gamma < \gamma_0 \mid \tilde{e}^{0\mu}(\gamma) < \bar{\alpha}\}$. By the induction hypothesis there is a unique $\bar{e}$ inserting $I^0 | a+1$ into $I^\mu | \bar{\alpha}+1$ s.t. $\tilde{\bar{e}} \restriction a = \tilde{e}^{0\mu} \restriction a$ and $\tilde{\bar{e}}(a) = \bar{\alpha}$. But then $\hat{e} = \tilde{e}^{\mu l} \tilde{\bar{e}}$ has the desired properties. QED (Case 2)

Case 3 i is a limit

Set $\hat{s} = \underset{j<i}{\text{lub}}\, s_j$. Then $I^i | s_j + 1 = I^j | s_j + 1$ for $j < i$. Hence it holds for $\alpha < \hat{s}$ by the induction hypothesis. But $l(\alpha) = i$ for $\hat{s} \leq \alpha$. Then there is $j \underset{T}{<} i$ s.t. $\alpha = \tilde{e}^{j i}(\bar{\alpha})$ and $\bar{\alpha} \geq t_j$. Hence $l^i(\bar{\alpha}) = j$, as before. But then $a = \{\gamma < \gamma_0 \mid e^{0 j}(\gamma) < \bar{\alpha}\}$ as before and, by the induction hypothesis there is $\bar{e}$ inserting $I^0 | a+1$ into $I^j | \bar{\alpha}+1$ s.t. $\tilde{\bar{e}} \restriction a = \tilde{e}^{0j} \restriction a$ and $\tilde{\bar{e}}(a) = \bar{\alpha}$. Hence $\hat{e} = \tilde{e}^{ij} \cdot \tilde{\bar{e}}$ has the desired properties. QED (Lemma 3)

Def For $i < lh(\mathcal{S})$, $\alpha \leq \eta_i$ set:

$a^i_\alpha = \text{lub}\{\xi < \eta_0 \mid e^{0l}(\xi) < \alpha\}$ where $l = l^i(\alpha)$

e^i_α = the unique e inserting $I^0 | a^i_\alpha + 1$ into $I^i | \alpha + 1$ s.t. $\tilde{e} \restriction a^i_\alpha = \tilde{e}^{0l} \restriction a^i_\alpha$, $\tilde{e}(a^i_\alpha) = \alpha$.

We leave it to the reader to prove:

Lemma 4

(a) If $l = l^i(\alpha)$, then $a^i_\alpha = a^l_\alpha$ and $\tilde{e}^i_\alpha = \tilde{e}^l_\alpha$

(Hence $\tilde{e}^i_\alpha = \tilde{e}^h_\alpha$ whenever $I^i | \alpha+1 = I^h | \alpha+1$.)

(b) If $e^{\mu, i}(\bar{\alpha}) = \alpha$, $\hat{\iota}_\mu \leq \bar{\alpha}$, $\hat{\iota}_i \leq \alpha$, then $l^\mu(\bar{\alpha}) = \mu$, $l^i(\alpha) = i$, $a^\mu_{\bar{\alpha}} = a^i_\alpha$, and $\tilde{e}^{\mu i} \cdot \tilde{e}^\mu_{\bar{\alpha}} = \tilde{e}^i_\alpha$.

(c) $\tilde{e}^i_{\eta_i} \restriction a^i_{\eta_i} = \tilde{e}^{0, \eta_i} \restriction a^i_{\eta_i}$, $\tilde{e}^i_{\eta_i}(a^i_{\eta_i}) = \eta_i$

(since $\hat{\iota}_i \leq \eta_i$)

(d) If there is no truncation on the main branch of $\mathcal{S} \restriction i+1$, Then $\tilde{e}^{0i} = \tilde{e}_{\eta_i}$ and $a_{\eta_i} = \eta_0$, (since then $\tilde{e}^{0i}(\eta_0) = \eta_i$).

Now fix an $i < lh(\mathcal{S})$ and set:

$I = \langle \langle M_\alpha \rangle, \langle \nu_\alpha \rangle, \langle \pi_{\alpha,\beta} \rangle, T \rangle =: I^0$

$I' = \langle \langle M'_\alpha \rangle, \langle \nu'_\alpha \rangle, \langle \pi'_{\alpha,\beta} \rangle, T' \rangle =: I^i$

$a = \langle a^i_\alpha \mid \alpha \leq \eta_i \rangle$, $e_\alpha = e^i_\alpha$ for $\alpha \leq \eta_i$

($\langle a, \langle e_\alpha \mid \alpha \leq \eta' \rangle \rangle$ is called the <u>history</u> of I' from I and is, as we shall see, solely determined by the pair $\langle I, I' \rangle$.)

Thm 5 Let I, I' etc. be as above. Then:

(1) $a : lh(I') \to lh(I)$ and e_α inserts $\pm|a_\alpha + 1$ into $I'|\alpha+1$ for $\alpha < lh(I')$. Moreover, $\tilde{e}_\alpha(a_\alpha) = \alpha$.

(2) Let $a_\alpha < \eta$. If $\tilde{\nu}_\alpha = \tilde{\sigma}^{e_\alpha}_{a_\alpha}(\nu_\alpha)$ exists, then $\alpha \leq \eta'$ and $\nu'_\alpha \leq \tilde{\nu}_\alpha$.

(3) Let $a_\alpha < \eta$, $\alpha+1 < lh(I')$, $\nu'_\alpha = \tilde{\nu}_\alpha$. Then

$a_{\alpha+1} = a_\alpha + 1$, $\tilde{e}_{\alpha+1} \upharpoonright a_\alpha + 1 = \tilde{e}_\alpha$.

We define the index $in(\alpha) = in^{I}(\alpha)$ for $\alpha < lh(I')$ by:

$$in(\alpha) = \begin{cases} 0 & \text{if } \alpha \text{ is as in (3)} \\ 1 & \text{if not} \end{cases}$$

(4) If $in(\alpha) = 1$, $\gamma = T'(\alpha+1)$, then $a_{\alpha+1} = a_\gamma$.

(Note: By (5) we then have: $\tilde{e}_{\alpha+1} \upharpoonright a_{\alpha+1} = \tilde{e}_\gamma \upharpoonright a_\gamma$.)

(5) If $\beta \leq_{T'} \alpha$, then $a_\beta \leq_T a_\alpha$ and $\tilde{e}_\beta \upharpoonright a_\beta = \tilde{e}_\alpha \upharpoonright a_\beta$.

Moreover, if α is a limit, then $a_\alpha = \sup\{a_\beta \mid \beta <_{T'} \alpha\}$.

(Hence $\tilde{e}_\alpha \upharpoonright a_\alpha = \bigcup_{\beta <_{T'} \alpha} \tilde{e}_\beta \upharpoonright a_\beta$.)

(6) If $\mathfrak{I}|i+1$ has a truncation on the main branch, then there is $\alpha \in (e_{\eta'}(a_{\eta'}), \eta']_{T'}$ which is a truncation point in I'.

Note $\eta' = \tilde{e}_{\eta'}(a_{\eta'})$ and:

$e_{\eta'}(a_{\eta'}) = \text{lub}\, \tilde{e}_{\eta'}{}'' a_{\eta'} = \text{lub}\, \tilde{e}^{0i}{}'' a_{\eta'} = e^{0i}(a_{\eta'})$

by Lemma 4 (c).

We prove Theorem 5 by induction on i.

<u>Case 1</u> $i=0$ is trivial

<u>Case 2</u> $i=h+1$

(1) is given.

(2) If $\alpha \leq s_h$, then $I^i|\alpha+1 = I^h|\alpha+1$. Hence $l = l^h(\alpha)$ and the result holds by the induction hypothesis, given that $\nu_\alpha^i \leq \nu_\alpha^h$ if ν_α^h exists. Otherwise $l(\alpha) = i$ and $\alpha = s_h + 1 + j$. But then, letting $\mu = \tilde{T}(h+1)$, we have $\tilde{e}^{\mu,i}(\bar{\alpha}) = \alpha$, where $\bar{\alpha} = t_h + j$. As in the proof of Lemma 3 (Case 2) we have $\mu = l(t_h) = l^\mu(t_h) = l^\mu(\bar{\alpha})$. As before we have: $a^\mu_{\bar{\alpha}} = a_\alpha$ and $\tilde{e}^{\mu i} e^\mu_{\bar{\alpha}} = \tilde{e}^i_\alpha$. Since $\tilde{\nu}_\alpha = \tilde{\sigma}^{e_\alpha^i}_{a_\alpha}(\nu_0)$ exists, so does $\tilde{\nu}_{\bar{\alpha}} = \tilde{\sigma}^{e^\mu_{\bar{\alpha}}}_{a_{\bar{\alpha}}}(\nu_0)$. By the induction hypothesis $\nu^\mu_{\bar{\alpha}} \leq \tilde{\sigma}^{e^\mu_{\bar{\alpha}}}_{a_{\bar{\alpha}}}(\nu_0)$ exists. Hence

$\nu_\alpha^i = \tilde{\sigma}^{\mu,i}_{\bar{\alpha}}(\nu^\mu_{\bar{\alpha}}) \leq \tilde{\sigma}^{\mu,i}_{\bar{\alpha}}(\tilde{\nu}^\mu_{\bar{\alpha}}) = \tilde{\nu}^i_\alpha$. QED (2)

(3) We assume $\nu_\alpha^i = \tilde{\nu}_\alpha^i$. If $\alpha < s_h$, then $\nu_\alpha^i = \nu_\alpha^h$ and $\tilde{\nu}_\alpha^i = \tilde{\nu}_\alpha^h$; hence $\nu_\alpha^h = \tilde{\nu}_\alpha^h$ and the result follows by the induction hypothesis. $\alpha = s_h$ is impossible, since then $\nu_\alpha^i < \nu_\alpha^h = \tilde{\nu}_\alpha^h = \tilde{\nu}_\alpha^i$. Now let $\alpha > s_h$. As before we have $\tilde{e}^{\mu,i}(\bar{\alpha}) = \alpha$, where $\mu = \tilde{T}(h+1)$ and $l^\mu(\bar{\alpha}) = \mu$.

We again have: $a^i_\alpha = a^\mu_{\bar\alpha}$, $\tilde{e}^i_\alpha = \tilde{e}^{\mu,i} e^\mu_{\bar\alpha}$.
Hence $\tilde{\nu}^\mu_{\bar\alpha}$ exists and $\tilde{\nu}^i_\alpha = \tilde{\sigma}^{\mu,i}_{\bar\alpha}(\tilde{\nu}^\mu_{\bar\alpha})$.
Hence $\nu^i_\alpha = \tilde{\sigma}^{\mu,i}_{\bar\alpha}(\nu^\mu_{\bar\alpha}) = \tilde{\sigma}^{\mu,i}_{\bar\alpha}(\tilde{\nu}^\mu_\alpha) = \tilde{\nu}^i_\alpha$, by the induction hypothesis. QED (3)

(4) At $\alpha < s_h$, the result follows by the induction hypothesis, since $I^i | s_h+1 = I^h | s_h+1$. Now let $\alpha = s_h$. Then $in(\alpha) = 1$, as established above. Letting $\mu = \tilde{T}(h+1)$ we have: $\mu = l(\alpha) = l^\mu(\alpha)$. But $\tilde{e}^{\mu,i}(t_h) = \alpha+1$, where $t_h = \gamma =: T^i(\alpha+1)$, Hence $a^\mu_\gamma = a_{\alpha+1}$ by Lemma 4 (b) and $a^i_\gamma = a^\mu_\gamma$ since $\mu = l^i(\gamma)$.

Now let $\alpha > s_h$. Then $i = h+1$ is not a truncation point in $\mathcal{S}$ (since otherwise $\eta_i = s_h+1$). Hence $\alpha = s_h+1+j$ where $\tilde{e}^{\mu i}(\bar\alpha) = \alpha$, $\bar\alpha = t_h + j$. Moreover $\tilde{e}^{\mu i}(\bar\alpha) = e^{\mu,i}(\bar\alpha)$ and hence:

$$\tilde{\sigma}^{\mu i}_{\bar\alpha} : M^\mu_{\bar\alpha} \xrightarrow[\Sigma^*]{} M^i_\alpha.$$

Moreover, $\bar\alpha < \eta_\mu \longleftrightarrow \alpha < \eta_i$. It follows easily that $in^\mu(\bar\alpha) = 1$. By the induction hypothesis we conclude:
$a^\mu_{\bar\alpha+1} = a^\mu_{\bar\gamma}$, where $\bar\gamma = T^\mu(\bar\alpha+1)$.

We know:

$a^i_{\alpha+1} = a^i_{\tilde e^{\mu i}(\bar\gamma)}$, since for $\zeta < \gamma^0$ we have;

$e^{0i}(\zeta) < \alpha+1 \leftrightarrow \tilde e^{0\mu}(\zeta) < \bar\alpha+1 \leftrightarrow \tilde e^{0\gamma}(\zeta) < \bar\gamma$

$\leftrightarrow \tilde e^{0i}(\zeta) < \tilde e^{\mu i}(\bar\gamma)$.

But $e^{\mu i}(\bar\gamma) \le_{T^i} \gamma \le_{T^i} \tilde e^{\mu i}(\bar\gamma)$ for $\gamma = T^i(\alpha+1)$, since $e^{\mu i}$ is an insertion.

If $\bar\gamma \neq t_h$, then $e^{\mu i}(\bar\gamma) = \tilde e^{\mu i}(\bar\gamma) = \gamma$ and we and we are done. If $\bar\gamma = t_i$, then either $\gamma = t_h = \tilde e^{\mu i}(\bar\gamma)$ or $\gamma = s_h + 1 = \tilde e^{\mu i}(\bar\gamma)$, since $(t_h, s_h]$ is in limbo at i.

In the latter case we are done. In the former case we have $a_{\alpha+1} = a_{\tilde e^{\mu i}(\bar\gamma)} = a_{\bar\gamma}$, since $a_{s_h+1} = a_{t_h}$ by Lemma 4 (b).

QED (4)

(5) If $\alpha \le s_h$, the conclusion follows by the ind. hyp., since $I^i | s_h+1 = I^h | s_h+1$.

Now let $\alpha \ge s_h+1$.

Case A $\quad \beta \ge s_h + 1$.

Then $l(\beta) = l(\alpha) = i$. Clearly there are $\bar\beta, \bar\alpha$ s.t. $\tilde e^{\mu, i}(\bar\beta, \bar\alpha) = \beta, \alpha$. Hence $\bar\beta < \bar\alpha$.

We know:

$e^{\mu i}(\bar\beta) \le_{T^i} \tilde e^{\mu i}(\bar\beta) = \beta \lneq_{T^i} \alpha$

$e^{\mu i}(\bar\alpha) \le_{T^i} \tilde e^{\mu i}(\bar\alpha) = \alpha$.

Hence $e^{\mu i}(\bar\beta) \leq_{T^i} e^{\mu i}(\bar\alpha)$. Hence $\bar\beta \leq_{T\mu} \bar\alpha$, since $e^{\mu i}$ is an insertion. Hence $a^\mu_{\bar\beta} \leq_{T_0} a^\mu_{\bar\alpha}$ by the ind. hyp. But $a_\beta = a^\mu_{\bar\beta}$, $a_\alpha = a^\mu_{\bar\alpha}$ by Lemma 4.
But $l(\beta) = l(\alpha) = i$. Hence:
$\tilde e_\alpha{}'' a_\beta = \tilde e^{0i}{}'' a_\beta = \tilde e_\beta{}'' a_\beta$ QED (Case A)

<u>Case B</u> Case A fails

Then $\beta \leq t_h$, since $(t_h, s_h]$ is in limbo at i.
Let $\tilde e^{\mu,i}(\bar\alpha) = \alpha$. Then $\beta \leq \bar\alpha$, since $t_h \leq \alpha$.
But: $\beta = e^{\mu i}(\beta) \leq_{T^i} \alpha$ and $e^{\mu i}(\bar\alpha) \leq_{T^i} \tilde e^{\mu i}(\alpha) = \alpha$.
Hence $e^{\mu i}(\beta) \leq_{T^i} e^{\mu i}(\bar\alpha)$ and
$\beta \leq_{T\mu} \bar\alpha$.
By Lemma 4 (a) we have: $a_\beta = a^\mu_\beta$.
By Lemma 4 (b) we have: $a_\alpha = a^\mu_{\bar\alpha}$,
since $\bar\alpha \geq t_h \geq \hat s_\mu$ by §1.
Let $k = l^i(\beta) = l^\mu(\beta)$. By the induction hypothesis: $\tilde e^\mu_{\bar\alpha} \upharpoonright a_\beta = \tilde e^\mu_\beta \upharpoonright a_\beta = \tilde e^{0,k} \upharpoonright a_\beta$.
But $\tilde e^{0,k}{}'' a_\beta \subset \beta \subset t_h$, where $\tilde e^{\mu,i} \upharpoonright t_h = id$.
Hence:

$$\tilde e_\alpha \upharpoonright a_\beta = \tilde e^{0,i} \upharpoonright a_\beta = \tilde e^{\mu i} \tilde e^{0\mu} \upharpoonright a_\beta$$
$$= \tilde e^{0\mu} \upharpoonright a_\beta \text{ since } \tilde e^{\mu,i} \upharpoonright t_h = id$$
$$= \tilde e^\mu_{\bar\alpha} \upharpoonright a_\beta = \tilde e^{0,k} \upharpoonright a_\beta = \tilde e_\beta \upharpoonright a_\beta$$

This proves the first part of (5)

Now let $\alpha > s_h + 1$ be a limit ordinal.

Set: $b = \{\gamma \mid s_h + 1 \leq \gamma \leq_T \alpha\}$. Then b is cofinal in α and:

$$e^{0,i}(\xi) < \alpha \longleftrightarrow \bigvee \gamma \in b \; e^{0,i}(\xi) < \gamma$$

where $l(\gamma) = i$ for $s_h + 1 \leq \gamma$. Hence:

$$a^0_\alpha = \bigcup_{\gamma \in b} a^i_\gamma = \bigcup_{\gamma \leq_T \alpha} a^i_\gamma. \quad \text{QED (5)}$$

(6) If $i = h+1$ is a truncation point, then $s_h = t_h$, $s_h + 1 = \eta_i$, and $\pi^i_{s_h, s_h+1} : M^h_{s_h} \| \mu \to M^i_{s_h+1}$ where $\mu < On_{M^h_{s_h}}$. Thus η_i is a truncation point in I^i, where $e^{0,i}(a_{\eta_i}) \leq_{T^i} \tilde{e}^{0,i}(a_{\eta_i}) = \eta_i$.

Now suppose that $h+1$ is not a truncation point in $\mathcal{S}$. There must then be a truncation on the main branch of $\mathcal{S}|\mu+1$. Hence there is $\alpha \in (e^{0,\mu}(a^\mu_{\eta_\mu}), \eta_\mu]$ which is a truncation point in I^μ. By §1 Lemma 1 (4) (and the note following it) we know that $e^{\mu,i}(\alpha)$ is a truncation point in I^i. But

$$e^{\mu,i}(\alpha) \in (e^{0,i}(a_{\eta_i}), \eta_i],$$

since $a^\mu_{\eta_\mu} = a_{\eta_i}$ and $e^{0,i}(a_{\eta_i}) =$

$$= e^{\mu i} e^{0,\mu}(a^\mu_{\eta_i}); \; \eta_i = \tilde{e}^{\mu i} \tilde{e}^{0,\mu}(a^\mu_{\eta_i})$$

$$= \tilde{e}^{\mu i}(\eta_{\mu i}) = \eta_i. \quad \text{QED (6)}$$

This completes Case 2

Case 3 $i = \lambda$ is a limit ordinal.

(1) is again given. We prove (2)–(6).

(2) Let $\hat{s} = \hat{s}_\lambda =: \operatorname{lub}_{i<\lambda} s_i$. Then $I^\lambda | s_i+1 = I^i | s_i+1$ for $i<\lambda$, so (2) holds by the induction hypothesis for $\alpha < \hat{s}$. Now let $\alpha \geq \hat{s}$. Then $l(\alpha) = \lambda$ by Lemma 1. Pick $i \underset{\tilde{T}}{<} \lambda$ s.t. $\tilde{e}^{i,\lambda}(\bar{\alpha}) = \alpha$. We may suppose w.l.o.g. that $i = h+1$ is a successor, so $\bar{\alpha} \geq \hat{s}_i = s_h+1$ since otherwise $\alpha = \bar{\alpha} \leq s_i < \hat{s}$ by §1 Lemma 4.1. Hence $l^i(\bar{\alpha}) = i$.

Since $\tilde{e}^{0,\lambda} = \tilde{e}^{i,\lambda} \cdot \tilde{e}^{0,i}$, it follows as usual that $a^i_{\bar{\alpha}} = a^\lambda_\alpha$. Since $\tilde{\nu}_\alpha = \tilde{\nu}^\lambda_\alpha = \tilde{\sigma}^{e_\alpha}_{a_\alpha}(\nu^0_{a_{\alpha\lambda}})$ exists, so does:

$$\tilde{\nu}^i_{\bar{\alpha}} = \tilde{\sigma}^{e^i_\alpha}_{a_\alpha}(\nu^0_{a_\alpha}).$$

By the induction hypothesis, we have: $\nu^i_{\bar{\alpha}} \leq \tilde{\nu}^i_{\bar{\alpha}}$ exists. We pick i large enough tha $[i,\lambda)_{\tilde{T}}$ has no truncation, then:

$$\tilde{\sigma}^{i,\lambda}_{\bar{\alpha}}(\nu^i_{\bar{\alpha}}) = \nu'_\alpha \leq \tilde{\sigma}^{i,\lambda}_\alpha(\tilde{\nu}^i_{\bar{\alpha}}) = \tilde{\nu}_\alpha.$$

QED(2)

(3) holds again for $\alpha < \hat{\alpha}$ by the ind. hyp. Let $\hat{\alpha} \leq \alpha$. Then we pick $\tilde{e}^{i,\lambda}(\bar{\alpha}) = \alpha$ as before. Since $in^{\lambda}(\alpha) = 0$, it follows easily that $in^{i}(\bar{\alpha}) = 0$. Hence by the ind. hyp. we have:

$$a^{i}_{\bar{\alpha}+1} = a^{i}_{\bar{\alpha}} + 1, \quad \tilde{e}^{i}_{\bar{\alpha}+1} \upharpoonright a^{i}_{\bar{\alpha}} + 1 = \tilde{e}^{i}_{\bar{\alpha}},$$

We can pick i large enough that $\tilde{e}^{i,\lambda}(\bar{\alpha}+1) = \alpha + 1$. (Otherwise η_{λ} would be ill founded). Hence:

$$a_{\alpha} = a^{i}_{\bar{\alpha}}, \quad a_{\alpha+1} = a^{i}_{\bar{\alpha}+1} = a_{\alpha} + 1;$$

But $\tilde{e}^{0i} \upharpoonright a_{\alpha} + 1 = \tilde{e}^{i}_{\bar{\alpha}+1} \upharpoonright a_{\alpha} + 1 = \tilde{a}_{\bar{\alpha}}$.

Hence $\tilde{e}^{0i}(a_{\alpha}) = \bar{\alpha}$ and $\tilde{e}^{0\lambda}(a_{\alpha}) = \tilde{e}^{i\lambda}(\bar{\alpha}) = \alpha$.

But then $\tilde{e}_{\alpha}(a_{\alpha}) = \alpha$, $\tilde{e}_{\alpha} \upharpoonright a_{\alpha} = \tilde{e}^{0\lambda} \upharpoonright a_{\alpha}$.

Hence $\tilde{e}_{\alpha+1} \upharpoonright a_{\alpha} + 1 = \tilde{e}^{0,\lambda} \upharpoonright a_{\alpha} + 1 = \tilde{e}_{\alpha}$.

QED (3)

(4) Holds for $\alpha < \hat{\alpha}$ by the ind. hyp. Let $\hat{\alpha} \leq \alpha$. Pick $i, \bar{\alpha}$ as above. Then $a_{\alpha} = a^{i}_{\bar{\alpha}}$, $a_{\alpha+1} = a^{i}_{\bar{\alpha}+1}$. If we pick i s.t. there is no truncation in $[i, \lambda)_{T}$, then $\tilde{\sigma}^{i,\lambda}(\eta^{i}) = \eta^{\lambda}$.

Clearly, $\tilde{\sigma}^{i\lambda}_{\bar{\alpha}}(\nu^{i}_{\bar{\alpha}}) = \nu^{\lambda}_{\alpha}$. Since $in^{\lambda}(\alpha) = 1$, we have $\alpha = \eta^{\lambda}$ or $\nu^{\lambda}_{\alpha} < \tilde{\sigma}^{e_{\alpha}}_{a_{\alpha}}(\nu^{0}_{a_{\alpha}}) = \tilde{\nu}_{\alpha}$.

But then $\bar{\alpha} = \eta^i$ or $\nu^i_{\bar{\alpha}} < \widetilde{e}_{a_\alpha}(\nu^0_{a_\alpha})$. Hence $in^i(\bar{\alpha}) = 1$.
But then $a_{\bar{\alpha}+1} = a_{\bar{\gamma}}$ where $\bar{\gamma} = T^i(\bar{\alpha}+1)$, by ind. hyp.
Clearly, if $\gamma = T^\lambda(\alpha+1)$, then $\tilde{e}^{i\lambda}(\bar{\gamma}) \geq \gamma$, since $e^{i\lambda}(\bar{\gamma}) \leq_{T^\lambda} \gamma \leq_{T^\lambda} \tilde{e}^{i\lambda}(\bar{\gamma})$. Hence we can pick i big enough that $\tilde{e}^{i\lambda}(\bar{\gamma}) = \gamma$, since otherwise η_λ would be ill founded.
Hence $a_\gamma = a^i_{\bar{\gamma}}$. It follows easily that $a_{\alpha+1} = a_\gamma$.
Hence $\tilde{e}^i_{\alpha+1} \restriction a_{\alpha+1} = \tilde{e}^{0\lambda} \restriction a_{\alpha+1} = \tilde{e}^{0\lambda} \restriction a_\gamma =$
$= \tilde{e}_\gamma \restriction a_\gamma$. QED (4)

(5) For $\alpha < \lambda$ it holds by the induction hypothesis. Let $\alpha \geq \lambda$. Hence $l(\alpha) = \lambda$.
Let $\beta <_{T^\lambda} \alpha$. We consider two cases.

<u>Case A</u> $\beta \geq \lambda$.

Pick $i = h+1 <_T \lambda$.
$\tilde{e}^{i\lambda}(\bar{\beta}, \bar{\alpha}) = \beta, \alpha$. Then $\bar{\beta} < \bar{\alpha}$. We have:
$e^{i\lambda}(\bar{\beta}) \leq_{T^\lambda} \tilde{e}^{i\lambda}(\bar{\beta}) = \beta \leq_{T^\lambda} \alpha$;
$e^{i\lambda}(\bar{\alpha}) \leq_{T^\lambda} \tilde{e}^{i\lambda}(\bar{\alpha}) = \alpha$.
Hence $e^{i\lambda}(\bar{\beta}) \leq_{T^\lambda} e^{i\lambda}(\bar{\alpha})$. Hence $\bar{\beta} \leq_{T^i} \bar{\alpha}$.
By the ind. hyp. $a^i_{\bar{\beta}} \leq_{T^0} a^i_{\bar{\alpha}}$. But
$a_\beta = a^i_{\bar{\beta}}$, $a_\alpha = a^i_{\bar{\alpha}}$ by Lemma 4.
Finally we note that, since $l(\beta) = l(\alpha) =$

we have: $\tilde{e}_\alpha \upharpoonright a_\beta = \tilde{e}^{0,\lambda} \upharpoonright a_\beta = \tilde{e}_\beta \upharpoonright a_\beta$.

<u>Case B</u> $\beta < s_\lambda$.

Pick $i = h+1 \underset{\mp}{<} \lambda$ s.t. $\beta < t_h$, $s_{h+1} \le \bar{\alpha}$ and $e^{i,\lambda}(\bar{\alpha}) = \alpha$. (Hence $l(\bar{\alpha}) = i$). Then:

$\beta = e^{i\lambda}(\beta) \le_{T\lambda} \alpha$; $e^{i\lambda}(\bar{\alpha}) \le_{T\lambda} \tilde{e}^{i\lambda}(\bar{\alpha}) = \alpha$.

Hence $e^{i\lambda}(\beta) \le_{T\lambda} e^{i\lambda}(\bar{\alpha})$. Hence

$\beta \le_{Ti} \bar{\alpha}$, since $e^{i\lambda}$ is an insertion.

By the ind. hyp. we, therefore, conclude:

$$a^i_\beta \le_{T_0} a^i_{\bar{\alpha}} \text{ and } \tilde{e}^i_{\bar{\alpha}} \upharpoonright a^i_{\bar{\beta}} = \tilde{e}^i_{\bar{\beta}} \upharpoonright a^i_{\bar{\beta}},$$

By Lemma 4(a): $a^i_\beta = a_\beta$; $\tilde{e}^i_\beta = \tilde{e}_\beta$.

By Lemma 4(b): $a^i_{\bar{\alpha}} = a_\alpha$ and $\tilde{e}_\alpha = \tilde{e}^{i,\lambda}\, \tilde{e}^i_{\bar{\alpha}}$.

But $\tilde{e}_\beta{}'' a_\beta \subset t_h$ and $\tilde{e}^{i,\lambda} \upharpoonright t_h = \mathrm{id}$.

Hence:

$\tilde{e}_\alpha \upharpoonright a_\beta = \tilde{e}^{i,\lambda} \cdot \tilde{e}_{\bar{\alpha}} \upharpoonright a_\beta = \tilde{e}^{i,\lambda} \cdot \tilde{e}_\beta \upharpoonright a_\beta =$

$= \tilde{e}_\beta \upharpoonright a_\beta$. QED (Case B).

Now let $\alpha \ge s_\lambda$ be a limit ordinal.

For $\beta <_{T\lambda} \alpha$ we have $\tilde{e}_\beta \upharpoonright a_\beta = \tilde{e}_\alpha \upharpoonright a_\beta$.

But $\tilde{e}_\beta \upharpoonright a_\beta = \tilde{e}^{0,l(\beta)} \upharpoonright a_\beta$, and

$\tilde{e}_\alpha \upharpoonright a_\beta = \tilde{e}^{0,\lambda} \upharpoonright a_\beta$. Hence

$\tilde{e}^{0,l(\beta)} \upharpoonright a_\beta = \tilde{e}^{0,\lambda} \upharpoonright a_\beta$. Thus

$$a_\alpha = \{\xi \mid \tilde{e}^{0,\lambda}(\xi) < \alpha\} = \bigcup_{\beta <_{T\lambda} \alpha} \{\xi \mid \tilde{e}^{0,l(\beta)} < \beta\}$$

$$= \sup_{\beta <_{T\lambda} \alpha} a_\beta.$$

QED (5)

(6) Pick $i = h+1 \leq_{\tilde{T}} \lambda$ s.t. there is no truncation point in $[0,\lambda]_{\tilde{T}}$. Then there is a truncation point on the main branch of $\mathcal{S}/\mu+1$, where $\mu = \hat{T}(h+1)$. Hence there is $\alpha \in (e^{0,\mu}(a^{\mu}_{\eta_\mu}), \eta_\mu]$ which is a truncation point in I^{μ}. As before, it follows that $e^{\mu,\lambda}(\alpha)$ is a truncation point in I^{λ}. But $e^{\mu,\lambda}(\alpha) \in (e^{0,\lambda}(a_{\eta_\lambda}), \eta_\lambda]_{I^{\lambda}}$.

QED (Theorem 5)

The general notion of inflation

Following Schlutzenberg we define:

Def Let I be a normal iteration of M of successor length $\eta+1$. Let I' be a normal iteration of M. We call I' an inflation of I iff there exists:

$$\langle a, \langle e_\alpha \mid \alpha < lh(I') \rangle\rangle$$

satisfying (1) – (5) in Theorem 5.

We note that $\langle a, \langle e_\alpha \mid \alpha < lh(I')\rangle\rangle$ is uniquely determined by the pair $\langle I, I'\rangle$, since (2) – (5) constitute a recursive definition of it.

We therefore call $\langle a, \langle e_\alpha \mid \alpha < lh(I')\rangle\rangle$ the history of I' from I and denote it by: $hist(I, I')$.

For the record we state:

Lemma 6 If I' is an inflation of I, then $\mathrm{hist}(I, I')$ is uniquely determined by the pair $\langle I, I' \rangle$.

. . . . .

Our definition of "inflation" differs slightly from Schlutzenberg's definition. For one thing, he allows I' to have points α which possess no ancestor a_α. For our purposes, however, it seemed unnecessary to include this case. He also used an axiom different from (5) in his definition of inflation. The following Lemma shows that that his axiom continues to hold under our definition.

<u>Lemma 7</u> Let $\mu \le a_\alpha$ s.t. $e_\alpha(\mu) \underset{T'}{\le} \beta \underset{T'}{\le} \tilde{e}_\alpha(\mu)$.

Then $a_\beta = \mu$. Moreover $\tilde{e}_\beta \restriction \mu = \tilde{e}_\alpha \restriction \mu$.

(Thus $\tilde{e}_\beta(\mu) = \beta$, $e_\beta(\mu) = e_\alpha(\mu) = \sup \tilde{e}_\alpha{}''\mu$.)

proof Induction on α

<u>Case 1</u> $\alpha = 0$. Then $0 = a_\alpha$ and $e_\alpha(0) = \tilde{e}_\alpha(0) = 0$.

<u>Case 2</u> $\alpha = \gamma + 1$

<u>Case 2.1</u> $\text{in}(\gamma) = 0$

Then $a_\alpha = a_\gamma + 1$, $\tilde{e}_\alpha \restriction \alpha = \tilde{e}_\gamma$.

For $\mu < a_\alpha$ it follows by the ind. hyp.

But for $\mu = a_\alpha$ we have:

$$e_\alpha(a_\alpha) = \tilde{e}_\alpha(a_\alpha) = \alpha_{\gamma}$$

(since $e_\alpha(a_\alpha) = \sup \tilde{e}_\alpha{}''(a_\alpha + 1) = \alpha + 1 = \tilde{e}_\alpha(a_\alpha)$)

<u>Case 2.2</u> $\text{in}(\gamma) = 1$.

Let $\tau = T'(\gamma + 1)$. Then $a_\alpha = a_\tau$, $\tilde{e}_\tau \restriction a_\tau = \tilde{e}_\alpha \restriction a_\tau$. For $\mu < a_\alpha$ it follows by the ind. hyp. Now let $\mu = a_\alpha$.

Then $e_\alpha(a_\alpha) = e_\tau(a_\tau)$, $\tilde{e}_\alpha(a_\alpha) = \alpha$, Let $e_\delta(a_\delta) \leq_{T'} \beta \leq_{T'} \tilde{e}_\delta(a_\delta) = \delta$.

If $\beta \leq \tau = \tilde{e}_\gamma(\bar{\tau})$, then it follows by the ind. hyp. If not, then $\beta = \alpha$ and $a_\beta = a_\alpha$. QED (Case 2.2)

<u>Case 3</u> α is a limit.

Then $a_\alpha = \sup\{a_\gamma \mid \gamma <_{T'} \alpha\}$ and
$\tilde{e}_\alpha \restriction a_\alpha = \bigcup_{\gamma <_{T'} \alpha} \tilde{e}_\gamma \restriction a_\gamma$.

If $\mu < a_\gamma$ for a $\gamma <_{T'} \alpha$, then
$e_\alpha(\mu) = e_\gamma(\mu) \leq_{T'} \beta \leq_{T'} \tilde{e}_\alpha(\mu) = \tilde{e}_\gamma(\mu)$
and the claim follows by the ind. hyp., since $\tilde{e}_\gamma \restriction a_\gamma = \tilde{e}_\alpha \restriction a_\gamma$.
If there is no such γ, then $\mu = a_\alpha$ and
$e_\alpha(a_\alpha) = \sup_{\gamma <_{T'} \alpha} \tilde{e}_\alpha(\gamma) = \sup_{\gamma <_{T'} \alpha} \tilde{e}_\gamma(\gamma) = \alpha = \tilde{e}_\alpha(a_\alpha)$,
where $\tilde{e}_\alpha \restriction a_\alpha = \bigcup_{\gamma <_{T'} \alpha} \tilde{e}_\gamma \restriction a_\gamma$).
(Hence $e_\alpha = \tilde{e}_\alpha$). QED (Lemma 7)

Extending Inflations

It is easily seen that:

Lemma 8 Let I' be an inflation of I, Let $\xi \le lh(I')$. Then $I' | \xi$ is an inflation of I.

proof

Axioms (1)–(5) still hold. QED (Lemma 8)

This means that an inflation I' can be much shorter than the original I. (For instance, the iteration $\langle \langle M \rangle, \emptyset, \langle id \rangle, \emptyset \rangle$ of length 1 is an inflation of I.) Hence it is useful to have lemmas enabling us to extend a given inflation in length. $\emptyset$ is the unique iteration of length 0 and it extends to $\langle \langle M \rangle, \emptyset, \langle id \rangle, \emptyset \rangle$ of length 1. We also have:

Lemma 9 Let I' be an inflation of I of successor length $\eta'+1$, Let $a = a_{\eta'}$ and assume that $\tilde{\nu} = \tilde{\sigma}_{\eta'}^{e_{\eta'}}(\nu_a^\circ)$ exists. Assume further that $\tilde{\nu} > \nu_\xi'$ for all $\xi < \eta'$. Extend I' to I'' of length $\eta'+2$, by appointing $\nu_{\eta'}'' = \tilde{\nu}$. Then I'' is an inflation of I with history

$\langle a', \langle e'_\alpha \mid \alpha \leq \eta'+1\rangle\rangle$, where: $a'_\gamma = a_\gamma$ for $\gamma \leq \eta'$, $a'_{\eta'+1} = a_{\eta'}+1$, $\tilde{e}'_\gamma = \tilde{e}_\gamma$ for $\gamma \leq \eta'$, $\tilde{e}_{\eta'+1} \upharpoonright a_{\eta'}+1 = \tilde{e}_{\eta'}$, and $\tilde{e}'_{\eta'+1}(a'_{\eta'+1}) = \eta'+1$.

proof.

We must show that (1) - (5) are satisfied by $\langle a', \langle e'_\alpha \mid \alpha \leq \eta'+1\rangle$.

The only problematical case is (5).

We must show that if $\gamma <_{T''} \eta'+1$, then $a_\gamma <_T a_{\eta'}+1$ and $\tilde{e}_\gamma \upharpoonright a_\gamma = \tilde{e}_{\eta'+1} \upharpoonright a_\gamma$. It suffices to prove this for $\gamma = T''(\eta'+1)$.

Let $\bar{\gamma} = T(a_{\eta'}+1)$. Then $e_{\eta'}(\bar{\gamma}) \leq_{T'} \gamma \leq_{T'} \tilde{e}_{\eta'}(\bar{\gamma})$. Hence $a_\gamma = \bar{\gamma}$ and $\tilde{e}_\gamma \upharpoonright a_\gamma = \tilde{e}_{\eta'} \upharpoonright a_\gamma$, where $\tilde{e}_{\eta'+1} \upharpoonright \eta'+1 = \tilde{e}_{\eta'}$.

QED (Lemma 9)

<u>Lemma 10</u> Let I' be an inflation of I of limit length η'. Let b be the unique cofinal well founded branch in I'. Extend I' to I'' of length $\eta'+1$, by appointing: $T''``\{\eta'\} = b$. Then I'' is an inflation of I with history $\langle a', \langle e'_\alpha \mid \alpha \leq \eta' \rangle\rangle$, where:

$a'_\gamma = a_\gamma$ for $\gamma < \eta'$, $a'_{\eta'} = \sup_{\beta \in b} a'_\beta$

$\tilde{e}'_\gamma = \tilde{e}_\gamma$ for $\gamma < \eta'$;

$\tilde{e}'_{\eta'} \restriction a'_{\eta'} = \bigcup_{\beta \in b} \tilde{e}_\beta \restriction a_\beta$;

$\tilde{e}'_{\eta'}(a'_{\eta'}) = \eta'$,

proof.

(1)–(5) are satisfied

Composing Inflations:

We now show that if I' is an inflation of I and I'' is an inflation of I', then I'' is an inflation of I.

Thm 11 Let I, I', I'' be normal iterations of M with: $lh(I) = \eta + 1$, $lh(I') = \eta' + 1$. Let I' be an inflation of I with:

$$\text{hit}(I, I') = \langle a, \langle e_\alpha \mid \alpha \leq \eta' \rangle\rangle.$$

Let I'' be an inflation of I' with:

$$\text{hit}(I', I'') = \langle a', \langle e'_\alpha \mid \alpha < lh(I'')\rangle\rangle.$$

Then I'' is an inflation of I with:

$$\text{hit}(I, I'') = \langle a'', \langle e''_\alpha \mid \alpha < lh(I'')\rangle\rangle,$$

where: $a''_\alpha = a_{a'_\alpha}$; $e''_\alpha = e'_\alpha \cdot e_{a'_\alpha}$

(hence $\tilde{e}''_\alpha = \tilde{e}'_\alpha \cdot \tilde{e}_{a'_\alpha}$.)

proof.

We verify (1) – (5)

(1) $a'' = a \cdot a'$ clearly maps $lh(I'')$ into $lh(I)$. Since e'_α inserts $I' | a'_\alpha + 1$ into $I'' | \alpha + 1$ and $e_{a'_\alpha}$ inserts $I | a''_\alpha + 1$ into $I' | a'_\alpha + 1$, then $e'_\alpha \cdot e_{a'_\alpha}$ inserts $I | a''_\alpha + 1$ into $I'' | \alpha + 1$. QED (1)

Now let:

$$I = \langle \langle m_\alpha \rangle, \langle \nu_\alpha \rangle, \langle \pi_{\alpha\beta} \rangle, T \rangle$$
$$I' = \langle \langle m'_\alpha \rangle, \langle \nu'_\alpha \rangle, \langle \pi'_{\alpha\beta} \rangle, T' \rangle$$
$$I'' = \langle \langle m''_\alpha \rangle, \langle \nu''_\alpha \rangle, \langle \pi''_{\alpha,\beta} \rangle, T'' \rangle$$

We recall that if e inserts I into I' and e' inserts I' into I'', then $e'e$ inserts I into I''. Moreover $\sigma^{e'e}_\xi = \sigma^{e'}_{e'(\xi)} \cdot \sigma^{e}_\xi$, $\tilde{\sigma}^{e'e}_\xi = \tilde{\sigma}^{e'}_{\tilde{e}'(\xi)} \cdot \tilde{\sigma}^{e}_\xi$ for $\xi \in I$. Thus, in particular:

$$\sigma^{e''_\alpha}_\xi = \sigma^{e'_\alpha \cdot e_{a'_\alpha}}_\xi = \sigma^{e'_\alpha}_{e'_\alpha(\xi)} \cdot \sigma^{e_{a'_\alpha}}_\xi$$

for $\xi < a''_\alpha$. Similarly for $\tilde{\sigma}^{e''_\alpha}$.

(2) If $\tilde{\nu}''_\alpha = \tilde{\sigma}^{e''_\alpha}_{a''_\alpha}(\nu_{a''_\alpha})$ exists, then

$$\tilde{\nu}''_\alpha = \tilde{\sigma}^{e'_\alpha}_\alpha \cdot \tilde{\sigma}^{e_{a'_\alpha}}_{a'_\alpha}(\nu_{a_{a'_\alpha}}) = \tilde{\sigma}^{e'_\alpha}_\alpha(\tilde{\nu}'_{a'_\alpha}).$$

But then $\nu'_{a'_\alpha} \leq \tilde{\nu}'_{a'_\alpha}$ and:

$$\nu''_\alpha \leq \tilde{\sigma}^{e'_\alpha}_\alpha(\nu'_{a'_\alpha}) \leq \tilde{\nu}''_\alpha. \quad \text{QED(2)}$$

Now let: $in(\alpha)$ = the index of α wrt I, I'
$in'(\alpha)$ = " " " α " I', I''
$in''(\alpha)$ = " " " α " I, I''.

(3) It is easily seen that, if $in''(\alpha) = 0$, then $in(a'_\alpha) = in'(\alpha) = 0$. Hence

$$a'_{\alpha+1} = a'_\alpha + 1, \quad a''_{\alpha+1} = a_{a'_{\alpha+1}} = a_{(a'_\alpha+1)} = a''_\alpha + 1,$$

Moreover:

$$\tilde{e}''_{\alpha+1} \upharpoonright a''_\alpha + 1 = \tilde{e}'_{\alpha+1} \cdot \tilde{e}_{a'_\alpha+1} \upharpoonright a_{a'_\alpha} + 1$$
$$= \tilde{e}'_{\alpha+1} \cdot \tilde{e}_{a'_\alpha}$$
$$= \tilde{e}'_{\alpha+1} \upharpoonright (a'_\alpha + 1) \cdot \tilde{e}_{a'_\alpha}$$
$$= \tilde{e}'_\alpha \cdot \tilde{e}_{a'_\alpha} = \tilde{e}''_\alpha .$$

QED(3)

(4) Assume $in''(\alpha) = 1$. Then $in'(\alpha) = 1$ or $in(a'_\alpha) = 1$.

Case 1 $in'(\alpha) = 1$

Let $\gamma = T''(\alpha+1)$. Then $a'_\gamma = a'_{\alpha+1}$. Hence

$$a''_\gamma = a_{a'_\gamma} = a_{a'_{\alpha+1}} = a''_{\alpha+1} .$$

Case 2 $in(a'_\alpha) = 1$ but $in'(\alpha) = 0$

Let $\gamma = T'(a'_\alpha + 1)$. Then

$$a_\gamma = a_{(a'_\alpha+1)} = a_{a'_{\alpha+1}} = a''_{\alpha+1} .$$

Let $\beta = T''(\alpha+1)$. Then

$e_\alpha(\gamma) \leq_{T''} \beta \leq_{T''} \tilde{e}_\alpha(\gamma)$. Hence by Lemma 7:

$$\gamma = a'_\beta, \quad a''_{\alpha+1} = a_\gamma = a_{a'_\beta} = a''_\beta .$$

QED(4)

(5) Let $\beta \leq_{T''} \alpha$. Then $a'_\beta \leq_{T'} a'_\alpha$ and

$a''_\beta = a_{a'_\beta} \leq_T a_{a'_\alpha} = a''_\alpha$. Now let $\mu < a''_\beta$.

Then: $\tilde{e}''_\alpha(\mu) = \tilde{e}'_\alpha \cdot \tilde{e}_{a'_\alpha}(\mu) = \tilde{e}'_\alpha \cdot \tilde{e}_{a'_\beta}(\mu)$ -

(since $\mu < a'_\beta \leq_{T'} a'_\alpha$) $= \tilde{e}'_\beta \cdot \tilde{e}_{a'_\beta}(\mu) = e''_\beta(\mu)$,

(since $\tilde{e}_{a'_\beta}(\mu) <_{T'} \tilde{e}_{a'_\beta}(a'_\beta) = \beta \leq_{T''} \alpha$)

This proves the first part of (5). Now let α be a limit. Then $a'_\alpha = \sup_{\beta \le_{T''} \alpha} a'_\beta$.

If $a'_\alpha = a'_\beta$ for a $\beta <_{T''} \alpha$, then

$a'_\alpha = a'_\beta$ and $a''_\alpha = a_{a'_\alpha} = a_{a'_\beta} = a''_\beta$

for sufficiently large $\beta \le_{T''} \alpha$.

Hence $a''_\alpha = \sup_{\beta <_{T''} \alpha} a''_\beta$.

Now suppose $a'_\alpha \neq a'_\beta$ for all $\beta <_{T''} \alpha$.

Then a'_α is a limit and

$$a''_\alpha = a_{a'_\alpha} = \sup_{\mu <_{T'} a'_\alpha} a_\mu = \sup_{\beta \le_{T''} \alpha} a_{a'_\beta} =$$

$$= \sup_{\beta <_{T''} \alpha} a''_\beta.$$

QED (5)

This proves Thm 11