Chapter 4

Properties of Mice

4.1 Solidity

In §2.5.3 we introduced the notion of *soundness*. Given a sound M, we were then able to define the n-th projectum $\rho_M^n (n < \omega)$. We then defined the n-th *reduct* $M^{n,a}$ with respect to a parameter a (consisting of a finite set of ordinals). We then defined the n-th set P_M^n of *good parameters* and the set R_M^n of *very good* parameters. (Soundness was, in fact, equivalent to the statement: $P^n = R^n$ for $n < \omega$). We then defined the n-th *standard parameter* $p_M^n \in R_M^n$ for $n < \omega$. This gave us the classical fine structure theory, which was used to analyze the constructible hierarchy and prove such theorems as $\Box$ in L. Mice, however, are not always sound. We therefore took a different approach in §2.6, which enabled us to define $\rho_M^n, M^{n,a}, P_M^n, R_M^n$ for all acceptable M. (In the absence of soundness we could, of course, have: $R_M^n \neq P_M^n$). In fact R_M^n could be empty, although P_M^n never is. P_M^n was defined in §2.6.

P_M^n is a subset of $[\mathrm{On}_M]^{<\omega}$ for acceptable $M = \langle J_\alpha^A, B\rangle$. Moreover, the reduct $M^{n,a}$ is defined for any $n < \omega$ and $a \in [\mathrm{On}_M]^{<\omega}$. The definition of P_M^n, M^n are recapitulated in §3.2.5, together with some of their consequences. R_M^n is defined exactly as before, taking $= R_M^n = \varnothing$ if n is not weakly sound. At the end of §2.6 we then proved a very strong downward extension lemma, which we restate here:

Lemma 4.1.1. *Let $n = m + 1$. Let $a \in [\mathrm{On}_M]^{<\omega}$. Let $N = M^{n,a}$. Let $\overline{\pi} : \overline{N} \longrightarrow_{\Sigma_j} N$ where $\overline{N}$ is a J-model and $j < \omega$. Then:*

(a) *There are unique $\overline{M}, \overline{a}$ such that $\overline{a} \in R_{\overline{M}}^n$ and $\overline{M}^{n,\overline{a}} = \overline{N}$.*

(b) *There is a unique* $\pi \supset \overline{\pi}$ *such that:*

$$\pi : \overline{M} \longrightarrow_{\Sigma_0^{(m)}} M \text{ strictly and } \pi(\bar{(a)}) = a.$$

(c) $\pi : \overline{M} \longrightarrow_{\Sigma_j^{(n)}} M.$

In §2.6. we also proved:

Lemma 4.1.2. *Let* $n = m + 1$*. Let* $a \in R_M^n$*. Then every element of* M *has the form* $F(\xi, a)$ *where* $\xi < \rho_M^n$ *and* F *is a good* $\Sigma_1^{(m)}$ *function.*

Corollary 4.1.3. *Let* $n, a, \overline{\pi}, \pi$ *be as in Lemma 4.1.1, wehere* $j > 0$*. Then*

$\mathrm{rng}(\pi) =$ *The set of* $F(\xi, a)$ *such that* F *is a good* $\Sigma_1^{(m)}$ *function and* $\xi \in \mathrm{rng}(\overline{\pi}) \cap \rho_M^n$

Proof.. Let Z be the set of such $F(\xi, a)$.

Claim 1. $\mathrm{rng}(\pi) \subset Z$.

Proof. Let $y = \pi(\overline{y})$. Then $\overline{y} = \overline{F}(\xi, \overline{a})$ where $\overline{F}$ is a good $\Sigma_1^{(n)}(\overline{M})$ function and $\xi < \rho_{\overline{M}}^n$ by Lemma 4.1.2. Hence $y = F(\pi(\xi), a)$, where F has the same good $\Sigma_1^{(n)}$ definition in M.

QED(Claim 1.)

Claim 2. $Z \subset \mathrm{rng}(\pi)$.

Proof. Let $y = F(\pi(\xi), a)$, where F is a good $\Sigma_1^{(m)}(M)$ function. Then the $\Sigma_1^{(n)}$ statement:

$$\bigvee y \; y = F(\pi(\xi), a)$$

holds in M. Hence, there is $\overline{y} \in \overline{M}$ such that $\overline{y} = \overline{F}(\xi, a)$ where $\overline{F}$ has the same good $\Sigma_1^{(m)}$ definition in $\overline{M}$. Hence

$$\pi(\overline{y}) = F(\pi(\xi), a) = y.$$

QED(Corollary 4.1.3)

Note. $\mathrm{rng}(\pi) \subset Z$ holds even if $j = 0$.

Lemma 4.1.1 shows that a great deal of the theory developed in §2.5.3 for sound structures actually generalizes to arbitrary acceptable structures. This is not true, however, for the concept of *standard parameter*.

In our earlier definition of standard parameter, we assumed the soundness of M (meaning that $P^n = R^n$ for $n < \omega$). We defined a well ordering $<_*$ of $[\mathrm{On}]^{<\omega}$ by:

$$a <_* b \longleftrightarrow \bigvee \xi (a \setminus \xi = b \setminus \xi \wedge \xi \in b \setminus a).$$

We then defined the n-th *standard parameter* p^n_M to be the $<_*$-least $a \in M$ with $a \in P^n$. This definition stil makes sense even in the absence of soundness. We know that $p^n \setminus \rho^i \in P^i$ for $i \leq n$. Hence by $<_*$-minimality we get: $p^n \setminus \rho^n = \varnothing$. For $i \leq n$ we clearly have $p^i \leq_* p^n \setminus \rho^i$ by $<_*$-minimality. However, it is hard to see how we could get more than this if our only assumption on M is acceptability.

Under the assumption of soundness we were able to prove:

$$p^n \setminus \rho^i = p^i \text{ for } i \leq n.$$

It turns out that this does still holds under the assumption that M is fully $\omega_1 + 1$ iterable. Moreover if $\pi : M \longrightarrow N$ is an iteration map, then $\pi(p^n_M) = P^n_N$. The property which makes the standard parameter so well behaved is called *solidity*. As a preliminary to defining this notion we first define:

Definition 4.1.1. Let $a \in M$ be a finite set of ordinals such that $\rho^\omega \cap a = \varnothing$ in M. Let $\nu \in a$. The ν-th *witness to* a in M (in symbols M^ν_a) is defined as follows:

Let $\rho^{i+1} \leq \nu < \rho^i$. Let $b = a \setminus (\nu + 1)$. Let $\overline{M} = M^{i,b}$ be the i-th reduct of M by b. Set: $X = h(\nu \cup (b \cap \overline{M}))$, i.e. $X =$ the closure of $\nu \cup (u \cap \overline{M})$ under $\Sigma_1(M)$ functions. Let:

$$\overline{\sigma} : \overline{W} \longleftrightarrow \overline{M}|X$$

be the transitivation of $\overline{M}|X$. By the extension of embedding lemma there are unique $W, n, \sigma \supset \overline{\sigma}$ such that:

$$\overline{W} = W^{i,\overline{b}}, \sigma : W \longrightarrow_{\Sigma^{(i)}_1} M, \sigma(\overline{b}) = b.$$

Set: $M^\nu_a = W$. σ is called the *canonical embedding* for a in M and is sometimes denoted by σ^ν_a.

Note. Using Lemma 4.1.3 it follows that $\mathrm{rng}(\pi)$ is the set of all $F(\vec{\xi}, b)$ such that $\xi_1, \ldots, \xi_n \subset \nu$, $b = a \setminus (\nu + 1)$ and F is good $\Sigma^{(i)}_1(M)$ function. This is a more conceptual definition of M^ν_a, σ.

Definition 4.1.2. M is *n-solid* iff $M^\nu_a \in M$ for $\nu \in a = p^n_M$ it is solid iff it is n-solid for all n.

p^n was defined as the $<_*$- least element of P^n. Offhand, this seems like a rather arbitrary way of choosing an element of P^n. Solidity, however, provides us with a structural reason for the choice. In order to make this clearer, let us define:

Definition 4.1.3. Let $a \in M$ be a finite set of ordinals. a is *solid for M* iff for all $\nu \in a$ we have

$$\rho_M^\omega \leq \nu \text{ and } M_a^\nu \in M$$

.

Lemma 4.1.4. *Let $a \in P^n$ such that $a \cap \rho^n = \varnothing$. If a is solid for M, then $a = p^n$.*

Proof. Suppose not. Then there is $q \in P^n$ such that $q <_* a$. Hence there is ν such that $q \setminus (\nu+1) = a \setminus (\nu+1)$ and $\nu \in a \setminus q$. But then $q \subset \nu \cup (a \setminus (\nu+1)) \subset \mathrm{rng}(\sigma)$ where $\sigma_a = \sigma_a^\nu$ is the canonical embedding. Let A be $\Sigma^{(n)}(M)$ in q such that $A \cap \rho^{n+1} \notin M$. Let $\overline{A}$ be $\Sigma_1^{(n)}(M_a^\nu)$ in $\overline{q} = \sigma^{-1}(q)$ by the same definition. Since $\sigma \restriction \nu = \mathrm{id}$ and $\rho^n \leq \nu$, we have:

$$A \cap \rho^n = \overline{A} \cap \rho^n \in M,$$

since $A \in \underset{\sim}{\Sigma}_1^n(M_a^\nu) \subset M$. Contradiction!

QED(Lemma 4.1.4)

The same proof also shows:

Lemma 4.1.5. *Let a be solid for M such that $a \cap \rho^n = \varnothing$ and $a \cup b \in P^n$ for some $b \subset \nu$ such that $ab \subset \nu$ for all $\nu \in a$. Then a is an upper segment of p^n (i.e. $a \setminus \nu = p^n \setminus \nu$ for all $\nu \in a$.)*

Hence:

Corollary 4.1.6. *If M is n-solid and $i < n$, then M is i-solid and $p^i = p^n \setminus \rho^i$.*

Proof. Set $a = p^n \setminus \rho^i$. Then $a \in P^i$ is M-solid. Hence $a = p^i$.

QED(Corollary 4.1.6)

We set $p_M^* =: \bigcup_{n<\omega} p_M^n$. Then $p^* = p^n$ where $\rho^n = \rho^\omega$.

p^* is called the *standard parameter* of M. It is clear that M is solid iff p^* is solid for M.

Definition 4.1.4. Let $a \in [\mathrm{On}_M]^{<\omega}, \nu \in a$ with $\rho^{i+1} \leq \nu < \rho^i$ in M. Let $b = a \setminus (\nu + 1)$. By a *generalized witness* to $\nu \in a$ we mean a pair $\langle N, c \rangle$ such that N is acceptable, $\nu \in N$ and for all $\xi_a, \ldots, \xi_r < \nu$ and all $\Sigma_1^{(i)}$ formulae φ we have:

$$M \models \varphi(\vec{\xi}, b) \longrightarrow N \models (\vec{\xi}, c).$$

Lemma 4.1.7. *Let $N \in M$ be a generalized witness to $\nu \in a$. Assume that $\nu \notin \mathrm{rng}(\sigma)$, where $\sigma = \sigma_a^\nu$ is the canonical embedding. Then $M_a^\nu \in M$.*

Proof. Let $W = M_a^\nu, \overline{W}, \overline{\sigma}$ be as in the definition of M_a^ν. Then $\overline{W} = W^{i,\bar{b}}$, where $\rho^{i+1} \leq \nu < \rho^i$ in M, $b = a \setminus (\nu + 1)$ and $\sigma(\bar{b}) = b$. Since $\sigma \restriction \nu = \mathrm{id}$, we have:

$$\overline{W} \models \varphi(\vec{\xi}, \bar{b}) \longrightarrow N \models \varphi(\vec{\xi}, c),$$

for $\xi_1, \ldots, \xi_r < \nu$ and $\Sigma_1^{(i)}$ formulae φ. We can then define a map $\tilde{\sigma}$: $W \longrightarrow_{\Sigma_1^{(i)}} N$ by:

Let $x = F(\vec{\xi}, \bar{b})$ where $\xi_1, \ldots, \xi_r < \nu$ and F is a good $\Sigma_1^{(i)}(W)$ function. Then, letting $\dot{F}$ be a good definition of F we have:

$$W \models \bigvee x(x = \dot{F}(\vec{\xi}, \bar{b})); \text{ hence } N \models \bigvee x(x = \dot{F}(\vec{\xi}, c)).$$

We set $\tilde{\sigma}(x) = y$, where $N \models y = \dot{F}(\vec{\xi}, c)$.

If we set: $\overline{N} = N^{i,c}$, we have:

$$\tilde{\sigma} \restriction \overline{W} : \overline{W} \longrightarrow_{\Sigma_0} \overline{N}.$$

Let $\gamma = \sup \tilde{\sigma}\text{"}\, \mathrm{On}_{\overline{N}}, \tilde{N} = \overline{N} | \gamma$. Then:

$$\tilde{\sigma} \restriction \overline{W} : \overline{W} \longrightarrow_{\Sigma_1} \tilde{N} \text{ cofinally.}$$

Note that, since $\sigma(\nu) > \nu$ and $\sigma \restriction \nu = \mathrm{id}$, we have: ν is regular in M_a^ν. Hence $\sigma(\nu)$ is regular in M and $H_{\sigma(\nu)}^M$ is a ZFC^- model. We now code $\overline{W}$ as follows. Each $x \in \overline{W}$ has the form: $h(j, \prec \xi, \bar{b} \succ)$ where $h = h_{\overline{W}}$ is the Skolem function of $\overline{W}$ and $\sigma < \nu$.

Set:

$$\dot{\in} = \{\prec\prec j, \xi \succ, \prec k, \zeta \succ\succ : h(j, \prec \xi, \bar{b} \succ) \in h(k, \langle \zeta, \bar{b} \rangle)\}$$
$$\dot{A} = \{\prec j, \xi \succ : h(j, \langle \xi, \bar{b} \rangle) \in A\}$$
$$\dot{B} = \{\prec j, \xi \succ : h(j, \langle \xi, \bar{b} \rangle) \in B\}$$

where $\overline{W} = \langle J_\gamma^A, B \rangle$. Let $D \subset \nu$ code $\langle \dot{\in}, \dot{A}, \dot{B} \rangle$. Then:

$$D \in \Sigma_\omega(\tilde{(}N)) \subset M,$$

since e.g.

$$\dot{\in} = \{\langle \prec j, \xi \succ, \prec k, \zeta \succ \rangle : h_{\tilde{N}}(j, \langle \xi, c \rangle) \in h_{\tilde{N}}(k, \langle \zeta, c \rangle)\}$$

But then $D \in H_{\sigma(\nu)}^M$ by acceptability. But $H_{\sigma(\nu)}^M$ is a ZFC^- model. Hence $\overline{W} \in H_{\sigma(\nu)}^M$ is recoverable from D in $H_{\sigma(\nu)}^M$. Hence $W \in H_{\sigma(\nu)}^M \subset N$ is recoverable from W in $H_{\sigma(\nu)}^M$.

QED(Lemma 4.1.7)

We note that:

Lemma 4.1.8. *Let $a \in P^n, \nu \in a, M_a^\nu \in M$. Then $\nu \notin \mathrm{rng}(\sigma_a^\nu)$.*

Proof. Suppose not. Then $a \in \mathrm{rng}(\sigma)$. Let A be $\Sigma_1(M)$ such that $A \cap \rho^n \notin M$. Let $\overline{A}$ be $\Sigma_1(M_a^\nu)$ in $\overline{a} = \sigma^{-1}(a)$ by the same definition. Then:

$$A \cap \rho^n = \overline{A} \cap \rho^n \in \underline{\Sigma}^*(M_a^\nu) \subset M.$$

Contradiction!

QED (Lemma 4.1.8)

But then:

Lemma 4.1.9. *Let $q \in P_M^n$. Let a be an upper segment of q which is solid for M. Let $\pi : M \longrightarrow_{\Sigma^*} N$ such that $\pi(q) \in P_N^n$. Then $\pi(a)$ is solid for N.*

Proof. Let $\nu \in a, W = M_a^\nu, \sigma = \sigma_a^\nu$. Set:

$$a' = \pi(a), \nu' = \pi(\nu), W' = N_{a'}^{\nu'}, \sigma' = \sigma_{a'}^{\nu'}.$$

We must show that $W' \in N$. We first show:

(1) $\nu' \notin \mathrm{rng}(\sigma')$.

Proof. Suppose not. Let $\rho^{i+1} \leq \nu < \rho^i$ in M. Then $\rho^{i+1} \leq \nu' < \rho^i$ in N. Then in N we have: $\nu' = F'(\xi, b')$ where $\xi < \nu', b' = a' \setminus (\nu' + 1)$, and F' is a good $\Sigma_1^{(i)}(N)$ function.

Let $\dot{F}$ be a good definition for F'. Then in N the $\Sigma_1^{(i)}$ statement holds:

$$\bigvee \xi' < \nu'(\nu' = \dot{F}(\xi', b')).$$

But then in M we have:

$$\bigvee \xi' < \nu(\nu = \dot{F}(\xi', b))$$

where $b = a \setminus (\nu + 1)$. Hence $\nu \in \mathrm{rng}(\sigma)$. Contradiction!

QED(1)

Now set: $W'' = \pi(W)$. In M we have:

$$\bigwedge \xi < \nu(M \models \varphi(\xi, b) \longrightarrow W \models \varphi(\xi, b))$$

for $\Sigma_1^{(i)}$ formulas φ. But this is a $\Pi_1^{(i)}$ statement in M about ν, b, W. Hence the corresponding statement holds in N:

$$\bigwedge \xi < \nu'(N \models \varphi(\xi, b') \longrightarrow W' \models \varphi(\xi, b'))$$

Hence W'' is a generalized witness for $\nu' \in a'$. Hence $W = N_a^{\nu'} \in N$.

QED(Lemma 4.1.9)

As a corollary we then have:

Lemma 4.1.10. *Let M be n-solid. Let $\pi : M \longrightarrow_{\Sigma^*} N$ such that $\pi(p_M^n) \in P_N^n$. Then N is n-solid and $\pi(P_M^n) = P_N^n$.*

Proof. Let $a = p_M^n$. Then $a' = \pi(a) \in P_N^n$ is solid for N by the previous lemma. Moreover, $a' \cap \rho_N^n = \varnothing$. Hence $a' = p_N^n$.

QED(Lemma 4.1.10)

This holds in particular if $\rho^n = \rho^\omega$ in M. But if $\pi : M \longrightarrow N$ is strongly Σ^*-preserving in the sense of §3.2.5, then $\rho^n = \rho^\omega$ in N and $\pi"(P_M^n) \subset P_M^n$. Hence:

Lemma 4.1.11. *Let M be solid. Let $\pi : M \longrightarrow N$ be strongly Σ^*-preserving. Then N is solid and $\pi(p^i_M) = p^i_N$ for $i < \omega$.*

QED(Lemma 4.1.11)

Corollary 4.1.12. *Let $I = \langle\langle M_i\rangle, \langle\nu_i\rangle, \langle\pi_{ij}\rangle, T\rangle$ be a normal iteration. Let $h = T(i+1)$ where $i+1 \leq_T j$. Assume that $(i+1, j]_T$ has no drop. If M^*_j is solid, then M_j is solid and $\pi_{h,j}(p^n_{M^*_i}) = p^n_{M_j}$ for $n < \omega_1$.*

Proof. $\pi_{h,j}$ is strongly Σ^*-preserving.

We now define:

Definition 4.1.5. Let M be acceptable. M is a *core* iff it is sound and solid. M is the *core of* N with *core map* iff M is a core and $\pi : M \longrightarrow_{\Sigma^*} N$ with $\pi(p^*_M) = p^*_N$ and $\pi \restriction \rho^\omega_M = \mathrm{id}$.

Clearly M can have at most one core and one core map.

Definition 4.1.6. Let $M = \langle J^E_\alpha, E_\alpha\rangle$ be a premouse. M is *presolid* iff $M||\xi$ is solid for all limit $\eta < \alpha$.

Lemma 4.1.13. *Let M be acceptable. The property "M is presolid" is uniformly $\Pi_1(M)$. Hence, if $\pi : M \longrightarrow_{\Sigma_1} N$, then N is presolid.*

Proof. The function:

$$\langle \Vdash_{M||\xi} : \xi \text{ is a limit ordinal}\rangle$$

is uniformly $\Sigma_1(M)$. But for each $i < \omega$ there is a first order statement φ_i which says that M is "solid above ρ^i", i.e.

$$M^\nu_{P^i_M} \in M \text{ for all } \nu \in p^i_M.$$

The map $i \mapsto \varphi_i$ is recursive. But M is presolid if and only if:

$$\bigwedge \xi \in M \bigwedge i(\xi \text{ is a limit } \longrightarrow \Vdash_{M||\xi} \varphi_i)$$

QED(Lemma 4.1.13)

We shall prove that every fully iterable premouse is solid. But if M is fully iterable, then so is every $M||\eta$. Hence M is presolid.

The *comparison Lemma* (Lemma 3.5.1) tells us that, if we coiterate two premice M^0, M^1 of cardinality less than a regular cardinal θ, then the coiteration will terminate below θ. If both mice are $\theta+1$-iterable, and we use successful strategies, then termination will not occur until we reach $i < \theta$ such that $M_i^0 \triangleleft M_i^1$ or $M_i^1 \triangleleft M_i^0$ ($M \triangleleft M'$ is defined as meaning $\bigvee \xi \leq \mathrm{On}_{M'}, M = M'||\xi$.) If $M_i^0 \triangleleft M_i^1$, we take this as making a statement about the original pair M^0, M^1 to the effect that M^1 contains at least as much information as M_0. However, we may have truncated on the man branch to M_i^1, in which case we have "thrown away" some of the information contained in M_1. If we also truncated on the main branch to M_0, it would be hard to see why the final result tell us anything about the original pair. We now show that, if M^0 and M^1 are both presolid, then this eventually cannot occur: If there is a truncation on the main branch of the M^1-side, there is no such truncation on the other side. (Hence no information was lost in passing from M^0 to M_i^0.) Moreover, we then have $M_i^0 \triangleleft M_1^1$.

Lemma 4.1.14. *Let $\theta > \omega$ be regular. Let $M^0, M^1 \in H_\theta$ be presolid premice which are normally $\theta+1$-iterable. Let:*

$$I^h = \langle \langle M_i^h \rangle, \langle \nu_i^h \rangle, \langle \pi_{ij}^h \rangle, T^h \rangle \ (h = 0, 1)$$

be the coiteration of length $i+1 < \theta$ by successful $\theta+1$ strategies S^0, S^1 (Hence $M_i^0 \triangleleft M_i^1$ or $M_i^1 \triangleleft M_i^0$.) Suppose that there is a truncation on the main branch of I^1. Then:

(a) $M_i^0 \triangleleft M_i^1$.

(b) *There is no truncation on the main branch of I^0.*

Proof. We first prove (a). Let $l_1 + 1 \leq i$ be the least point of truncation in T^1"$\{i\}$. Let $h_1 = T(l_1 + 1)$. Let $Q^1 = M_{l_1}^{1*}$. Then Q^1 is sound and solid. Let $\pi^1 = \pi_{h_1,i}^1$. By Lemma 4.1.12, M_i' is solid and $\pi^1(p_{Q^1}) = p_{M_i^1}$. Hence $Q^1 = \mathrm{core}(M_i^1)$ and π^1 is the core map. But $\pi^1 \neq \mathrm{id}$. Hence M_i^1 is not sound. If $M_i^0 \not\trianglelefteq M_i^1$, we would have: $M_i^1 = M_i^0||\eta$ for an $\eta \in M_i^0$. But $M_i^0||\eta$ is sound. Contradiction! This proves (a).

We now prove (b). Suppose not. Let $l_0 + 1$ be the last truncation point in $T^{00}\{i\}$. Let $h_0 = T^0(l_0 + 1)$. Let Q^0, π^0 be defined as before. Then $Q^0 = \mathrm{core}(M_i^0)$ and $\pi^0 \neq \mathrm{id}$ is the core map. Hence M_i^0 is not sound. Hence, as before, we have: $M_i^1 \triangleleft M_i^0$. Hence $M_i^0 = M_i^1$ and $Q = Q^0 = Q^1$ is the core of $M_i = M_i^0 = M_i^1$ with core map $\pi = \pi^0 = \pi^1$. Set:

$$F^h =: E_{\nu_{l_h}}^{M_{l_h}^h} \ (h = 0, 1).$$

It follows easily that there is κ defined by:

$$\kappa = \kappa_{l_h}^h = \mathrm{crit}(F^h) = \mathrm{crit}(\pi) \ (h = 0, 1)$$

Thus $\mathbb{P}(\kappa_\alpha) \cap M^h_{l_h} = \mathbb{P}(\kappa) \cap Q$. But:

$$\alpha \in F^h[X] \longleftrightarrow \alpha \in \pi(X)$$

for $X \in \mathbb{P}(\kappa) \cap Q, \alpha < \lambda_h = F^h(\kappa)$. Hence $l_0 \neq l_1$, since otherwise $\lambda_0 = \lambda_1$ and $F^0 = F^1$. Contradiction!, since ν_{l_h} is the first point fo difference. Now let e.g. $l_0 < l_1$. Then ν_{l_0} is regular in M^0_j for $l_0 < j \leq i$. But then it is regular in $M^1_{l_1}||\nu_{l_1}$, since $M^1_{l_1}||\nu_{l_1} = M^0_{l_1}||\nu_{l_1}$ and $\nu_{l_1} > \nu_{l_0}$.

But $F^0 = F^1|\lambda_{l_0}$ is a full extender. Hence $F^0 \in M_{l_1}||\lambda_{l_1}$ by the initial segment condition. But then $\tilde{\pi} \in M_{l_1}||\lambda_l$, where $\tilde{\pi}$ is the canonical extension of F^0. But $\tilde{\pi}$ maps $\overline{\sigma} = \kappa^{+Q}$ cofinally to ν_{l_0}. Hence ν_{l_0} is not regular in $M^1_{l_1}||\nu_{l_1}$. Contradiction!

Lemma 4.1.14

We remark in passing that:

Lemma 4.1.15. *Each J_α is solid.*

Proof. Suppose not. Let $M = J_\alpha, \nu \in a = p^i_M$, where $\rho^{i+1} \leq \nu < \rho^i$ in M. Let $M^\nu_a = J_{\overline{\alpha}}$ and let $\pi : J_{\overline{\alpha}} \longrightarrow J_\alpha$ be the canonical embedding. Then $\overline{\alpha} = \alpha$, since $J_{\overline{\alpha}} \notin J_\alpha$. Let $b = a\setminus(\nu+1), \overline{b} = \overline{\pi}^{-1}(b)$. Set $\overline{a} = (a \cap \nu) \cup \overline{b}$. Then $\overline{a} \in P^i$ in M_i. But $\pi"(\overline{a}) = (a \cap \nu) \cup b <_* a$ where π is monotone. Hence $\overline{a} <_* a$. Hence $\overline{a} \notin P^i$ by the $<_*$-minimality of a. Contradiction!

QED(Lemma 4.1.15)

By virtually the same proof:

Lemma 4.1.16. *Let $M = J^A_\alpha$ be a constructible extension of J^A_β (i.e. $A \subset J^A_\beta$, where $\beta \leq \alpha$). Let $\rho^\omega_M \geq \beta$. Then M is solid.*

The solidity Theorem

We intend to prove:

Theorem 4.1.17. *Let M be a premouse which is fully ω_1+1-iterable. Then M is solid.*

A consequence of this is:

Corollary 4.1.18. *Let M be a 1-small premouse which is normally ω_1+1-iterable. Then M is solid.*

Proof. If M is restrained, then it has the minimal uniqueness property and is therefore fully ω_1+1-iterable by Theorem 3.6.1 amd Theorem 3.6.2. But if M is not restrained it is solid by Lemma 4.1.16.

QED(Corollary 4.1.18)

It will take a long time for us to prove Theorem 4.1.17. A first step is to notice that, if $M \in H_\kappa$, where $\kappa > \omega_1$ is regular and $\pi : H \prec H_\kappa$, with $\pi(\overline{M}) = M$, where H is transitive and countable, then M is solid iff $\overline{M}$ is solid, by absoluteness. Moreover, $\overline{M}$ is fully ω_1+1-iterable by Lemma 3.5.7. Hence it suffices to prove our Theorem under the assumption: M is countable. This assumption will turn out to be very useful, since we will employ the Neeman-Steel Lemma. It clearly suffices to prove:

(*) If M is presolid, then it is solid.

To see this, let M be unsolid and let η be least such that $M||\eta$ is not solid. Then $M||\eta$ is also fully ω_1+1-iterable and ν is also presolid. Hence $M||\eta$ is solid. Contradiction!

Now let N be presolid but not solid. Then there is a least $\lambda \in p_N^*$ such that $N_a^\lambda \notin N$, where $a = p_N^*$. Set: $M = N_a^\lambda$ and let $\sigma : M \longrightarrow_{\Sigma_1^{(n)}} N$, $\sigma \restriction \lambda = \mathrm{id}$ where $\rho_N^{n+1} \leq \lambda < \rho_N^n$ and $a\smallsetminus(\lambda+1) \in \mathrm{rng}(\sigma)$. We would like to show: $M \in N$, thus getting a contradiction. How can we do this? A natural approach is to coiterate M with N. Let $\langle I^0, I^1\rangle$ be the coiteration, I^0 being the iteration of M. If we are lucky, it might turn out that $M_\mu \in N_\mu$, where μ is the terminal point of the coiteration. If we are ever luckier, it may turn out that no point below λ was moved in pairing from M to M_μ -i.e. $\mathrm{crit}(\pi^0_{0,\mu}) \geq \lambda$. In this case it is easy to recover M from M_μ, so we have: $M \in N_\mu$, and there is some hope that $M \in N$. There are many "ifs" in this scenario, the most problematical being the assumption that $\mathrm{crit}(\pi^0_{0,\mu}) \geq \lambda$. In an attempt to remedy this, we could instead do a "phalanx" iteration, iterating the pair $\langle N, M\rangle$ against M. If, at some $i < \mu$, we have $F = E_{\nu_i}^{M_i^0} \neq \varnothing$, we ask whether $\kappa_i^0 < \lambda$. If so we apply F to N. Otherwise we apply it in the usual way to M_h, where h is least such that $\kappa_i^0 < \lambda_h$. For the sake of simplicity we take: $N = M_0^0, M = M_1^0$. ν_i is only defined for $i \geq 1$. The tree of I^0 is then "double rooted", the two roots being 0 and 1. (In the normal iteration of a premouse, 0 is the single root, lying below every $i \geq 0$). Here, $i < \mu$ will be above 0 or 1, but not both.

If we are lucky it will turns out the final point μ lies above 1 in T^0. This will then ensure that $\mathrm{crit}(\pi^0_{0,\mu}) \geq \lambda$. It turns out that this -still improbable seeming- approach works. It is due to John Steel.

In the following section we develop the theory of Phalanxes.

4.2 Phalanx Iteration

In this section we develop the technical tools which we shall use in proving that fully iterable mice are solid. Our main concern in this book is with one small mice, which are known to be of type 1, if active. We shall therefore restrict ourselves here to structures which are of type 1 or 2. When we use the term "mouse" or "premouse", we mean a premouse M such that neither it nor any of its segments $M||\eta$ are of type 3.

We have hitherto used the word "iteration" to refer to the iteration of a single premouse M. Occasionally, however, we shall iterate not a single premouse, but rather an array of premice called a *phalanx*. We define:

By a *phalanx* of length $\eta + 1$ we mean:

$$\mathbb{M} = \langle \langle M_i : i \leq \eta \rangle, \langle \lambda_i : i < \eta \rangle \rangle$$

such that:

(a) M_i is a premouse $(i \leq \eta)$

(b) $\lambda_i \in M_i$ and $J^{E^{M_i}}_{\lambda_i} = J^{E^{M_j}}_{\lambda_i}$, $(i < j \leq \eta)$

(c) $\lambda_i < \lambda_j$ $(i < j < \eta)$

(d) $\lambda_i > \omega$ is a cardinal in M_j $(i < j \leq \eta)$.

A *normal iteration* of the phalanx $\mathbb{M}$ has the form

$$I = \langle \langle M_i : i < \mu \rangle, \langle \nu_i : i + 1 \in (\eta, \mu) \rangle, \langle \pi_{i,j} : i \leq_T j \rangle, T \rangle$$

where $\mu > \eta$ is the *length* of I. $\mathbb{M} = I|\eta + 1$ is the first segment of the iteration. Each $i \leq \eta$ is a minimal point in the tree T. As usual, η_i is chosen such that $\lambda_h < \lambda_i$ for $h < i$. If h is minimal such that $\kappa_i < \lambda_h$ then $h = T(i+1)$ and $E^{M_i}_{\nu_i}$ is applied to an apropiately defined $M^*_i = M_h||\gamma$. But here a problem arises. The natural definition of M^*_i is:

> $M^*_i = M_h||\gamma$, where $\gamma \leq \mathrm{On}_{M_h}$ is maximal such that $\tau_i < \gamma$ is a cardinal in $M_h||\gamma$.

But is there such a γ? If λ_h is a limit cardinal in M_i, then $\tau_i < \lambda_h$ and hence λ_h is such a γ. For $i < \eta$ we have left the possibility open, however, that λ_h is a successor cardinal in M_i. We could then have: $\tau_i = \lambda_h$. In this case κ_i is the largest cardinal in $J_{\lambda_i}^{E^{M_h}}$. If $E_{\lambda_h} \neq \varnothing$ in M_h, it follows that $\rho^1_{M_h||\lambda_h} \leq \kappa_i < \tau_i$. Hence there is no γ with the desired property and M_i^* is undefined.

In practice, phalanxes are either defined with restrictions which prevent this eventuality, or -in the worst case- a more imaginative definition of M_i^* is applied. If $h = T(i+1)$ and M_i^* is given, then $M_{i+1}, T_{h,i+1}$ are, as usual, defined by:

$$\pi_{h,i+1} : M_i^* \longrightarrow^{(n)}_{E_{\nu_i}} M_{i+1},$$

where $n \leq \omega$ is maximal such that $\kappa_i < \rho^n_{M_i^*}$. In iterations of a single premouse, we were able to show that E_{ν_i} is always close to M_i^*, but there is no reason to expect this in arbitrary phalanx iterations.

We will not attempt to present a general theory of phalanxes, since in this section we use only phalanxes of length 2. We write $\langle N, M, \lambda\rangle$ as an abbreviation for the phalanx $\mathbb{M}$ of length 2 with $M_0 = N, M_1 = M$, and $\lambda_0 = \lambda$. We define:

Definition 4.2.1. The phalanx $\langle N, M, \lambda\rangle$ is *witnessed* (or verified) by σ iff the following hold:

(a) $\sigma : M \longrightarrow_{\Sigma_0^{(n)}} N$ for all $n < \omega$ such that $\lambda < \rho^n_M$

(b) $\lambda = \mathrm{crit}(\sigma)$

(c) σ is cardinal preserving and regularity preserving, i.e. if τ is a cardinal (regular) in M then $\sigma(\tau)$ is cardinal (regular) in N.

Note. (c) is superfluous if σ is Σ_1-preserving, since being a cardinal or regular is a Π_1 property.

Lemma 4.2.1. *Let $\langle N, M, \lambda\rangle$ be witnessed by σ. Then the following hold:*

(1) *Let $\alpha \in M$. Then α is a cardinal (regular) in M if and only if $\sigma(\alpha)$ is a cardinal (regular) in N.*

(2) *λ is regular in M.*

Proof. Suppose not. Then there is $f \in M$ such that $f : \gamma \longrightarrow \lambda$ and $\gamma < \lambda = \mathrm{lub}\, f''\gamma$. Hence $\sigma(\gamma) = \gamma$, $\sigma(f(\xi)) = f(\xi)$ for $\xi < \gamma$. Hence $\sigma(f) = f$ and $\sigma(\lambda) = \mathrm{lub}\, f''\gamma = \lambda$ in N. But $\sigma(\lambda) > \lambda$. Contradiction!

By acceptability it follows that:

(3) *If λ is a limit cardinal in M, then it is a limit cardinal in N. But if $\lambda = \gamma^+$ in M, then $\sigma(\lambda) = \gamma^+$ in N.*

Hence:

(4) $E^M_\lambda = \emptyset$.

Proof. This is trivial if λ is a limit cardinal in M. If $\lambda = \gamma^+$ in M, then $\rho^1_{M||\lambda} \leq \gamma$. Hence λ is not a cardinal in M. Contradiction! QED(4)

Hence:

(5) *Let $\kappa < \lambda$ be a cardinal in M. Set $\tau = \kappa^{+M}$. There is $\gamma \in N$ such that $\gamma > \tau$ and τ is a cardinal in $N||\gamma$.*

Proof. If $\tau < \lambda$, take $\lambda = \gamma$. Otherwise $\tau = \lambda$. But $E^N_\lambda = E^M_\lambda = \emptyset$ and λ is a cardinal in M. Hence $M||\lambda+\omega = N||\lambda+\omega = J^{E^M_\lambda}_{\lambda+\omega}$ and the assertion holds with $\gamma = \lambda+\omega$.

QED(Lemma 4.2.1)

Note. It will follow from (5) that if $h = T(i+1)$ is a normal iteration of $\langle N, M, \lambda\rangle$, then M^*_i is defined.

Following our earlier sketch, we define:

Definition 4.2.2. Let $\langle N, M, \lambda\rangle$ be a phalanx which is witnessed by σ. By a *normal iteration* of $\langle N, M, \lambda\rangle$ of length $\eta \geq 2$ we mean:

$$I = \langle\langle M_i : i < \mu\rangle, \langle \nu_i : i+1 \in (\eta, \mu)\rangle, \langle \pi_{i,j} : i \leq_T j\rangle, T\rangle$$

such that:

(a) T is a tree on η with $iTj \longrightarrow i < j$. Moreover $T``\{0\} = T``\{1\} = \varnothing$.

(b) M_i is a premouse for $i < \eta$. Moreover $M_0 = N, M_1 = N$.

(c) If $1 \leq i, i+1 < \eta$, then $M_i||\nu_i = \langle J^E_{\nu_i}, E_{\nu_i}\rangle$ with $E_{\nu_i} \neq \varnothing$. We define $\kappa_i, \tau_i, \lambda_i$ as usual. We also set: $\lambda_0 = \lambda$. We require: $\nu_i > \nu_h$ if $1 \leq h < i$ and $\lambda_h > \lambda$. (Hence $\lambda_i > \lambda_h$ for $h < i$).

(d) Let $i > 0$. Let h be least such that $h = i$ or $h < i$ and $\kappa_i < \lambda_h$. Then $h = T(i+1)$ and $J^{E^{M_h}}_{\tau_i} = J^{E^{M_i}}_{\tau_i}$.

(e) $\pi_{i,j}$ is a partial map of M_i to M_j for $i \leq_T j$. Moreover $\pi_{i,i} = \mathrm{id}$, $\pi_{i,j}\pi_{h,i} = \pi_{h,j}$.

(f) Let $h = T(i+1)$. Set: $M_i^* = M_h||\gamma$, where $\gamma \leq \mathrm{On}_{M_h}$ is maximal such that $\tau_i < \gamma$ is a cardinal in $M_h||\gamma$. (We call it a *drop point* in I if $M_i^* \neq M_k$). Then:

$$\pi_{h,i+1} : M_i^* \longrightarrow^{(n)}_{E_{\nu_i}} M_{i'+1}, \text{ where } n \leq \omega \text{ is maximal s.t.}$$
$$\lambda_h \leq \rho^n_{M_i^*} (\text{where } \lambda_0 = \lambda)$$

(g) If $i \leq_T j$ and $(i, j]_T$ has no drop point, then π_{ij} is a total function on M_i.

(h) Let $\mu < \eta$ be a limit ordinal. Then $T"\mu$ is a club in μ and contains at most finitely many drop points. Moreover, if $i < \mu$ and $(i, \mu)_T$ is drop free, then:

$$M_\mu, \langle \pi_{j,\mu} : i \leq_T j <_T \mu \rangle$$

is the transitivized direct limit of

$$\langle M_j : i \leq_T j \leq_T \mu \rangle, \langle \pi_{j,k} : i \leq_T j \leq_T k <_T \mu \rangle.$$

As usual we call M_μ, $\langle \pi_{j,\mu} : j <_T \mu \rangle$ the limit of $\langle M_i : i <_T \mu \rangle$, $\langle \pi_{j,k} : i \leq_T j \leq_T k <_T \mu \rangle$, since the missing points are given by:

$$\pi_{h,j} = \pi_{i,j}\pi_{h,i} \text{ for } h <_T i \leq_T j <_T \mu \rangle$$

This completes the definition. Note that the existence of M_i^* is guaranteed by Lemma 4.2.1(5). We define:

Definition 4.2.3. $i+1$ is an anomaly in I if $i > 0$ and $\tau_i = \lambda$ (hence $0 = T(i+1)$).

Anomalies will cause us some problems. Just as in the case of ordinary normal iterations, we can extend an iteration of length $\eta + 1$ to a *potential iteration* of length $\eta + 2$ by appointing ν_η such that:

$$E^{M_\eta}_{\nu_\eta} \neq \varnothing, : \nu_\eta > \nu_i \text{ for } i \leq i < \eta, \lambda_\eta > \lambda.$$

This determines M_η^*. In ordinary iterations we know that E_{ν_η} is close to M_η^*. In the present situation this may fail, however, if $\eta + 1$ is an anomaly. We, nonetheless, get the following analogue of Theorem 3.4.4:

Theorem 4.2.2. *Let I be a potential normal iteration of $\langle N, M, \lambda \rangle$ of length $i+1$. If $i+1$ is not an anomaly, then $E^{M_i}_{\nu_i}$ is close to M_i^*. If $i+1$ is an anomaly, then $E^{M_i}_{\nu_i,\alpha} \in N$ for $\alpha < \lambda_0$.*

We essentially repeat our earlier proof (but with one additional step). We show that if $A \subset \tau_i$ is $\underline{\Sigma}_1(M_i||\nu_i)$, then it is $\underline{\Sigma}_1(M_i^*)$ if $i+1$ is not an anomaly, and otherwise $A \in N$. Let I be a counterexample of length $i+1$ where i is chosen minimally. Let $h = T(i+1)$. Let $A \subset \tau_i$ be a counterexample. Then:

(1) $h < i$.

We then prove:

(2) $\nu_i = \mathrm{On}_{M_i}, \rho^1_{M_i} \leq \tau_i$.

The first equation is proven exactly as before. The second follows as before if $i+1$ is not an anomaly, since then $\tau_i < \lambda_h$. Now let $i+1$ be an anomaly. Assume $\rho^1_{M_i} > \tau_i$ and let $A \subset \tau_i$ be $\underline{\Sigma}(M_i)$. Then $A \in M_1$, since either $i = 1$ or $A \in J^{E^{M_i}}_{\lambda_1} = J^{E^{M_1}}_{\lambda_1}$ where λ_1 is a cardinal in M_i. Hence $A = \sigma(A) \cap \lambda \in N$. Contradiction!

QED(2)

In an extra step we then prove:

Claim. $i > 1$.

Proof. Suppose not. Then $i = 1$ and $h = 0$. Let:

$$\pi : J^E_{\tau_1} \longrightarrow J^E_{\nu_1},\ \pi' : J^{E'}_{\tau'_1} \longrightarrow J^{E'}_{\nu'_1}$$

be the extensions of M, N respectively. Then π, π' are cofinal and $\sigma\pi = \pi'\sigma$. If $\tau_1 < \lambda$ then $\sigma \restriction \tau_1 + 1 = \mathrm{id}$ and σ takes M cofinally to N. Hence σ in Σ_1-preserving. If A is $\Sigma_1(M)$ in p, then A is also $\Sigma_1(N)$ in $\sigma(p)$, where $N = M_1^*$. Contradiction!

Now let $\tau_1 = \lambda$. Then $i+1$ is an anomaly. Then σ takes ν_1, non cofinally to ν'_1, since $\pi'(\lambda) > \pi(\xi) = \sigma\pi(\xi)$ for $\xi < \lambda$. Let $\tilde{\nu} =: \sup \sigma"\nu_1$. Then:

$$\sigma : M \longrightarrow_{\Sigma_1} \tilde{M} \text{ cofinally,}$$

where $\tilde{M} = \langle J^{E'}_{\tilde{\nu}}, E'_{\nu'_1} \cap J^{E'}_{\tilde{\nu}} \rangle$. Let A' be $\Sigma_1(\tilde{M})$ in $\sigma(p)$ by the same definition as A in p. Then $A' \in N$ and $A = A' \cap \lambda \in N$. Contradiction!

QED(Claim)

(3) i is not a limit ordinal.

Proof. Suppose not. Then as before, we can pick $l <_T i$ such that $\pi_{l,i}$ is a total function on M_l and $l > h$. Hence $\pi_{l,i}$ is Σ_1-preserving. Let $M_i = \langle J^E_{\nu_i}, F \rangle$. We can also pick l big enough that $p \in \mathrm{rng}(\pi_{l,i})$, where A is $\Sigma_1(M_i)$ in p. Hence $A \in \underline{\Sigma}_1(M_l)$, where $M_l = \langle J^{\tilde{E}}_{\tilde{\nu}}, \tilde{F} \rangle$, where $\tilde{\nu} = \mathrm{On}_{M_l} \geq \nu_l$. Extend $I|l+1$ to a potential iteration I' of length $l+2$ by setting: $\nu'_l = \tilde{\nu}$. Since $l > h$, it follows easily that:

$$\kappa'_l = \kappa_i, \tau'_l = \tau_i, h = T'(l+1), M_i^* = M_l'^*.$$

By the minimality of i it follows that $A \in \Sigma_1(M_l^*)$ if $i+1$ is not an anomaly and otherwise $A \in N$. Contradiction!

QED(3)

We then let: $i = j+1, \xi = \tau(i)$. By the claim we have: $j \leq 1$.

But:

$$\pi_{\xi,i} : M_j^* \longrightarrow^{(n)}_{E^{M_i}_{\nu_j}} M_i = \langle J^E_{\nu_i}, E_{\nu_i} \rangle.$$

If $n = 0$, this map is cofinal. Hence in any case $\pi_{\xi,i}$ is Σ_1-preserving. Hence:

(4) $M_j^* = \langle J^{\overline{E}}_{\overline{\nu}}, \overline{E}_{\overline{\nu}} \rangle$ where $\overline{E}_{\overline{\nu}} \neq \varnothing$.

Hence:

(5) $\tau_i < \kappa_j$.

Proof. $\kappa_i < \lambda_h \leq \lambda_j$ where λ_j is inaccessible in M_i (since $j \geq 1$). Hence $\tau_i < \lambda_j$. Moreover, $\kappa_i, \tau_i \in \text{rng}(\pi_{\xi,i})$ by (4). But:

$$\text{rng}(\pi_{\xi,i}) \cap [\lambda_j, \lambda_j) = \varnothing.$$

QED(5)

Exactly as before we get:

(6) $\pi_{\xi,i} : M_j^* \longrightarrow_{E_{\nu_j}} M_i$ is a Σ_0 ultrapower. But then:

(7) i is not an anomaly.

Proof. Let $A \subset \tau_i$ be $\Sigma_1(M_i)$ in the parameter p. By (6) we have: $p = \pi_{\xi,i}(f)(\alpha)$, where $f \in M_j^*, \alpha < \lambda_j$.

Then:

$$A(\zeta) \longleftrightarrow \bigvee u \in M_j^* \bigvee y \in \pi_{\zeta,i}(u) A'(y, \zeta, p)$$

But then:

$$A(\zeta) \longleftrightarrow \bigvee u \in M_j^* \{\gamma < \kappa_j : \overline{A}'(y, \zeta, f(\gamma))\} \in (E_{\nu_j})_\alpha.$$

But since $j < i$ and $j+1$ is an anomaly, we have by the minimality of i that $(E_{\nu_j})_\alpha \in N$. Hence $A \in N$. Contradiction!

QED(7)

Since $j+1$ is not an anomaly, we have $(E_{\nu_j})_\alpha \in \underline{\Sigma}_1(M_j^*)$. Hence $A \in \underline{\Sigma}_1(M_j^*)$. Hence we have shown:

(8) $\mathbb{P}(\tau_i) \cap \underline{\Sigma}_1(M_i) \subset \underline{\Sigma}_1(M_j^*)$.

We know that $M_j^* = M_\xi || \overline{\nu} = \langle J^{\overline{E}}_{\overline{\nu}}, \overline{E}_{\overline{\nu}} \rangle$. Moreover, $\overline{\nu} > \nu_l$ for $l < \xi$, since $\lambda_l \leq \kappa_j < \lambda_{\overline{\xi}} < \overline{\nu}$; hence $\nu_l < \lambda_\xi < \overline{\nu}$. Thus we can extend $I|\xi+1$

to a potential iteration I' of length $\xi + 2$ by setting: $\nu'_\xi = \overline{\nu}$. Since $\tau_i < \kappa_j$, we then have: $\kappa_i = \kappa'_\xi, \tau_i = \tau'_\xi$. Hence:

$$h = T(i+1) = T'(\xi+1) \text{ and } M_i^* = (M_\xi^*)'.$$

Suppose that $i+1$ is not an anomaly in I. Then neither is $\xi+1$ in I'. By the minimality of i we conclude:

$$\mathbb{P}(\tau_i) \cap \underline{\Sigma}_1(M_\xi || \overline{\nu}) \subset \underline{\Sigma}_1(M_i^*)$$

where $M_\xi || \overline{\nu} = M_j^*$. Hence by (8):

$$\mathbb{P}(\tau_i) \cap \underline{\Sigma}_1(M_i) \subset \underline{\Sigma}_1(M_i^*).$$

Contradiction!

Now let $i+1$ be an anomaly. Then so is $\xi+1$ in I'. But then just as before:

$$\mathbb{P}(\tau_i) \cap \underline{\Sigma}_1(M_i) \subset \mathbb{P}(\tau_i) \cap \underline{\Sigma}_1(M_\xi || \overline{\nu}) \subset N.$$

Contradiction! QED(Theorem 4.2.2)

We now prove:

Lemma 4.2.3. *Let $h = T(i+1)$ in I, where I is a normal iteration of $\langle N, M, \lambda \rangle$. Then:*

$$\pi_{h,i+1} : M_i^* \longrightarrow_{\Sigma^*} M_{i+1} \textit{ strongly.}$$

Proof. If $i+1$ is not an anomaly, then $E_{\nu_i}^{M_i}$ is close to M_i^* and the result is immediate. Now let $i+1$ be an anomaly. Then $h = 0, M_i^* = N||\eta$ for an $\eta < \tau'_i = \sigma(\lambda)$, since $\tau_i = \lambda$. $\rho^\omega_{M_i^*} \leq \kappa_i$, since τ_i is not a cardinal in $N|\eta + \omega = J^{E^N}_{\eta+\omega}$. But then $\rho^\omega_{M_i^*} = \kappa_i$, since κ_i is a cardinal in N. Let $\rho^n_{M_i^*} > \kappa_i \geq \rho^{n+1}_{M_i^*}$, where $n < \omega$. Let $\pi = \pi_{h,i+1}$. Since M_{i+1} is the $\Sigma_0^{(n)}$ ultrapower of M_i^*, we know:

$$\pi\text{"}\rho^n_{M_i^*} \subset \rho^n_{M_{i+1}^*} \text{ and } \pi(\rho^j_{M_i^*}) = \rho^j_{M_{i+1}} \text{ for } j < n.$$

Since E_{ν_i} is weakly amenable, Lemma 3.2.16 gives us:

(1) $\sup \pi\text{"}\rho^n_{M_i^*} = \rho^n_{M_{i+1}}$ and π is $\Sigma_1^{(n)}$-preserving.

We now prove:

(2) Let $H =: |J^{E^{M_i}}_{\nu_i}| = |J^{E^{M_{i+1}}}_{\nu_i}|$. Then $\mathbb{P}(H) \cap \Sigma_1^{(n)}(M_{i+1}) \subset N$.

Proof. Let B be $\Sigma_1^{(n)}(M_{i+1})$ in q such that $B \subset H$. Let $q = \pi(f)(\alpha)$ where $f \in \Gamma^*(\kappa_i, M_i^*), \alpha < \lambda_i$. Let:

$$B(x) \longleftrightarrow \bigvee y \in H^n_{M_{i+1}} B'(y, x, q)$$

where B' in $\Sigma_0^{(n)}(M_{i+1})$. Let $\overline{B}'$ be $\Sigma_0^{(n)}(M_i^*)$ by the same definition. Then:

$$\begin{aligned} B(x) &\longleftrightarrow \bigvee u \in H^n_{M_i^*} \bigvee y \in \pi(u) B'(y, x, \pi(f)(\alpha)) \\ &\longleftrightarrow \bigvee u \in H^n_{M_i^*} \{\gamma < \kappa_i : \bigvee y \in u\, \overline{B}'(y, x, f(\gamma))\} \in (E^{M_i}_{\nu_i})_\alpha \end{aligned}$$

But $(E^{M_i}_{\nu_i})_\alpha \in N$. Hence $B \in N$.

QED(2)

Clearly, if $A \subset H$ is $\underline{\Sigma}^*(M_{i+1})$, then it is $\underline{\Sigma}_\omega(\langle H, B \rangle)$ where B is $\underline{\Sigma}_1^{(n)}(M_{i+1})$. Hence $A \in N$ and $\langle H, A \rangle$ is amenable, since $H = J^{E^{M_i^*}}_{\kappa_i} = J^{E^N}_{\kappa_i}$, and κ_i is regular in N. But then $\rho^\omega_{M_{i+1}} = \rho^\omega_{M_i^*} = \kappa_i$. It follows that:

(3) π is Σ^*-preserving.

Proof. By induction on j we show that if $R(\vec{x}^j, \vec{z})$ is $\Sigma_1^{(i)}(M_i^*)$ and $R'(\vec{x}^j, \vec{z})$ are $\Sigma_1^j(M_{i+1})$ by the same definition (where $\vec{z} = z_1^{h_1}, \ldots, z_m^{h_m}$ with $h_1, \ldots h_m < j$), then:

$$R(\vec{x}, \vec{z}) \longleftrightarrow R'(\pi(\vec{x}), \pi(\vec{z})).$$

For $j \leq n$ this holds by (1). Now let it hold for $j = m \geq n$. We show that it holds for $j = m + 1$. Then:

$$R(\vec{x}, \vec{z}) \longleftrightarrow H_{\vec{z}} \models \varphi[\vec{x}]$$

where φ is Σ_1 and:

$$H_{\vec{z}} = \langle H, \overline{Q}^1_{\vec{z}}, \ldots, \overline{Q}^P_{\vec{z}} \rangle$$

where $Q^l(\vec{w}, \vec{z})$ is $\Sigma_1^{(m)}(M_i^*)$ and:

$$\overline{Q}^l = \{\langle \vec{w} \rangle \in H : Q^l(\vec{w}, \vec{z})\} \text{ for } l = 1, \ldots, p.$$

Now let Q' be $\Sigma_1^{(m)}(M_{i+1})$ by the same definition and let $H'_{\vec{x}}$ be defined like $H_{\vec{x}}$ with $Q^{l'}$ in place of Q^l $(l = 1, \ldots, p)$. By the induction hypothesis we then have:

$$\begin{aligned} R(\vec{x}, \vec{z}) &\longleftrightarrow H_{\vec{z}} \models \varphi(\vec{x}) \\ &\longleftrightarrow H_{\pi(\vec{z})} \models \varphi(\vec{x}) \\ &\longleftrightarrow R'(\vec{x}, \pi(\vec{z})) \longleftrightarrow R'(\pi(\vec{x}), \pi(\vec{z})) \end{aligned}$$

since $\pi(\vec{x}) = \vec{x}$.

QED(3)

But this embedding π is also strong, since if $\rho^{m+1} = \kappa$ and A confirms $a \in P^m$ in M_i^*, then if A' is $\Sigma_{i+1}^{(m)}$ in $\pi(a)$ by the same definition, we have: $A \cap H = A' \cap H$, where $M_i^* \cap \mathbb{P}(H) = M_{i+1} \cap \mathbb{P}(H)$. Hence $A' \cap H \notin M_{i+1}$.

QED(Lemma 4.2.3)

But then:

Lemma 4.2.4. *Let $h = T(i+1)$, where $i+1 \leq_T j$ and $(i+1, j]$ has no drop point. Then:*

$$\pi_{h,j} : M_i^* \longrightarrow_{\Sigma^*} M_j \textit{ strongly.}$$

Proof. By Lemma 3.2.27 and Lemma 3.2.28.

QED(Lemma 4.2.4)

Exactly as in Corollary 4.1.12, we conclude that if M_i^* is solid and $i = j+1$, then so is M_j and $\pi(p_i^m) = p_j^m$ for $m < \omega$.

We intend to do comparison iterations in which $\langle N, M, \lambda\rangle$ is coiterated with a premouse. For this we shall again need padded iteration. Our definition of a normal iteration of $\langle N, M, \lambda\rangle$ encompassed only strict iteration, but we can easily change that:

Definition 4.2.4. Let $\langle N, M, \lambda\rangle$ be a phalanx which is witnessed by σ. By a *padded normal iteration* of $\langle N, M, \lambda\rangle$ *of length* $\mu \geq 1$ we mean:

$$I = \langle\langle M_i : i < \mu\rangle, \langle \nu_i : i \in A\rangle, \langle \pi_{i,j} : i \leq_T j\rangle, T\rangle.$$

Where:

(1) $A = \{i :<\leq i+1 < \mu\}$ is the set of *active points*.

(2) (a)-(b) of the previous definition hold. However (f), (d) require that $i \in A$. Moreover:

   (i) Let $1 \leq h < j < \mu$ such that $[h, j) \cap A = \varnothing$. Then:
   - $h <_T j, M_h = M_j, \pi_{h,j} = \mathrm{id}$.
   - $i \leq h \longrightarrow (i \leq_T h \longleftrightarrow i <_T j)$ for $i < \mu$.
   - $j \leq i \longrightarrow (j \leq_T i \longleftrightarrow h <_T i)$ for $i < \mu$.
     (In particular, if $2 \leq i+1 < \mu, i \notin A$. Then $i = T(i+1), M_i = M_{i+1}, \pi_{i,i+1} = \mathrm{id}$).

Note. 0 plays a special role, behaving like an active point in that λ_0 exists, but ν_0 does not exist.

Our previous results go through *mutatis mutandis*. We shall say more about that later.

Definition 4.2.5. Let M^0 be a premouse and $M^1 = \langle M, N, \lambda\rangle$ a phalanx iteration witnessed by σ. By a *coiteration* of M^0, M^1 of length $\mu \geq 1$ with *coindices* $\langle \nu_i : 1 \leq i < \mu\rangle$ we mean a pair $\langle I^0, I^1\rangle$ such that:

(a) $I^h = \langle\langle M_i^h\rangle, \langle \nu_i^h : i \in A^h\rangle, \langle \pi_{i,j}^h\rangle, T^h\rangle$ is a padded normal iteration of M^h $(h = 0, 1)$.

(b) $M_0^0 = M_1^0$.

(c) $\nu_i =$ the least ν such that $E_\nu^{M_i^0} \neq E_\nu^{M_i^1}$.

(d) If $E_{\nu_i}^{M_i^n} \neq \varnothing$, then $i \in A^h$ and $\nu_i^h = \nu_j$. Otherwise $i \notin A_i^h$.

Note. We always have $M_0^0 = M_1^0$ whereas: $M_0^1 = N, M_1^1 = M$.

Definition 4.2.6. Let $M^0, M^1 \in H_\kappa$, where $\kappa > \omega$ is regular. Let S^h be a successful iteration strategy for M^h $(h = 0, 1)$. The $\langle S^0, S^1\rangle$*-coiteration of* length $\mu \leq \kappa + 1$ with coindices $\langle \nu_i : 1 \leq i < \mu\rangle$ is the coiteration $\langle I^0, I^1\rangle$ such that:

- I^h is S^h-conforming.
- Either $\mu = \kappa + 1$ or $\mu = i + 1 < \kappa$ and ν_i does not exist (i.e. $M_1^0 \lhd M_i^1$ or $M_0^1 \lhd M_i^0$).

  Note that $\lhd$ was defined by:

$$P \lhd Q \longleftrightarrow P = Q || \operatorname{On}_P$$

We leave it to the reader to show that the coiteration exists. This is spelled out in §3.5 for coiteration of premice. We obtain the following analogue of Lemma 3.5.1:

Lemma 4.2.5. *The coiteration of $M : M^1$ terminates below κ_1.*

The proof is virtually unchanged. We leave the details to the reader. Using Lemma 4.2.4, we get the following analogue of Lemma 4.1.14:

Lemma 4.2.6. *Let N, M^0 be presolid. (Hence M^1 is presolid). Let $\langle I^0, I^1\rangle$ be the coiteration of M^0, M^1 terminating at $j < \kappa$. Suppose there is a drop on the main branch of I^h. Then there is no drop on the main branch of I^{i-h}. Moreover, $M_i^{i-h} \lhd M_i^h$.*

The proof is virtually the same.

At the end of §4.1 we sketched an approach to proving that fully iterable mice are solid. The basic idea was to coiterate $\langle N, M, \lambda\rangle$ with N, where N is fully iterable and σ witnesses $\langle N, M, \lambda\rangle$. In order to do this, we must know that $\langle N, M, \lambda\rangle$ is normally iterable. (The notions "iteration strategy", "successful iteration strategy" and "iterability" are defined in the obvious way for phalanxes $\langle N, M, \lambda\rangle$. We leave this to the reader.) We prove:

Lemma 4.2.7. *If $\langle N, M, \lambda\rangle$ is witnessed by σ and N is normally iterable, then $\langle N, M, \lambda\rangle$ is normally iterable.*

For the sake of simplicity we shall first prove this under a *special assumption*, which eliminates the possibility of anomalies:

$$\text{(SA) } \lambda \text{ is a limit cardinal in } M.$$

Later we shall prove it without SA.

In §3.4.5 we showed that if $\sigma : M \longrightarrow_{\Sigma^*} N$ and N is normally iterable, then M is normally iterable. Given a successful iteration strategy for N, we defined a successful strategy for M, based on the principle of *copying* the iteration of M onto N. In this case, we "copy" an iteration of $\langle N, M, \lambda\rangle$ onto an iteration of N. It suffices to prove it for strict iterations. Let

$$I = \langle\langle M_i\rangle, \langle\nu_i\rangle, \langle\pi_{ij}\rangle, T\rangle$$

be a strict normal iteration of $\langle N, M, \sigma\rangle$. Its copy will be an iteration of N:

$$I' = \langle\langle N_i\rangle, \langle\nu'_i\rangle, \langle\pi'_{ij}\rangle, T'\rangle$$

of the same length. We will have $N_0 = N_1 = N$. (Thus I' is a padded iteration, even if I is not). There will be *copying maps* $\sigma_i (i < \mathrm{lh}(I))$ with:

$$\sigma_i : M_i \longrightarrow N_i, \sigma_0 = \mathrm{id} \restriction N, \sigma_1 = \sigma.$$

We shall have $\nu'_i \cong \sigma_i(\nu_i)$ for $1 \leq i$. The tree T was "double rooted" with 0, 1 as its two initial points, T', on the other hand, has the sole initial point 0. We can define T' from T by:

$$iT'j \longleftrightarrow (iTj \ \vee \ i < 2 \leq j)$$

In I each point $i < \mu$ has a unique origin $h \in \{0,1\}$ such that $h \leq_T i$. Denote this by: $\mathrm{or}(i)$. Using the function or we can define T from T' by:

$$iTj \longleftrightarrow (iTj \wedge \mathrm{or}(i) = \mathrm{or}(j))$$

Thus, each infinite branch b' in I' uniquely determines an infinite branch b in I defined by:

$$b = \bigcup_{i \in b' \setminus 2} \{\mathrm{or}(i), i\}$$

However, we cannot expect the copying map to always be Σ^*-preserving, since $\sigma_1 = \sigma$ is assumed to be $\Sigma_0^{(n)}$-preserving only for $\rho_M^n > \lambda$. In this connection it is useful to define:

$$\mathrm{depth}(M, \lambda) =: \text{ the maximal } n \leq \omega \text{ s.t. } \rho_M^n > \lambda.$$

Modifying our definition of "copy" in §3.4.5 appropiately we now define:

Definition 4.2.7. Let $\langle N, M, \lambda \rangle$ be witnessed by σ. Let

$$I = \langle \langle M_i \rangle, \langle \nu_i \rangle, \langle \pi_{ij} \rangle, T \rangle$$

be a normal iteration of $\langle N, M, \lambda \rangle$ of length μ. Let:

$$I' = \langle \langle N_i \rangle, \langle \nu_i' \rangle, \langle \pi_{ij}' \rangle, T' \rangle$$

be a normal iteration of N of the same length. I' is a *copy of I onto N* with *copying maps* $\sigma_i (i < \mu)$ iff the following hold:

(a) $\sigma_i : M_i \longrightarrow_{\Sigma_*} N_i, \sigma_0 = \mathrm{id} \restriction N, \sigma_1 = \sigma, N_0 = N_1 = N$.

(b) $iT'j \longleftrightarrow (iTj \vee i < 2 \leq j)$

(c) $\sigma_i \restriction \lambda_h = \sigma_h \restriction \lambda_h$ for $h \leq i < \mu$

(d) $\sigma_i \pi_{hi} = \pi_{hi}' \sigma_h$ for $i \leq_T h$.

(e) $\nu_i' \cong \sigma_i(\nu_i)$

(f) Let $1 \leq_T i$. If $(1, i]_T$ has no drop point in I, then σ_i is $\Sigma_0^{(n)}$-preserving for all n such that $\lambda \leq \rho_M^n$. If $(1, i]_T$ has a drop point in I. Then σ_i is Σ^*-preserving.

(g) If $0 \leq_T i$ then σ_i is Σ^*-preserving.

Note: $N_0 = N_1$, since $0 \notin A$.

Our notion of copy is very close to that defined in §3.4.5. The main difference is that σ_i need not always be Σ^*-preserving. Nonetheless we can imitate the theory developed in §3.4.5. Lemma 3.4.14 holds literally as before. In interpreting the statement, however, we must keep in mind that if $i \in A$ and $T(i+1) = 0$, then $T'(i+1) = 1$. In this case $\tau_i < \lambda$ is a cardinal in N. Hence $M_i^* = N$. Moreover $\tau_i' = \sigma(\tau_i) = \tau_i$. Hence τ_i' is a cardinal in $N^* = N$ and $N_i^* = N$. In all other cases $T'(i+1) = T(i+1)$. Clearly $\pi'_{0j} = \pi'_{ij}$ for all $j \geq 1$. Lemma 3.4.14 then becomes:

Lemma 4.2.8. *Let $I, I', \langle \sigma_i : i < \mu \rangle$ be as in the above definition. Let $h = T(i+1)$. Then:*

(i) *If $i+1$ is a drop point in I, then it is a drop point in I' and $N_i^* = \sigma_h(M_i^*)$.*

(ii) *If $i+1$ is not a drop point in I, then it is not a drop point in I' and $N_i^* = N_h$.*

(iii) *If $F = E_{\nu_i}^{M_i}, F' = E_{\nu_i'}^{N_i}$. Then:*

$$\langle \sigma_h \upharpoonright M_i^*, \sigma_i \upharpoonright \lambda_i \rangle : \langle M_i^*, F \rangle \longrightarrow \langle N_i^*, F' \rangle$$

(iv) $\sigma_{i+1}(\pi_{h,i+1}(f)(\alpha)) = \pi'_{h,i+1}\sigma_h(f)(\sigma_i(\alpha))$ *for $f \in \Gamma^*(\kappa_i, M_i^*), \alpha < \lambda_i$.*

(v) $\sigma_j(\nu_i) \cong \nu_i'$ *for $j > i$.*

(vi) σ_i *is cardinal preserving.*

Note. In the general case, where anomalies can occur, Lemma 3.4.14 will not translate as easily.

Proof. In §3.4.5 we proved this under the assumption that each σ_i is Σ^*-preserving. We must now show that the weaker degree of preservation which we have posited suffices. The proof of (i)-(ii) are virtually unchanged. We now show that Σ_0-preservation is sufficient to prove (iii). Set: $\overline{M} = M_i||\nu_i, \overline{N} = N_i||\nu_i'$. Then $\sigma_i \upharpoonright \overline{M}$ is a Σ_0 preserving map to $\overline{N}$. Let $\alpha < \lambda, X \in \mathbb{P}(\kappa_i) \cap \overline{M}$. The statement $\alpha \in F(X)$ is uniformly $\Sigma_1(\overline{M})$ in α, X. But it is also $\Pi_1(\overline{M})$ since:

$$\alpha \in F(X) \longleftrightarrow \alpha \notin F(\kappa_i \setminus X)$$

Hence:

$$\alpha \in F(X) \longleftrightarrow \sigma(\alpha) \in F'(\sigma(X))$$

by Σ_0-preservation. Finally we note that $\sigma_i \upharpoonright (M_i \upharpoonright \lambda_i)$ embeds $M_i||\lambda_i$ elementarily into $\sigma_i(M_i||\lambda_i) = N_i||\lambda_i'$. Hence:

$$\sigma_i(\prec \vec{\alpha} \succ) = \prec \sigma_i(\vec{\alpha}) \succ \text{ for } \alpha_1, \ldots, \alpha_n < \lambda_i.$$

Thus all goes through as before, which proves (iii).

In our previous proof of (iv) we need that $\sigma_h \upharpoonright M_i^*$ is Σ^*-preserving. This can fail if $1 \leq_T h$ and $[1, h]_T$ has no drop point. But then σ_h is $\Sigma_0^{(n)}$-preserving for $\lambda < \rho^M$ in M, where $\lambda \leq \kappa_i$. Hence the preservation is sufficient. Finally, (v) is proven exactly as before.

(vi) is clear if σ_i is Σ_1-preserving. If not, then $1 \leq i$ and $(1, i]$ has no drop. Hence $\pi_{1,i}$ is cofinal, since only Σ_0-ultraproducts were involved. If α is a cardinal in M_i, then $\alpha \leq \beta$ for a β which is a cardinal in M. By acceptability it suffices to note that $\sigma_i \pi_{1i}(\beta) = \pi'_{1i}\sigma(\beta)$ is a cardinal in N_i.

QED(Lemma 4.2.8)

Exactly as before we get the analogue of Lemma 3.4.15:

Lemma 4.2.9. *There is at most one copy I' of I induced by σ. Moreover, the copy maps are unique.*

As before we define:

Definition 4.2.8. Let $\langle N, M, \lambda\rangle$ be a phalanx witnessed by σ. $\langle I, I', \langle\sigma\rangle\rangle$ is a *duplication induced by σ* iff I is a normal iteration of $\langle N, M, \lambda\rangle$ and I' is the copy of I induced by σ with copy maps $\langle \sigma_i : i < \mu\rangle$.

We also define:

Definition 4.2.9. $\langle I, I', \langle\sigma_i : i \leq \mu\rangle\rangle$ is a *potential duplication of length $\mu + 2$ induced by σ* iff:

- $\langle I|\mu + 1, I'|\mu + 1, \langle\sigma_i : i \leq \mu\rangle\rangle$ is a duplication of length $\mu + 1$ induced by σ.
- I is a potential iteration of length $\mu + 2$.
- I' is a potential iteration of length $\mu + 2$.
- $\sigma_\mu(\nu_\mu) = \nu'_\mu$.

To say that an actual duplication of length $\mu + 2$ is the *realization* of a potential duplication means the obvious thing. If it exists, we call the potential duplication *realizable*.

Our analogue of Theorem 3.4.16 is somewhat more complex. We define:

Definition 4.2.10. i is an exceptional point ($i \in \mathrm{EX}$) iff:

$$1 \leq_T i, (1, i]_T \text{ has no drop point, and } \rho^1 \leq \lambda \text{ in } M.$$

Note. Suppose $\rho^1 \leq \lambda$ in M. For $j \in \mathrm{EX}$ we have: $\rho^1_{M_j} \leq \lambda$, as can be seen by induction on j.

Our analogue of Theorem 3.4.16 reads:

Lemma 4.2.10. *Let $\langle I, I', \langle\sigma_i\rangle\rangle$ be a potential duplication of length $i+2$, where $h = T(i+1)$. Suppose that $i+1 \notin EX$. Then:*

$$\langle \sigma_h \restriction M_i^*, \sigma_i \restriction \lambda_i \rangle : \langle M_i^*, F\rangle \longrightarrow^* \langle N_i^*, F'\rangle$$

where $F = E^{M_i}_{\nu_i}, F' = E^{N_i}_{\nu_i'}$.

Before proving this we note some of its consequences. Just as in §3.4.5 it provides exact criteria for determining whether the copying process can be carried one step further. We have the following analogue of Lemma 3.4.17:

Lemma 4.2.11. *Let $\langle I, I', \langle\sigma_i : i \leq \mu\rangle\rangle$ be a potential duplication of length $\mu+2$ (where $\mu \geq 1$). It is realizable iff N_μ^* is $*$-extendible by $E^{N_\mu}_{\nu_\mu'}$.*

Proof. If N_μ^ν is not $*$-extendable, then no realization can exist, so suppose that it is. Form the realization $\hat{I}'$ of I' by setting:

$$\pi'_{h,i+1} : N_\mu^* \longrightarrow^*_{F'} N_{\mu+1},$$

where $h = T(\mu+1), F' = E^{N_\mu}_{\nu_\mu'}$. We consider three cases:

Case 1. $\sigma_h \restriction M_\mu^*$ is Σ^*-preserving.

Bu Lemma 4.3.2 we have:

$$\langle \sigma_h \restriction M_\mu^*, \sigma_\mu \restriction \lambda_\mu \rangle \; \langle M_\mu^*, F\rangle \longrightarrow^* \langle N_\mu^*, F'\rangle,$$

where $\sigma_h \restriction M_h^*$ is Σ^*-preserving. By Lemma 3.2.23 this gives us:

$$\pi_{h,\mu+1} : M_\mu^* \longrightarrow^*_F M_{\mu+1},$$

and a unique:

$$\sigma_{\mu+1} : M_{\mu+1} \longrightarrow_{\Sigma^*} N_{\mu+1}$$

such that $\sigma_{mu+1}\pi_{h,\mu+1} = \pi'_{h,\mu+1}\sigma_h, \sigma_{\mu+1} \restriction \lambda_\mu = \sigma_\mu \restriction \lambda_\mu$.

The remaining verification are straightforward.

Case 2. Case 1 fails and $\eta + 1 \notin \mathrm{EX}$.

By Lemma 4.3.2 we again have:

$$\langle \sigma_h, \sigma_\mu \upharpoonright \lambda_\mu \rangle : \langle M_h, F \rangle \longrightarrow^* \langle N_h, F' \rangle.$$

Moreover σ_h is $\Sigma_0^{(m)}$-preserving, where $m \leq \omega$ is maximal such that $\lambda < \rho^m$ in M. Now let $n \leq \omega$ be maximal such that $\kappa_i < \rho^n$ in M_h. Then $n \leq m$, since $\lambda \leq \kappa_i$. By Lemma 3.2.19 M_h is n-extendible by F. But then it is $*$-extendible, since F is close to M_h. Set:

$$\pi_{h,\mu+1} : M_h \longrightarrow_F^* M_{\mu+1}.$$

Since σ is $\Sigma_0^{(m)}$-preserving, it follows by Lemma 3.2.19 that there is a unique:

$$\sigma_{\mu+1} : M_{\mu+1} \longrightarrow_{\Sigma_0^{(n)}} N_{mu+1},$$

such that $\sigma'_{\mu+1}\pi_{h,\mu+1} = \pi'_{h,\mu+1}\sigma_h$ and $\sigma'\lambda_\mu = \sigma_n \upharpoonright \lambda_\kappa$. But σ' is, in fact, $\Sigma_0^{(m)}$-preserving. If $n = m$, this is trivial. If $n < m$, it follows by Lemma 3.2.24. We let $\sigma_{\mu+1} = \sigma'$. The remaining verification are straightforward.

QED(Case 2)

Case 3. The above cases fail.

Then $\mu + 1 \in \mathrm{EX}$ and $\rho^1 \leq \lambda$ in M. Thus $\rho^1 \leq \lambda \leq \kappa_i$ in M_h. By Lemma 4.2.8 we have:

$$\langle \sigma_h, \sigma_\mu \upharpoonright \lambda_\mu \rangle : \langle M_h, F \rangle \longrightarrow \langle N_h, F' \rangle.$$

Hence by Lemma 3.2.19, there are π, σ' with:

$$\pi : M_h \longrightarrow_F M_{\mu+1}, \sigma' : M_{\mu+1} \longrightarrow_{\Sigma_0} N_{\mu+1}$$

such that $\sigma'\pi = \pi'_{h,\mu+1}\sigma_h$ and $\sigma' \upharpoonright \lambda_\mu = \sigma_\mu \upharpoonright \lambda_\mu$. But $M_{\mu+1}$ is the $*$-ultrapower of M_h, since $\rho^1_{M_h} \leq \kappa_i$ and F is close to M_h. We set: $\pi_{h,\mu+1} = \pi, \sigma_{\mu+1} = \sigma'$. The remaining verifications are straightforward.

QED(Lemma 4.3.3)

Our analogue of Lemma 3.4.18 reads:

Lemma 4.2.12. *Let $\langle I, I', \langle\sigma_i : i < \mu\rangle\rangle$ be a duplication of limit length μ. Let b' be a well founded cofinal branch in I'. Let $b = \bigcup_{i \in b' \setminus 2}\{\mathrm{or}(i), i\}$ be the induced cofinal branch in I. Our duplication extends to one of length $\mu + 1$ with:*

$$T\text{"}\{\mu\} = b, T\text{"}\{\mu\} = b'$$

and $\sigma_\mu \pi_{i,\mu} = \pi'_{i\mu}\sigma_i$ for $i \in b$.

The proof is left to the reader.

With these two lemmas we can prove Lemma 4.2.7:

Fix a successful normal iteration strategy for N. We construct a strategy S^* for $\langle N, M, \lambda\rangle$ as follows: Let I be a normal iteration of $\langle N, M, \lambda\rangle$ of limit length μ. If I has no S-conforming copy, then $S^*(I)$ is undefined. Otherwise, let I' be an S-conforming copy. Let $S(I') = b'$ be the cofinal well founded branch given by S. Set $S^*(I) = b$, where b is the induced branch in I. Clearly if I is S^*-conforming, then the S-conforming copy I' exists. If I is of length $\mu + 1 (\mu \geq 1)$, then by Lemma 4.3.3, if $\nu \in M_\mu, \nu > \nu_i$ for $i < \mu$, then I extends to an S^*-conforming iteration of length $\mu + 2$ with $\nu_\mu = \nu$. By Lemma 4.3.4, if I is of limit length μ, then $S^*(I)$ exists. Hence S^* is successful.

QED(Lemma 4.2.7)

We still must prove Lemma 4.3.2. This, in fact turns out to be a repetition of Lemma 3.4.16 in §3.4. As before we derive it from:

Lemma 4.2.13. *Let $\langle I, I', \langle\sigma_j\rangle\rangle$ be a potential duplication of length $i + 1$ where $h = T(i + 1)$. Suppose that $i + 1 \notin EX$. Let $A \subset \tau_i$ be $\Sigma_1(M_i||\nu_i)$ in a parameter p. Let $A' \subset \tau'_i$ be $\Sigma_1(N_i||\nu'_i)$ in $\sigma_i(p)$ by the same definition. Then A is $\Sigma_1(M^*_i)$ in a parameter q and A' is $\Sigma_1(N^*_i)$ in $\sigma_h(q)$ by the same definition.*

Proof. The proof is a virtual repetition of the proof of Lemma 3.4.20 in §3.4. As before we take $\langle I, I', \langle\sigma_j\rangle\rangle$ as being a counterexample of length $i + 1$, where i is chosen minimally for such counterexamples. The proof is exactly the same as before. The only difference is that σ_j may not be Σ^*-preserving if $j \in \mathrm{EX}$. But in the case where we need it, we will have that σ_j is $\Sigma_0^{(1)}$-preserving, which suffices.

QED(Lemma 4.3.5).

Hence Lemma 4.2.7 is proven.

However, we have only proven this on the special assumption that λ is a limit cardinal in M. We now consider the case: $\lambda = \kappa^+$ in M. This will require a radical change in the proof. Set:

$$N^* =: N||\gamma \text{ where } \gamma \text{ is maximal such that } \lambda \text{ is a cardinal in } N||\gamma.$$

Then $\lambda = \kappa^{+N^*} < \sigma(\lambda) = \kappa^{+N}$. An anomaly occurs at $i+1$ whenever $\tau_i = \lambda$. Then $0 = T(i+1)$ and $\kappa = \kappa_i$. Clearly $N^* = M_j^*$. Thus M_{i+1} is the ultraproduct of N^* by $F = E_{\nu_i}^{M_i}$ and N_{i+1} is the ultraproduct of N_i^* by $F' = E_{\nu_i}^{N_i}$. In order to define σ_{i+1}, we require:

$$\sigma(M_i^*) = N_i^*.$$

This is false however, since $\sigma_i \restriction \lambda_0 = \sigma \restriction \lambda_i$ where $\tau_i < \lambda_i$. Hence:

$$\tau_i' = \sigma_i(\tau_i) = \sigma(\tau_i) = \tau^{+N}.$$

Hence $N_i^* = N \ni \sigma(N^*)$.

The answer to this conundrum is to construct two sequences I' and $\hat{I}$. The sequence:

$$\hat{I} = \langle\langle \hat{N}_i\rangle, \langle \hat{\nu}_i : i \in A\rangle, \langle \pi_{ij} : \hat{i \leq_T} j\rangle, \hat{T}\rangle$$

will be a padded iteration of N of length μ in which many points may be inactive. The second sequence:

$$I' = \langle\langle N_i\rangle, \langle {\nu'}_i : i \in A\rangle, \langle \pi_{ij}' : i \leq_T j\rangle, T'\rangle$$

will have most of the properties it had before, but, in the presence of anomalies, it will not be an iteration . If no anomalies occurs, we will have: $I' = \hat{I}$. If $i+1$ is an anomaly, then $\pi_{0,i+1}$ will not be an ultrapower and N_i will be a proper segment of $\hat{N}_i = \hat{N}_{i+1}$. (Hence i is passive in $\hat{I}$). To see how this works, let $i+1$ be the first anomaly to occur in I, then $I'|_{i+1} = \hat{I}|_{i+1}$, but at $i+1$ we shall diverge. Under our old definition we would have taken $N_i^* = N$ and $\pi_{i,i+1}' = \pi''$, where:

$$\pi'' : N \longrightarrow_F^* N'',\ F = E_{\nu_i'}^{N_i}.$$

We instead take:

$$N_i^* = N^*,\ N_{i+1} = \pi''(N^*),\ \pi_{i,i+1} = \pi'' \restriction N^*.$$

Note that $\pi''(N^*) = \pi'(N^*)$, where π' is the extension of $\langle J_{\nu_i}^{E^{M_i}}, F\rangle$. But then N_{i+1} is a proper segment of $J_{\nu_i}^{E^{N_i}}$ hence of $N_i = \hat{N}_i$.

We can then define:

$$\sigma_{i+1} : M_{i+1} \longrightarrow N_{i+1}$$

by:

$$\sigma_{i+1}(\pi_{0,i+1}(f)(\alpha)) =: \pi'(f)(\sigma_i(\alpha))$$

for $f \in \Gamma^*(\kappa, N^*), \alpha < \lambda_i$. σ_{i+1} will then be $\Sigma_0^{(n)}-$preserving, where $n \leq \omega$ s maximal such that $\kappa < \rho^n$ in N^*. To see that this is so, let φ be a $\Sigma_0^{(n)}$ formula. Let $f_1, \ldots, f_n \in \Gamma^*(\kappa, N^*)$ and let $\alpha_1, \cdots, \alpha_n < \lambda_i$. Let:

$$x_j = \pi_{0,i+1}(f_j)(\alpha_j), y_j = \pi'(f_j)(\sigma_i(\alpha_j)) \ \ (j = 1, \ldots, n).$$

Let $X =: \{\prec \xi_1, \ldots, \xi_m \succ : N^* \models \varphi[f_1(\xi_1), \ldots, f_n(\xi_n)]\}$. Then $\sigma_i F(X) = F'(X)$, since $\sigma_i \restriction H_\lambda^M = \sigma_0 \restriction H_\lambda^M = \text{id}$. Hence:

$$\begin{aligned} M_{i+1} \models \varphi[\vec{X}] &\longleftrightarrow \prec \vec{\alpha} \succ \in F(X) \\ &\longleftrightarrow \prec \sigma_i(\vec{\alpha}) \succ \in F'(X) = \pi'(X) \\ &\longleftrightarrow \sigma(N^*) \models \varphi[\vec{y}]. \end{aligned}$$

Since we had no need to form an ultraproduct at $i+1$, we set: $\hat{N}_{i+1} = \hat{N}_i$. i is then an inactive point in $\hat{I}$ and N_{i+1} is a proper segment of $\hat{N}_{i+1}$.

We continue in this fashion: The active points in $\hat{I}$ are just the points $i > 0$ such that $i + 1 < \mu$ is not an anomaly. If i is active, we set $\hat{\nu}_i = \nu'_i$. (This does not, however, mean that $\hat{N}_i = N'_i$.) If i is any non anomalous point, we will have: $N_i = \hat{N}_i$. If $h < i$ is also non anomalous, thus $\pi'_{hi} = \hat{\pi}_{hi}$. If i is an anomaly, we will have: N_i is a proper segment of $\hat{N}_i$. If μ is a limit ordinal it then turns out that any cofinal well founded branch b' in I', which, in turn, gives us such a branch b in I. This enables us to prove iterability.

We now redo our definition of "copy" as follows:

Definition 4.2.11. Let $I = \langle\langle M_i\rangle, \langle\nu_i\rangle, \langle\pi_{ij}\rangle, T\rangle$ be a strict normal iteration of $\langle N, M, \lambda\rangle$, where $\langle N, M, \lambda\rangle$ is a phalanx witnessed by σ.

$$I' = \langle\langle M_i\rangle, \langle\nu'_i\rangle, \langle\pi'_{ij}\rangle, T'\rangle$$

is a copy of I with *copy maps* $\langle\sigma_i : i < \mu\rangle$ induced by σ if and only if the following hold:

(I) (a) T' is a tree such that $iT'j \longrightarrow i < j$.

(b) Let μ be the length of I. Then N_i is a premouse and

$$\sigma_i : M_i \longrightarrow_{\Sigma_0} N_i \text{ for } i < \mu$$

(c) $\pi'_{ij}(i \leq_T j)$ is a commutative system of partial maps from N_i to N_j.

(II) (a)-(f) of our previous definition hold. Moreover:

(g) Let $0 \leq_T j$. If $(0, i]_T$ have no anomaly, then σ_i is Σ^*-preserving.

(h) Let $h = T(i+1)$. Set:

$$N_i^* = \begin{cases} \sigma_h(M_i^*) & \text{if } M_i^* \in M_h \\ N_h & \text{if not} \end{cases}$$

Then $\pi'_{h,i+1} : N_i^* \longrightarrow_{\Sigma^*} N_{i+1}$.

(i) Let h, i be as above. If $i+1$ is not an anomaly, then:

$$\pi'_{h,i+1} : N_i^* \longrightarrow^*_{F'} N_{i+1}$$

where $F' = E^{N_i}_{\nu'_i}$.

(j) Let $i+1$ be an anomaly. (Hence $\tau_i = \lambda = \kappa^{+M}$, where $\kappa = \kappa_i$ is a cardinal in M, hence in N.)

We then have:

$$M_i^* = N^* =: N||\gamma,$$

where γ is maximal such that λ is a cardinal in $N||\gamma$. Let π be the extension of $N_i||\nu_i = \langle J^E_{\nu'}, F' \rangle$. Then:

$$N_{i+1} = \pi(N^*) \text{ and } \pi'_{0,i+1} = \pi \upharpoonright N^*.$$

Moreover, $\sigma_{i+1} : M_{i+1} \longrightarrow N_{i+1}$ is defined by:

$$\sigma_{i+1}(\pi_{0,i+1}(f)(\alpha)) = \pi'(f)(\sigma_i(\alpha))$$

where $f \in \Gamma^*(\kappa, N^*), \alpha < \lambda_i$. (Hence σ_{i+1} is $\Sigma_0^{(n)}$-preserving for $\kappa < \rho^n_{N^*}$.)

(k) Let $h \leq_T i$, where h is an anomaly. If $(h, i]_T$ has no drop point, then σ_i is $\Sigma_0^{(n)}$-preserving for $\kappa < \rho^n$ in N^*. If $(h, i]_T$ has a drop point, then σ_i is Σ^*-preserving.

(III) There is a *background iteration*:

$$\hat{I} = \langle \langle \hat{N}_i \rangle, \langle \hat{\nu}_i \rangle, \langle \hat{\pi}_{ij} \rangle, \hat{T} \rangle$$

with the properties.

(a) $\hat{I}$ is a padded normal iteration of length μ.

(b) $i < \mu$ is active in $\hat{I}$ iff $0 < i+1 < \mu$ and $i + \mu$ is not an anomaly in I. In this case: $\hat{\nu}_i = \nu'_i$.

(c) If i is not an anomaly in I, then $\hat{N}_i = N'_i$. Moreover, if $h < i$ is also not an anomaly, then:

$$h <_{\hat{T}} i \longleftrightarrow h <_{T'} i,\ \hat{\pi}_{h,i} = \pi'_{h,i} \text{ if } h <_{T'} i.$$

This completes the definition. In the special case that λ is a limit cardinal in M, we of course have: $I' = \hat{I}$ and the new definition coincides with the old one. We note some simple consequence of our definition:

Lemma 4.2.14. *The following hold:*

(1) *If $i < j < \mu$, then $\sigma_j(\lambda_i) = \lambda_i$. (Hence $\lambda'_i < \lambda'_j$ for $j + 1 < \mu$.)*

Proof. By induction on j. For $j = 0$ it is vacuously true. Now let it hold for j.

$$\sigma_{j+1}(\lambda_j) = \sigma_{j+1}\sigma_{h,i+1}(\kappa_j) = \pi'_{h,j+1}\sigma_h(\kappa_j) = \pi'_{h,j+1}(\kappa'_j) = \lambda_j.$$

(Here $\sigma_h(\kappa_j) = \sigma_j(\kappa_j) = \lambda'_j$, since $\kappa_j < \lambda_h$ and $\sigma_j||\lambda_h = \sigma_h \restriction \lambda_h$.)

For $i < j$ we then have:

$$\sigma_{j+1}(\lambda_i) = \sigma_j(\lambda'_i)(\text{since } \lambda_i < \lambda_j).$$

QED(1)

(2) *σ_i is a cardinal preserving for $i < \mu$.*

Proof. If σ_i is Σ_1-preserving, this is trivial, so suppose not. Then one of two cases hold:

Case 1. $1 \leq_T i, (1, i]_T$ has no drop, and $\rho^1 \leq \lambda$ in M.

Then $\pi_{hj} : M_h \longrightarrow_{\Sigma^*} M_j$ is cofinal for all $h \leq_T j \leq_T i_\eta$ since each of the ultrapower involved is a Σ_0-ultrapower. Hence, if α is a cardinal in M_i, then $\alpha \leq \pi_{1,i}(\beta)$ where β is a cardinal in M_1. By acceptability it suffices to show that $\sigma_i\pi_{1,i}(\beta)$ is a cardinal in N_i. But $\sigma_i\pi_{1,i}(\beta) = \pi'_{1t}\sigma(\beta)$, where σ and π'_{1i} are cardinal preserving.

Case 2. $h \leq_T i$ where h is an anomaly, $(h, i]_T$ has no drop and $\rho^1 \leq k = k_i$ in N^*.

The proof is a virtual repeat of the proof in Case 1, with $(0, i]_T$ in place of $(1, i]_T$.

QED(2)

(3) I' behaves like an iteration at limits. More precisely:

Let $\eta < \kappa$ be a limit ordinal. Let $i_0 <_T \eta$ such that $b = (i_0, \eta)_T$ is free of drops. Then

$$N_\eta, \langle \pi_{i\eta} : i \in b \rangle$$

is the direct limit of:

$$\langle N_i : i \in b \rangle, \langle \pi_{ij} : i \leq j \text{ in } b \rangle.$$

Proof. No $i \in b \cup \{\eta\}$ is an anomaly since every anomaly is a drop point. Hence:

$$N'_i = \hat{N}_i, \pi'_{i,j} = \hat{\pi}_{i,j} \text{ for } i \leq j \text{ in } b \cup \{\eta\}.$$

Since I is an iteration, the conclusion is immediate.

QED(3)

(4) Let $i < \mu$. If $i + 1$ is an anomaly, then:

(a) N_{i+1} is a proper segment of $N_i||\nu'_i$. (Hence $\nu'_{i+1} < \nu'_i$).

(b) $\rho^\omega = \lambda'_i$ in N_{i+1}.

Proof. (a) is immediate by II (i) in the definition of "copy". But $N_{i+1} = \pi(N^*)$ where π is the extension of $N_i||\nu'_i$. By definition, $N^* = N||\gamma$, where $\gamma < \sigma(\lambda) = \kappa^{+N}$ is the maximal γ such that $\tau_i = \lambda$ is a cardinal in $N||\gamma$. Hence $\rho^\omega = \kappa$ in N^*. But then $\rho^\omega = \lambda'_i$ in N_{i+1}.

QED(4)

(5) Let $i < \mu$. There is a finite n such that $i + n + 1$ is not an anomaly. (This includes the case: $i + n + 1 = \mu$.)

Proof. If not then $\nu_{i+n+1} < \nu_{i+n}$ for $n < \mu$ by(4). Contradiction!

(6) Let $i < \mu$. There is a maximal $j \leq i$ such that j is not an anomaly.

Proof. Suppose not. Then $i \neq 0$ is an anomaly and for each $j < i$ there is $j' \in (j, i)$ which is an anomaly. But then i is a limit ordinal, hence not an anomaly.

By(5) and (6) we can define:

Definition 4.2.12. Let $i < \mu$. We define:

- $l(i) =$ the maximal $j \leq i$ such that j is not an anomaly.
- $r(i)$ the least $j \geq i$ such that $j + 1$ is not an anomaly.

Definition 4.2.13. An interval $[l, r]$ in μ is called *passive* iff i is an anomaly for $l < i \leq r$. A passive interval is called *full* if it is not properly contained in another passive interval.

It is then trivial that:

(7) $[l(i), r(i)] =$ the unique full I such that $i \in I$.

(8) Let $[l, r]$ be a full passive interval. Then, for all $i \in [l, r]$:

(a) $N_l = N_i$.

(b) If $j \leq l$ and $j \leq_{\hat{T}} i$, then $j \leq_{\hat{T}} l$.

(c) If $j \geq r$ and $i \leq_{\hat{T}} j$, then $r \leq_{\hat{T}} j$.

Proof. This follows by induction on j, using the general fact about padded iterations that if j is not active, then:

- $\hat{N}_j = \hat{N}_{j+1}$
- $h \leq_{\hat{T}} j \longleftrightarrow h <_{\hat{T}} j + 1$
- $j <_{\hat{T}} h \longleftrightarrow j + 1 \leq_{\hat{T}} h$. QED(8)

(9) Let b be a branch of limit length in $\hat{I}$. There are cofinally many $i \in b$ such that i is not an anomaly.

Proof. Let $j \in b$. Pick $i \in b$ such that $i > r(j)$. Then $l(i) > r(j)$, since $r(j)+1 \leq i$ is not an anomaly. Hence $l(i) \in b$ and $l(i) > j$ is not an anomaly.

QED(9)

We define N_i^* for $i < \mu$ exactly as if I' were an iteration: Let $h = T'(i + 1)$. Then:

$$N_i^* =: N_i||\gamma \text{ where } \gamma \text{ is maximal such that } \tau_i' \text{ is a cardinal in } N_i||\gamma.$$

We then get the following version of Lemma 4.2.8.

Lemma 4.2.15. *Let I' be a copy of I induced by σ. Let $h = T(i + 1)$. If $i + 1$ is not an anomaly. Then the conclusion (i)-(vi) of Lemma 4.2.8 hold. If $i + 1$ is an anomaly, then (v), (vi) continue to hold.*

Proof. If $i + 1$ is not an anomaly, the proof are exactly as before. Now let $i + 1$ be an anomaly. (iv) is immediate by II (j) in the definition of "copy". But then (vi) follows as before.

QED(Lemma 4.2.15)

Lemma 3.3.20 is strengthened to:

Lemma 4.2.16. *I has at most one copy I'. Moreover the background iteration $\hat{I}$ is unique.*

Proof. The first part is proven exactly as before (we imagine I'' to be a second copy and show by induction on i that $I'|i = I''|i$). The second part is proven similarly, assuming $\hat{I}'$ to be a second background iteration.

QED(Lemma 4.2.16)

The concept *duplication induced by* σ is defined exactly as before. Now let:

$$D = \langle I, I', \langle \sigma_i : i \leq \eta \rangle \rangle$$

be a duplication of length $\eta + 1$. We turn this into a *potential duplication* D of length $\eta + 2$ by appointing a ν_ξ such that $\nu_\xi > \nu_i$ for $0 < i < \eta$.

By a *realization* of $\tilde{D}$ of length $\eta + 2$ by appointing a ν_η such that $\nu_\eta < \nu_i$ for $0 < i < \eta$. By a *realization* of $\tilde{D}$, we mean a duplication $\mathring{D} = \langle \mathring{I}, \mathring{J}, \langle \dot{\sigma}_i : i \leq \eta + 1 \rangle \rangle$ of length $\eta + 2$ such that $\mathring{D}|\eta + 1 = D$ and $\dot{\nu}_\eta = \nu_\eta$. It follows easily that $\tilde{D}$ has at most one realization.

Our analogue, Lemma 4.3.2, of Lemma 3.4.16 will continue to hold as stated if we enhance the definition of *exceptional point* as follows:

Definition 4.2.14. i is an *exceptional point* ($i \in EX$) iff either:

$$1 \leq_T i, (1, i]_T \text{ has no drop, and } \rho^1 \leq \lambda \text{ in } M$$

or there is an anomaly $h \leq_T i$ such that:

$$(0, i]_T \text{ has no drop, and } \rho^1 \leq \kappa \text{ in } N^*.$$

With this change Lemma 4.3.2 goes through exactly as before. As before, we derive this form Lemma 4.3.5. The proof is as before. As before the condition $i + 1 \notin EX$ guarantees that the map σ_i will always have sufficient preservation when we need it.

When we worked under the special assumption Lemma 4.3.3 was our analogue of Lemma 3.4.17. In the presence of anomalies the situation is somewhat more complex. We first note:

Lemma 4.2.17. *Let $\tilde{D} = \langle I, I', \langle \sigma_i : i \leq \eta \rangle \rangle$ be a potential duplication of length $\eta + 2$. If $\eta + 1$ is an anomaly, then $\tilde{D}$ is realizable.*

Proof. Form $N_{\eta+1}, \pi_{0,\eta+1} : N^* \longrightarrow N_{\eta+1}$ and $\sigma_{\eta+1}$ as in II(j). Set: $\tilde{N}_{\eta+1} = N_\eta$. The verification of I, II, III is straightforward.

QED(Lemma 4.2.17)

Now suppose that $\eta + 1$ is not an anomaly. Let $h = T(\eta + 1)$. Then η is an active point is any realization of $\hat{I}$, so we set: $\hat{\nu}_\eta = \nu'_\eta$. In order to realize $\tilde{D}$, we must apply $F = E^{M_\eta}_{\nu_\eta}$ to M^*_η, getting:

$$\pi_{h,\eta} : M^*_\eta \longrightarrow^*_F M_{\eta+1}.$$

Similarly we apply $F' = E^{N_\eta}_{\nu'_\eta}$ to N^*_η getting:

$$\pi'_{h,\eta} : N^*_\eta \longrightarrow^*_{F'} N_{\eta+1}.$$

We then set:

$$\sigma_{\eta+1}(\pi_{h\eta}(f)(\alpha)) = \pi'_{h\eta}\sigma_h(f)(\sigma_\eta(\alpha))$$

for $f \in \Gamma^*(\kappa_{\dot{\eta}}, M^*_{\dot{\eta}}), \alpha < \lambda_\eta$.

We must also extend $\hat{I}$. Since $\hat{\nu}_\eta = \nu_\eta$ and N_η is an initial segment of $\hat{N}_\eta$, we have:

$$F' = E^{\hat{N}_\eta}_{\hat{\nu}_\eta}.$$

Now let: $k = \hat{T}(\eta + 1)$. (k can be different from h!) III constrains us to set:

$$\hat{\pi}_{k,\eta+1} : \hat{N}^*_\eta \longrightarrow^*_F \hat{N}_{\eta+1}.$$

However, III also mandates that $\hat{N}_{\eta+1} = N_{\eta+1}$. Happily, we can prove:

Lemma 4.2.18. *Let $\tilde{D} = \langle I, I', \langle\sigma_i : i \leq \eta\rangle\rangle$ be as above, where $\eta + 1$ is not an anomaly. Then:*

(a) $N^*_\eta = \hat{N}^*_\eta$.

(b) *$\tilde{D}$ is realizable iff N^*_η is $*$-extendible by F'.*

Proof. We first prove (a). Let $h = T'(\eta + 1)$. Set:

$$l = l(h), r = r(h).$$

Then $h \in [l, r]$ where l is not an anomaly, $j + 1$ is an anomaly for $l \leq j < r$, and $r + 1$ is not an anomaly. h is least such that $\kappa'_\eta < \lambda'$ or $h = \eta$. $k = T'(\eta + 1)$ is least such that $k + 1$ is not an anomaly and $\kappa'_\eta < \lambda'_k$. Since j is not an anomaly for $l < j \leq r$, we conclude that $k = r$. Then $N_l = \hat{N}_j$ for $l \leq j \leq r$.

Case 1. $h = l$.

Then $\hat{N}_h = N_h$ and:
$$N^*_\eta = \hat{N}_\eta = N_h||\gamma$$
where γ is maximal such that τ'_η is a cardinal in $N_h||\gamma$. QED(Case 1)

Case 2. $l < h$.

Then $h = j + 1$ where $l \leq j$. N_h is a proper segment of $\hat{N}_h$. We again have: $N^*_\eta = N_h||\gamma$ where $\gamma \leq \mathrm{On}_{N_h}$ is maximal such that τ'_η is a cardinal in $N_h||\gamma$. We have $r = \hat{T}(\eta + 1)$ and $\hat{N}^*_\eta = \hat{N}_r||\hat{\gamma}$, where $\hat{\gamma} \leq \mathrm{On}_{\hat{N}_r}$ is maximal such that τ'_η is a cardinal in $\hat{N}_r||\hat{\gamma}$. But $\rho^\omega_{N_h} = \kappa_j$, where $h = j + 1$ by Lemma 4.2.14 (4). Since $\lambda'_j \leq \kappa'_\eta < \tau'_\eta < \lambda'_h$ and N_h is a proper segment of $\hat{N}_h = \hat{N}_r$, we conclude that $\hat{\gamma} \leq \mathrm{On}_{N_h}$. Hence $\gamma = \hat{\gamma}$ and $N^*_\eta = \hat{N}^*_\eta$. QED(a)

We now prove (b). If $\hat{N}^*_\eta$ is not extendable by F', then no realization can exists, so assume otherwise. This gives us $N_{\eta+1}$ and $\pi'_{h,\eta+1}$, where $\hat{N}_{\eta+1} = N_{\eta+1}$ and $\hat{\pi}_{k,\eta+1} = \pi'_{h,\eta+1}$, where $k = T'(\eta + 1)$. $\sigma_{\eta+1}$ is again defined by:

$$\sigma_{\eta+1}(\pi_{h,\eta+1}(f)(\alpha)) = \pi'_{h,\eta+1}\sigma_h(f)(\sigma_\eta(\alpha))$$

for $f \in \Gamma^*(\kappa_\eta, M^*_\eta), \alpha < \lambda_\eta$. The verification of I, II, III is much as before. However Case 2 splits into two subcases:

Case 2.1. $1 \leq_T \eta + 1$.

This is exactly as before.

Case 2.2. $0 \leq_T \eta + 1$.

Then there is $j \leq_T h$ such that j is an anomaly and $(0, \eta + 1]_T$ has no drop. Moreover, $\rho^1 > \kappa$ in N^*. Then σ_h is a $\Sigma^{(m)}_0$-preserving where $m \leq \omega$ is maximal such that $\kappa < \rho^m$ in N^*. The rest of the proof is as before.

Case 3 also splits into two subcases:

Case 3.1. $1 \leq_T \eta + 1$.

We argue as before.

Case 3.2. $0 \leq_T \eta + 1$.

Then $j \leq_t h$, where j is an anomaly and $\rho^1 \leq \kappa$ in N^*. Hence $\rho^1 \leq \kappa_h$ in M_h and we argue as before. QED(Lemma 4.2.18)

Using Lemma 4.2.14 (9) we get:

Lemma 4.2.19. *Let $D = \langle I, I', \langle \sigma_i \rangle \rangle$ be a duplication of limit length μ. Let $\hat{b}$ be a cofinal well founded branch in $\hat{I}$. Let X be the set of $i \in \hat{b}$ which are not an anomaly. Let:*

$$b' = \{j : \bigvee i \in X j <_T i\}, b = \{j : \bigvee i \in X j <_T i\}.$$

Then D has a unique extension to a $\tilde{D}$ of length $\mu + 1$ such that:

$$\hat{T}\text{"}\{\mu\} = \hat{b}, T'\text{"}\{\mu\} = b', T\text{"}\{\mu\} = b.$$

The proof is left to the reader.

Now let S be a successful normal iteration strategy for N. We define an iteration strategy S^* for $\langle N, M, \lambda \rangle$ as follows:

Let I be an iteration of $\langle N, M, \lambda \rangle$ of limit length μ. We ask whether there is a duplication $\langle I, I', \langle \sigma_0 \rangle \rangle$ induced by σ^*. If not, then $S^*(I)$ is undefined. Otherwise, we ask whether $S(\hat{I})$ is defined. If not, then $S^*(I)$ is undefined. If not, then $S^*(I)$ is undefined. If $\hat{b} = S(\hat{I})$, define b', b as above and set: $S^*(I) = b$. It is easily seen that if I is any S^*-conforming normal iteration of $\langle N, M, \lambda \rangle$, then the duplication $\langle I, I', \langle \sigma_i \rangle \rangle$ exists. Moreover $\hat{I}$ is S-conforming. In particular, if I is of limit length, then $S(I)$ is defined. Moreover, if I is of length $\eta + 1$, and $\nu > \nu_i$ for $i < \eta$, then by Lemma 4.2.18, we can extend I to an $\tilde{I}$ of length $\eta + 2$ by setting: $\nu_\eta = \nu$. Hence S is a successful iteration strategy.

This proves Lemma 4.2.7 at last!

We note however, that our strategy S^* is defined only for strict iteration of $\langle N, M, \lambda \rangle$. We can remedy this in the usual way. Let:

$$I = \langle \langle M_i \rangle, \langle \nu_i : i \in A \rangle, \langle \pi_{ij} \rangle, T \rangle$$

be a padded iteration of $\langle N, M, \lambda \rangle$, of length μ. Let h be the monotone enumeration of:

$$\{i : i = 0 \vee i \in A \vee i + 1 = \mu\}.$$

The *strict pullback* of I is then:

$$\dot{I} = \langle \langle \dot{M}_i \rangle, \langle \dot{\nu}_i \rangle, \langle \dot{\pi}_{ij} \rangle, \hat{T} \rangle$$

where:

$$\dot{M}_i = M_{h(i)}, \dot{\nu}_i = \nu_{h(i)}, \dot{\pi}_{ij} = \pi_{h(i),h(i)}$$

and:

$$i \hat{T} j \longleftrightarrow h(i) T h(j).$$

$\dot{I}$ is a strict iteration and contains all essential information about I. We extend S^* to a strategy on padded iteration as follows: Let I be a padded iteration of limit length μ. If A is cofinal in μ, we form $\dot{I}$, which is then also of limit length. We set:

$$S^*(I) = b, \text{ where } S^*(\dot{I}) = \dot{b},$$

and $b = \{i : \bigvee j(i \leq_T h(j))\}$. If A is not cofinal in μ, there is $j < \mu$ such that $A \cap [j, \mu) = \varnothing$. We set:

$$S^*(I) = \{i < \mu : iTj \vee j \leq i\}.$$

It follows that I is S^*-conforming iff $\dot{I}$ is S^*-conforming.

Since $\dot{I}$ is strict, we have $I', \hat{I}, \langle \sigma_i : i < \dot{\mu} \rangle$, (where $\dot{\mu}$ is the length of $\dot{I}$). We shall make use of this machinery in analyzing what happens when we coiterate N against $\langle N, M, \sigma \rangle$. This will yield the "simplicity lemma" stated below.

Note. We could, of course, have defined $I', \hat{I}$ and $\langle \sigma_i : i < \mu \rangle$ for arbitrary padded I, but this will not be necessary.

Building upon what we have done thus far, we prove the following "simplicity lemma", which will play a central role in our further deliberations:

Lemma 4.2.20. *Let N be a countable premouse which is presolid and fully $\omega_1{+}1$ iterable. Let $\langle N, M, \sigma \rangle$ be witnessed by σ. Set $Q^0 = N, Q^1 = \langle N, M, \sigma \rangle$. There exist successful $\omega_1 + 1$ normal iteration strategies S^0, S^1 for Q^0, Q^1 respectively such that $\langle I^0, I^1 \rangle$ is the coiteration of Q^0, Q^1 by S^0, S^1 respectively with coiteration indices ν_i, then the coiteration terminates at $\mu < \omega_1$ with:*

$$I_0 = \langle \langle Q_i \rangle, \langle \nu_i \rangle, \langle \pi^0_{ij} \rangle, T^0 \rangle$$

$$I_1 = \langle \langle M_i \rangle, \langle \nu_i \rangle, \langle \pi^1_{ij} \rangle, T^1 \rangle$$

such that:

(a) $M_\mu \triangleleft Q_\mu$.

(b) $1 \leq_{T^1} \mu$ *in* I^1.

(c) *There is no drop point* $i + 1 \leq_{T^1} \mu$ *in* I^1.

In the next section we shall use this to derive the solidity lemma, which says that all mice are solid. We shall also us eit to derive a number of other structural facts about mice.

We now prove the simplicity lemma.

Let N be countable, presolid and fully $\omega_1 + 1$.iterable. Let $\langle N, M, \lambda \rangle$ be a phalanx witnessed by σ. (Recall that this entails $\lambda \in M$ and $\lambda = \mathrm{crit}(\sigma)$. Moreover, σ is $\Sigma_0^{(n)}$-preserving whenever $\lambda < \rho_M^n$). Fix an enumeration $e = \langle e(n) : n < \omega \rangle$ of $\mathrm{On} \cap N$. Suppose that $\sigma : N \longrightarrow_{\Sigma^*} N'$. We can define a sequence $e'_i \in N' (i < \omega)$ as follows. By induction on $i < \omega$ we define:

$$\begin{aligned} e'_i = {} & \text{the least } \eta \in N' \text{ s.t. there is some } \sigma' : N \longrightarrow_{\Sigma^*} N' \\ & \text{with } \sigma'(e_h) = e'_h \text{ for } h < i \text{ and } \eta = \sigma'(e_i). \end{aligned}$$

It is not hard to see that there is exactly one $\sigma' : N \longrightarrow_{\Sigma^*} N$ such that $\sigma'(e_i) = e'_i$ for $i < \omega$. We then call σ' the *e-minimal* embedding of N into N'. The *Neeman-Steel Lemma* (Theorem 3.5.8) says that N has an *e-minimal normal iteration strategy* S with the following properties:

- S is a successul $\omega_1 + 1$ normal iteration strategy for N.
- Let N' be an iterate of N by an S-conforming iteration I. Let $\sigma : N \longrightarrow_{\Sigma^*} M \lhd N'$. Then I has no drop on its main branch $M = N'$ and the iteration map $\pi : N \longrightarrow N'$ is the e-minimal embedding.

Hence, in particular, if M is a proper segment of N' or the main branch of I has a drop, then there is no Σ^*-preserving embedding from N to M.

From now on let e be a fixed enumeration of On_N and let S be an e-minimal strategy for N. Let S^* be the induced strategy for $\langle N, M, \lambda \rangle$. Coiterate $Q_0 = N$ against $M_0 = \langle N, M, \lambda \rangle$ using the strategies S, S^* respectively. Let $\langle I^0, I^1 \rangle$ be the coiteration with:

$$I^1 = \langle \langle M_i \rangle, \langle \nu_i^0 \rangle, \langle \pi_{ij}^0 \rangle, T^0 \rangle$$

$$I^0 = \langle \langle Q_i \rangle, \langle \nu_i^1 \rangle, \langle \pi_{ij}^1 \rangle, T^1 \rangle$$

and coiteration indices $\langle \nu_i : 1 \leq i \leq \mu \rangle$ where $\mu + 1 < \omega_1$ is the length of the coiteration.

We note some facts:

(A) If N' is any S-iterate of N (i.e. the result of an S-conforming iteration), then there is no Σ^*-preserving map of N into a proper segment of N'.

(B) Call N' a *truncating* S-iterate of N iff it results from an S-conforming iteration with a truncation on its main branch. If N' is a truncating S-iterate, then there is no Σ^*-preserving embedding of N into N'.

(C) If N' is a *non truncating* S-iterate of N, then the iteration map $\pi : N \longrightarrow N'$ is the unique e-minimal map.

Now form the *strict pullback* $\dot{I}$ of I^1 as before. Let I be of length $\mu + 1$. $\dot{I}$ will then be of length $\dot{\mu} + 1$. Let $I', \hat{I}, \langle \sigma_i : i \leq \dot{\mu} \rangle$ be defined as before. Set: $N' =: N'_{\dot{\mu}}$, $\hat{N} =: \hat{N}_{\dot{\mu}}$, $\sigma' = \sigma'_{\dot{\mu}}$. The following facts are easily established:

(D) $\hat{N}$ is an S-iterate of N. Moreover: $\sigma' : M_\mu \longrightarrow_{\Sigma_0} N'$ where $N' \triangleleft \hat{N}$.

(E) If there is a drop point $i + 1 \leq_{T^1} \mu$ which is not an anomaly in I^1, then there is $i + 1 \leq_{T^0} \dot{\mu}$ which is not an anomaly in $\dot{I}$. Hence $\hat{N}$ is a truncating iterate of N and $\sigma' : M_\mu \longrightarrow_{\Sigma^*} \hat{N}$.

(F) If there is no anomaly $i + 1 \leq_{T^1} \mu$ in I, then there is no anomaly $i + 1 \leq_{\dot{T}} \dot{\mu}$ in $\dot{I}$.

(G) Suppose $0 \leq_{T^1} \mu$ and no $i + 1 \leq \mu$ is an anomaly. Hence the same situation holds in $\dot{I}$. Then $\hat{N}$ is an S-iterate of N by the iteration map $\sigma' \pi'_{0,\mu}$ (since $\dot{\sigma}_{\dot{\mu}} \dot{\pi}_{0,\dot{\mu}} = \hat{\pi}_{0,\dot{\mu}}$).

We now prove the simplicity lemma. We do this by eliminating all other possibilities.

Claim 1. Q_μ is not a proper segment of M_μ.

Proof. Suppose not. Then Q_μ is a non-truncating iterate of N with iteration map $\pi^0_{0,\mu}$. Hence $\sigma' \pi^0_{0,\mu} : N \longrightarrow_{\Sigma^*} \sigma_\mu(Q_\mu)$, where $\sigma_\mu(Q_\mu)$ is a proper segment of $\hat{N}$ and $\hat{N}$ is an S-iterate of N. Contradiction!

QED(Claim 1)

Claim 2. There is no truncation point $i + 1 \leq_{T^1} \mu$ such that $i + 1$ is not an anomaly in I^1.

Proof. Suppose not. Then $\sigma' : M_\mu \longrightarrow_{\Sigma^*} \hat{N}$, where $\hat{N}$ is a truncating S-iterate of N. I^0 is truncation free on its main branch, since I^1 is not. Hence $Q^0_\mu \triangleleft M_\mu$. Hence, $Q^0_\mu \triangleleft M'_\mu$ by Claim 1. Hence:

$$\sigma' \pi^0_{0,1} : N \longrightarrow_{\Sigma^*} \hat{N},$$

where $\hat{N}$ is a truncating iterate of N. Contradiction!

QED(Claim 2)

Claim 3. No $i + 1 \leq_{T^1} \mu$ is an anomaly in I^1.

Proof. Suppose not. Then $\kappa_i = \kappa$ and $\tau_i = \lambda$. Hence $\tau_i < \sigma(\lambda) = \kappa^{+N}$. Thus $M_i^* = N^*$, where $N^* = N||\eta$, η being maximal such that λ is a cardinal in $N||\eta$. By Claim 2, there is no drop point $j+1 \leq_{T^1} \mu$ such that $i < j$. Hence:

$$\pi'_{0,\mu} : N^* \longrightarrow_{\Sigma^*} M_\mu.$$

$\kappa = \rho^\omega$ in N^*, since $\rho^\omega \leq \kappa$ by the definition of N^*, but $\rho^\omega \geq \kappa$ since $N^* \in N$ and κ is a cardinal in N. But $\kappa_i = \text{crit}(\pi^1_{0,\mu})$. Hence $\kappa = \rho^\omega$ in M_μ.

$Q_\mu = M_\mu$ as above. Moreover the iteration I^0 is truncation free on its main branch, since I^1 is not. Thus:

$$\pi^0_{0,\mu} : N \longrightarrow_{\Sigma^*} M_\mu$$

Hence $\kappa^0_i \geq \rho^\omega_N$ for $i+1 \leq_{T^0} \mu$, since otherwise $\rho^\omega_{M_\mu} \geq \lambda_i > \kappa$. Hence:

$$\rho^\omega_N = \rho^\omega_{Q_\mu} = \kappa$$

and:

$$\mathbb{P}(\kappa) \cap N = \mathbb{P}(\kappa) \cap Q_\mu = \mathbb{P}(\kappa) \cap M_\mu = \mathbb{P}(\kappa) \cap N^*.$$

This is clearly a contradiction, since $N^* \in N$ and $\text{card}(N^*) = \kappa$ in N. Hence by a diagonal argument there is $A \in \mathbb{P}(\kappa) \cap N$ such that $A \notin N^*$.

QED(Claim 3)

It remain only to show:

Claim 4. $1 \leq_{T^1} \mu$.

Proof. Suppose not. Then $o <_{T^1} \mu$. By Claim 3 there is no anomaly on the main branch of I^1. Hence, if $\kappa_i < \lambda$ and $i+1 \leq_{T^1} \mu$, we have $\tau_i < \lambda$. But then $M^*_{\nu^1_i} = N$. By claim 2 there is no drop on the main branch of I^1. Hence:

$$\pi^1_{0,\mu} : N \longrightarrow_{\Sigma^*} M_\mu.$$

$M_\mu \triangleleft Q_\mu$ by Claim 1. Hence $M_\mu = Q_\mu$, since otherwise $\pi^1_{0,\mu}$ would map N into a proper segment of an S-iterator of N. Thus we have:

$$\pi^0_{0,\mu}; N \longrightarrow_{\Sigma^*} M_\mu.$$

Set: $\pi^0 = \pi^0_{0,\mu}, \pi^1 = \pi^1_{0,\mu}$. We claim:

Claim. $\pi^0 = \pi^1$.

Proof. Suppose not. Let i be least such that $\pi^0(e_i) \neq \pi^1(e_i)$. Then $\pi^1(e_i) > \pi^0(e_i)$ since the map π^0, being an S-iteration map, is e-minimal. But $\sigma'\pi^1$

is the S-iteration map from N to $\hat{N}$. Hence $\sigma'\pi^1(e_i) < \sigma'\pi^0(e_i)$, since $\sigma'\pi^0 : N \longrightarrow_{\Sigma^*} \hat{N}$. Hence $\pi^1(e_i) < \pi^0(e_i)$. Contradiction!

QED(Claim)

Let $i_h + 1 \leq_{T^h} \mu$ with $o = T^h(i_h + 1)$ for $h = 0, 1$. Then $\kappa_{i_0} = \kappa_{i_1} = \text{crit}(\pi)$, where $\pi = \pi^0_{0,\mu} = \pi^1_{0,\mu}$. Set:

$$F^0 = E^{Q_0}_{\nu_{i_0}}, F^1 = E^{M_0}_{\nu_{i_1}}.$$

Then:

$$F^h(X) = \pi^h_{0,i_h+1}(X) \text{ for } X \in \mathbb{P}(\kappa_{i_h}) \cap N.$$

Thus:

$$\alpha \in F^h(X) \longleftrightarrow \alpha \in \pi(X) \text{ for } \alpha < \lambda_{i_h},$$

since $\pi = \pi^h_{i_h+1,\mu} \circ \pi^h_{0,i_h+1}$. But then $\nu_{i_0} \not< \nu_{i1}$, since otherwise $F^0 \in J^{E^{M_{i_1}}}_{\nu_{i_1}}$ by the initial segment condition, whereas ν_{i_0} is a cardinal in $J^{E^{M_{i_1}}}_{\nu_{i_1}}$. Contradiction! Similarly $\nu_{i_1} \not< \nu_{i_0}$. Thus $i_0 = i_1 = i$ and $F^0 = F^1$. But then ν_i is not a coiteration index! Contradiction.

QED(Claim 4)

This proves the simplicity lemma.

4.3 Solidity and Condensation

In this section we employ the simplicity lemma to establish some deep structural properties of mice. In §4.3.1 we prove the **Solidity Lemma** which says that every mouse is solid. In §4.3.2 we expand upon this showing that any mouse N has a unique core $\overline{N}$ and core map σ defined by the properties:

- $\overline{N}$ is sound.
- $\sigma :\longrightarrow_{\Sigma^*} N$.
- $\rho^\omega_{\overline{N}} = \rho^\omega_N$ and $\sigma \upharpoonright \rho^\omega_N :=$ id.
- $\sigma(p^i_{\overline{N}}) = p^i_N$ for all i.

In §4.3.3 we consider the *c*ondensation properties of mice. The condensation lemma for L says that if $\pi : M \longrightarrow_{\Sigma_1} J_\alpha$ and M is transitive, then $M \triangleleft J_\alpha$. Could the same hold for an arbitrary sound mouse in place of J_α? In that generality it certainly does not hold, but we discover some interesting instances of condensation which do hold.

We continue to restrict ourselves to premice M such that $M||\alpha$ is not of type 3 for any α. By a mouse we mean such a premouse which is fully iterable. (Though we can take this as being relativized to a regular cardinal $\kappa > \omega$, i.e. $\mathrm{card}(M) < \kappa$ and M is fully $\kappa + 1$-iterable.)

4.3.1 Solidity

The *Solidity lemma* says that every mouse is solid. We prove it in the slightly stronger form:

Theorem 4.3.1. *Let N be a fully $\omega_1 + 1$-iterable premouse. Then N is solid.*

We first note that we may w.l.o.g. assume N to be countable. Suppose not. Then there is a fully $\omega_1 + 1$ iterable N which is unsolid, even though all countable premice with this property are solid. Let $N \in H_\theta$, where θ is a regular cardinal. Let $\sigma : \overline{H} \prec H_\theta$, $\sigma(\overline{N}) = N$, where $\overline{H}$ is transitive and countable. Then $\overline{H}$ is a ZFC^- model. Since $\sigma \upharpoonright \overline{N} : \overline{N} \prec N$, it follows by a copying argument that $\overline{N}$ is a $\omega_1 + 1$ fully iterable (cf. Lemma 3.5.6.). Hence $\overline{N}$ is solid. By absoluteness, $\overline{N}$ is solid in the sense of $\overline{H}$. Hence N is solid in the sense of H_θ. Hence N is solid. Contradiction!

Now let $a = p^n_N$ for some $n < \omega$. Let $\lambda \in a$. Let $M = N^\lambda_a$ be the λ–th witness to a as defined in §4.1. For the reader's convenience we repeat that definition here. Let:

$$\rho^{l+1} \leq \lambda < \rho^l \text{ in } N;\ b =: a \setminus (\lambda + 1)$$

Let $\overline{N} = N^{l,b}$ be the l-th reduct of N by b. Set:

$$X = h(\lambda \cup b) \text{ where } h = h_{\overline{N}} \text{ is the } \Sigma_1\text{-Skolem function of } \overline{N}.$$

Then $X = h``(\omega \times (\lambda \times \{b\}))$ is the smallest Σ_1-closed submodel of $\overline{N}$ containing $\lambda \cup b$. Let:

$$\overline{\sigma} : \overline{M} \longleftrightarrow \overline{N}|X \text{ where } \overline{M} \text{ is transitive.}$$

By the extension of embedding lemma, there are unique $M, \sigma, \overline{b}$ such that $\sigma \supset \overline{\sigma}$ and:

$$\overline{M} = M^{l,b},\ \sigma : M \longrightarrow_{\Sigma_1'} N \text{ and } \sigma(\overline{b}) = b.$$

Then $N^\lambda_a =: M$ and $\sigma^\lambda_a =: \sigma$.

It is easily seen that σ witnesses the phalanx $\langle N, M, \lambda\rangle$. Employing the simplicity lemma, we coiterate $\langle N, M, \lambda\rangle$ against N, getting $\langle I^N, I^M\rangle$, terminating at η, where:

- $I^N = \langle\langle N_i\rangle, \langle\nu_i^N\rangle, \langle\pi_{ij}^N\rangle, T^N\rangle$ is the iteration of N.
- $I^M = \langle\langle M_i\rangle, \langle\nu_i^M\rangle, \langle\pi_{ij}^M\rangle, T^M\rangle$ is the iteration of $\langle N, M\rangle$.
- $\langle\nu_i : i < \eta\rangle$ is the sequence of coiteration indices. We know that:
- $M\eta \triangleleft N_\eta$.
- I^M has no truncation on its main branch.
- $1 \leq_{T^M} \eta$.

It follows that $\kappa_i \geq \lambda$ for $i <_{T^M} \eta$. Moreover $\nu_i > \lambda$ for $i < \eta$, since $M|\lambda = N|\lambda$.

We consider three cases:

Case 1. $M_\eta = N_\eta$ and I^N has no truncation on its main branch.

We know that $\rho_M^{l+1} \leq \lambda$, since every $x \in M$ is $\Sigma_1^{(l)}(M)$ in $\lambda \cup \bar{b}$. But $\kappa_i \geq \lambda$ for $i <_{T^M} \eta$.

Hence:

(1) $\mathbb{P}(\lambda) \cap M = \mathbb{P}(\lambda) \cap M_\eta$ and $\rho_M^h = \rho_{M_\eta}^h$ for $h > i$. But then $\kappa_j \geq \rho_N^{l+1}$ for $j <_{T^N} \eta$, since otherwise:

$$\kappa_i < \sup \pi_{h,j+1}^N{}''\rho_N^{l+1} \leq \rho_{N_\eta}^{l+1} = \rho_{M_\eta}^{l+1} \leq \lambda < \kappa_j$$

where $h = T^N(j+1)$. Hence for $h > l$ we have:

(2) $\rho_M^h = \rho_N^h$ and $\mathbb{P}(\rho^h) \cap M = \mathbb{P}(\rho^h) \cap N$.

Recall, however, that $a = p_N^n$, where $m > l$. Since every $x \in M$ is $\Sigma_1^{(i)}(M)$ in $\lambda \cup \bar{b}$, there is a finite $c \subset \lambda$ such that $c \cup \bar{b} \in P_M^n$. Let $\overline{A}$ be $\Sigma_1^{(n)}(M)$ in $c \cup \bar{b}$ such that $\overline{A} \cap \rho^n \notin M$. Let A be $\Sigma_1^{(n)}(N)$ in $c \cup b$ by the same definition. Then:

$$\overline{A} \cap \rho^n = A \cap \rho^n \in N,$$

since $c \cup b <_* a = p_N^n$. Thus,

$$\mathbb{P}(\rho^n) \cap M \neq \mathbb{P}(\rho^n) \cap N,$$

contradiction! QED(Case 1)

Case 2. M_η is a proper segment of N_η.

Then M_η is sound. Hence M did not get moved in the iteration and $M = M_\eta$. But then N is not moved and $N = N_\eta, \eta = 0$, since otherwise ν_1 is a cardinal in N_η. But then $\lambda < \nu_1 \leq \mathrm{On}_M$ and $\rho_M^\omega \leq \lambda < \nu_1$, where M is a proper segment of N_η. Hence ν_1 is not a cardinal in N_η. Contradiction!

QED(Case 2)

Case 3. The above cases fail.

Then $M_\eta = N_\eta$ and I^N has a truncation on its main branch. We shall again prove: $M \in N$.

We first note the following:

Fact. Let Q. be acceptable. Let $\pi : Q \longrightarrow_F^* Q'$, where $\rho^{i+1} \leq \kappa < \rho^i$ in $Q, \kappa = \mathrm{crit}(F)$. Then:

$$\underline{\Sigma}_1^{(n)}(Q') \cap \mathbb{P}(\kappa) = \underline{\Sigma}_1^{(n)}(Q) \cap \mathbb{P}(\kappa) \text{ for } n \geq i.$$

Note. It follows easily that:

$$\underline{\Sigma}_1^{(n)}(Q') \cap \mathbb{P}(H) = \underline{\Sigma}_1^{(n)}(Q) \cap \mathbb{P}(H)$$

where $H = H_\kappa^Q = H_\kappa^{Q'}$.

We prove the fact. The direction $\supset$ is straightforward, so we prove $\subset$ by induction on $n \geq i$. The first case is $n = i$. Let $A \subset \kappa$ be $\Sigma_1^{(i)}(Q')$ in the parameter a. Then:

$$A_\xi \longleftrightarrow \bigvee z \in H_{Q'}^i \, B'(z, \xi, a)$$

where B' is $\Sigma_1^{(1)}(Q')$. But then π takes H_Q' cofinally to $H_{Q'}^i$. Hence:

$$A_\xi \longleftrightarrow \bigvee u \in H_Q^{i'} \bigvee z \in \pi(u) B'(\tau, \xi, a).$$

Let $a = \pi(f)\alpha$ where $f \in \Gamma^*(\kappa, Q)$ and $\alpha < \lambda(F) = F(\kappa)$. Let B be $\Sigma_0^{(i)}(Q)$ by the same definition as B'. Then:

$$A_\xi \longleftrightarrow \bigvee u \in H_Q^i \{\zeta < \kappa : \bigvee z \in uB(z, \xi, f(\alpha))\} \in F_\alpha,$$

where $F_\alpha \in \underline{\Sigma}_1(Q)$ by closeness.

This proves the case $n = i$. The induction step uses the fact that $\rho^n_Q = \rho^n_{Q'}$, for $n > i$. (Hence $H^n_Q = H^n_{Q'}$.)

Let $n = m + 1 > i$ and let it hold at m. Let $A \subset \kappa$ be $\underline{\Sigma}^{(m)}_1(Q')$. Then:

$$A_\xi \longleftrightarrow \langle H^n_{Q'}, B^1_\xi, \dots, B^r_\xi\rangle \vdash \varphi$$

where φ is a Σ_1 sentence and:

$$B^h_\zeta = \{z \in H^n_Q : \langle \xi, z\rangle \in B^h\}\ (h = 1, \dots, r)$$

and B^h is $\underline{\Sigma}^{(m)}_1(Q')$. We may assume w.l.o.g. that $B^h \subset H$. But then B^h is $\Sigma^{(m)}_1(Q)$. Hence A is $\underline{\Sigma}^{(n)}_1(Q)$.

QED(Fact)

Recall that $\rho^{l+1} \leq \lambda < \rho^l$ in M. Using this we get:

(1) There is a $\underline{\Sigma}^{(l)}_1(M)$ set $B \subset \lambda$ which codes M (in particular, if Q is a transitive ZFC^- model and $B \in Q$, then $M \in Q$.)

Proof. Recall from the definition of M that:

$$\overline{M} = M^{l,b} = h_{\overline{M}}(\omega \times (\lambda \times \{\overline{c}\})),\text{ where } \overline{c} = \overline{b} \cap \rho^l_M.$$

Thus we can set:

$$\dot{M} = \{\prec i, \xi \succ \in M : i < \omega, \xi < \lambda,\text{ and } h_{\overline{M}}(i, \langle \xi, \overline{c}\rangle)\text{ is defined}\}.$$

For $\prec i, \xi \succ \in \dot{M}$ set: $h(\prec i, \xi \succ) = h_{\overline{M}}(i, \prec \xi, \overline{c} \succ)$. Let $M = \langle J^E_\alpha, F\rangle$. We set:

- $\dot{\in} =: \{\langle x, y\rangle \in \dot{M}^2 : h(x) \in h(y)\}$
- $\dot{I} =: \{\langle x, y\rangle \in \dot{M}^2 : h(x) = h(y)\}$
- $\dot{E} =: \{x \in \dot{M} : h(x) \in E\}$
- $\dot{F} =: \{\ x \in \dot{M} : h(x) \in F\}$

Then:

$$\langle \dot{M}, \dot{\in}, \dot{E}, \dot{F}\rangle / I \cong \langle J^E_\alpha, F\rangle = M.$$

Let B be a simple coding of $\langle \dot{M}, \dot{\in}, \dot{E}, \dot{F}\rangle$, e.g. we could take it as the set of $\prec \xi, j \succ$ such that one of the following holds:

- $j = 0 \wedge \xi \dot{\in} \dot{M}$
- $j = 1 \wedge \xi = \prec \xi_u, \xi_1 \succ$ with $\xi_0 \dot{\in} \xi_1$
- $j = 2 \wedge \xi = \prec \xi_0, \xi_1 \succ$ with $\xi_0 I \xi_1$

- $j = 3 \wedge \xi \in \dot{E}$
- $j = 4 \wedge \xi \in \dot{F}$.

It is clear that if $B \in Q$ and Q is a transitive ZFC^- model, then $\overline{M}$ is recoverable from B in Q by absoluteness. Hence $\overline{M} \in Q$. But $\overline{M} = M^{l,\bar{b}}$ and M is recoverable from $\overline{M}$ in Q by absoluteness. Hence $M \in Q$.

QED(1)

Let $j+1$ be the final truncation point on the main branch of I^N. Then:

(2) B is $\underline{\Sigma}_1^{(l)}(N_{j+1})$.

Proof. Let B be $\Sigma_1^{(l)}(M)$ in the parameter p. Let B' be $\Sigma_1^{(\theta)}(M_\eta)$ in $\pi(p)$ by the same definition, where $\pi = \pi_{1,\eta}^M$. Then $B = \lambda \cap B'$ is $\underline{\Sigma}_1^{(l)}(N_\eta)$. Let i be the least $i \geq_T j+1$ in I^N set. B is $\Sigma_1^{(l)}(N_i)$. i is not a limit ordinal, since otherwise $\text{lub}\{\kappa_h : h \leq_{T^N} i\} = \text{lub}\{k_h : h < i\} > \lambda$ and there is $h \leq_{T^N} i$ such that $\kappa_h > \lambda$ and $a \in \text{rng}(\pi_{hi}^N)$, where B is $\Sigma_1^{(l)}(N_i)$ in the parameter a. Hence B is $\underline{\Sigma}_1^{(l)}(N_h)$. Contradiction! But then $i = k+1$. Let $t = T^N(k+1)$. If $k > j$, then $t \geq j+1$ and $\kappa_k \geq \lambda_j \geq \lambda > \rho_M^{l+1} = \rho_{N_\xi}^{l+1} = \rho_{N_t}^{l+1}$. By the above Fact we conclude that $B \in \underline{\Sigma}_1^{(l)}(N_t)$ where $t < i$. Contradiction! Hence $i = j+1$. QED(2)

We consider two cases:

Case 3.1. $\kappa_j \geq \lambda$.

By the Fact, we conclude that B is $\underline{\Sigma}_1^{(i)}(N_j^*)$ is a proper segment of N_t, where $t = T^N(j+1)$. Hence $B \in \underline{\Sigma}_1^{(i)}(N_j^*) \subset N$. But then $B \cap \mathbb{P}(\lambda) \cap N \subset J_{\sigma(\lambda)}^{E^N}$, since $\sigma(\lambda) > \lambda$ is regular in N. Hence $J_{\sigma(\lambda)}^{E^N}$ is a ZFC^- model and $M \in J_{\sigma(N)}^{E^N} \subset N$.

QED(Case 3.1)

Case 3.2. Case 3.1 fails.

Then $\kappa_j < \lambda$. But $\tau_j \geq \lambda$, since otherwise $\tau_j < \lambda$ is a cardinal in M, hence in N. Hence $N_j^* = N$ and no truncation would take place at $j+1$. Contradiction! Thus:

$$\lambda = \tau =: \tau_j,\ N_j^* = N^* = N||\gamma,\ \kappa_j = \kappa,$$

where κ is the cardinal predecessor of λ in M and $\gamma > \lambda$ is maximal such that τ is a cardinal in $N||\gamma$. Then:

(1) $\pi : N^* \longrightarrow_F^* N_{j+1}$ where $\pi = \pi_{0,j+1}^N$, $F = E_{\nu_j}^{N_j}$

Since:

$$\pi_{j+1,\eta} : N_{j+1} \longrightarrow_{\Sigma^*} M_\eta \text{ and } \mathrm{crit}(\pi_{j+1,\eta}) > \lambda,$$

we know that:

(2) $\rho^{l+1} < \lambda < \rho^l$ in N_{j+1}

By the definition of N^* we have: $\rho^\omega_{N^*} < \lambda$. But $\rho^\omega_{N^*} \geq \kappa$, since κ is a cardinal in N and $N^* \in N$. Hence:

(3) $\rho^\omega_{N^*} = \kappa$.

Now let: $\rho^{i+1} \leq \kappa < \rho^i$ in N^*. Then:

$$\rho^{i+1} \leq \kappa < \lambda \leq \rho^i \text{ in } N_{j+1},$$

since:

$$\lambda < \sup \pi\text{"}\lambda = \lambda(F) \leq \sup \pi\text{"}\rho^i_{N^*} = \rho^i_{N_{j+1}}.$$

Hence $i = l$ and:

(4) $\rho^{l+1} = \kappa < \rho^l$ in N_{j+1}.

We now claim:

(5) $B \in \mathrm{Def}(N^*)$, i.e. B is definable in parameters from N^*. Hence $B \in N$.

Proof. For $\xi < \lambda$ define a map $g_\xi : \kappa \longrightarrow \kappa$ as follows:

For $\alpha < \kappa$ set:

- $X_\alpha =$ the smallest $X \prec J^{E^{N^*}}_\lambda$ such that $\alpha \cup \{\xi\} \in X$.
- $C_\xi = \{\alpha < \kappa : X_\xi \circ k \subset \alpha\}$.

For $\alpha \in C_\xi$, let $\sigma_\xi : Q_\xi \overset{\sim}{\longleftrightarrow} X_\xi$ be the transitivator of X_ξ. Set:

$$g_\xi(\alpha) =: \begin{cases} \sigma_\xi^{-1}(\xi) & \text{if } \alpha \in C_\xi \\ \varnothing & \text{if not} \end{cases}$$

It is easily seen that:

$$\pi(g_\xi)(\kappa) = \xi \text{ where } \pi = \pi^N_{0,j+1}.$$

Since B is $\underline{\Sigma}^{(l)}_1(N_{j+1})$ we have:

$$B_\zeta \longleftrightarrow \bigvee z \in J^{E^{N_{j+1}}}_{\rho^l_{N_{j+1}}} B'(z, \zeta, a)$$

for some $a \in N_{j+1}$. But π takes cofinally to $\rho^l_{N_{j+1}}$. Hence:

$$B_\zeta \longleftrightarrow \bigvee u \in J^{E^{N^v}}_{\rho_{N^*}} \bigvee z \in \pi(u) B'(z, \zeta, u).$$

Let $f \in \Gamma^*(\kappa, N^*)$ such that $a = \pi(f)(\alpha), \alpha < \lambda$. We know that $\xi = \pi(g_\xi)(\kappa)$ for $\xi < \lambda$. But then the statement B_ζ is equivalent to

$$\bigvee u \in J^{E^{N^v}}_{\rho^f_{N^*}} \{\langle \mu, \delta \rangle : \bigvee x \in u B''(x, g_\zeta(\mu), f(\delta))\} \in F_{\langle K, \alpha \rangle}$$

where $F = E^{N_j}_{\nu_j}$ and B'' is $\Sigma^{(l)}_0(N^*)$ by the same definition. But $F_{\langle \kappa, \alpha \rangle}$ is $\underline{\Sigma}_1(N^*)$ by closeness. QED(5)

But then $B \in \text{Def}(N^*) \subset J^{E^N}_{\sigma(\lambda)} \subset N$. Hence $M \in N$.

QED(Lemma 4.3.1)

4.3.2 Soundness and Cores

Let N be any acceptable structure. Let $m < \omega$. In §2.5 we defined the set R^n_N of very good n-parameters. The definition is equivalent to:

> $a \in R^n$ iff a is a finite set of ordinals and for $i < n$, each $x \in N||\rho^i$ has the form $F(\xi, a)$ where F is a $\Sigma^{(i)}_1(N)$ map and $\xi < \rho^{i+1}$.

We said that N is n-sound iff $R^n_N = P^n_N$. It follows easily that N is n-sound iff $p^n \in R^n$, where $p^n = p^n_N$ is the $<_*$-least $p \in P^n$. We called N **sound** iff it is n-sound for all n. It followed that, if N is sound, then $\rho^n \setminus \rho^i = p^i$ for $i \leq n < \omega$.

We have now shown that, if N is a mouse then $p^n \setminus \rho^i = p^i$ for $i \leq n < \omega$, regardless of soundness. We set: $p^* = \bigcup_{n<\omega} p^n$. Then $p^* = p^n$ whenever $\rho^n = \rho^\omega$ in N. We know:

Lemma 4.3.2. *If N is a mouse and $\pi : \overline{N} \longrightarrow_{\Sigma^*} N$ strongly, then $\overline{N}$ is a mouse and $\pi(p^*_{\overline{N}}) = p^*_{N^*}$.*

Proof. $\overline{N}$ is a mouse by a copying argument. Hence $\overline{N}$ is solid. But then $\pi(p^i_{\overline{N}}) = P^i_N$ for all $i < \omega$, by Lemma 4.1.11.

QED(Lemma 4.3.2)

We know generalize the notion R^n_N as follows:

Definition 4.3.1. Let $\rho^\omega_N \leq \mu \in N, a \in R^{(\mu)}_N$ iff a is aa finite set of ordinals and for some n,

- $\rho^n \leq \mu < \rho^{n-1}$ in N.

- Every $x \in N||\rho^{n-1}$ has the form $F(\vec{\xi}, a)$, where $\xi_1, \ldots, \xi_r < \mu$ and F is $\Sigma_1^{(n-1)}(N)$.
- If $j < n-1$, then $a \in R_N^j$.

We also set:

Definition 4.3.2. N is sound above μ iff for some $n, \rho^n \leq \mu < \rho^{n-1}$ in N and whenever $p \in P_N^n$ then $p \setminus \mu \in R_N^{(\mu)}$.

(It again follows that N is sound above μ iff $p_N^n \setminus \mu \in R_N^{(\mu)}$.) We prove:

Lemma 4.3.3. *Let N be a mouse. Let $\rho_N^\omega \leq \mu \in N$. There is a unique pair σ, M such that:*

- $\sigma : M \longrightarrow_{\Sigma^*} N$
- *M is a mouse which is sound above μ*
- $\sigma \restriction \mu = \mathrm{id}$ *and* $\sigma(p_M^*) = p_N^*$.

Before proving this, we develop some of its consequences.

Definition 4.3.3. Let N be a mouse. If M, σ are as above, we call M the **μ-th core of** N, denoted by: $\mathrm{core}_\mu(N)$, and σ the **μ-th core map**, denoted by σ_μ^N.

We also set: $\mathrm{core}(N) = \mathrm{core}_{\rho_N^\omega}(N)$ and $\sigma^N = \sigma_{\rho_N^\omega}^N$, $M = \mathrm{core}(N)$ is the **core** of N, and σ^N is the **core map**.

We leave it to the reader to prove:

Corollary 4.3.4. *Let N be a mouse. Then:*

- $\mathrm{core}_\mu(\mathrm{core}_\mu(N)) = \mathrm{core}_\mu(N)$.
- *N is sound above μ iff $N = \mathrm{core}_\mu(N)$.*
- *Let $M = \mathrm{core}_\mu(N), \overline{\mu} \leq \mu, \overline{M} = \mathrm{core}_{\overline{\mu}}(M)$. Then $\overline{M} = \mathrm{core}_{\overline{\mu}}(M)$ and $\sigma_\mu^N \sigma_{\overline{\mu}}^M = \sigma_{\overline{\mu}}^N$.*

We now turn to the proof of Lemma 4.3.3. By Löwenheim-Skolem argument it suffices to prove it for countable N. We first prove uniqueness. Suppose not. Let M, π and M', π' both have the property. If $x \in M$, then $x =$

$F(\vec{\xi}, P^*_N)$ where F is good and $\xi_1, \ldots, \xi_r < \mu$, since M is sound above μ. Hence:

$$\pi(x) = \tilde{F}(\vec{\xi}, P^*_N)$$

where $\tilde{F}$ has the same good definition over N. But then in N the Σ^* statement holds:

$$\bigvee y\, y = \tilde{F}(\vec{\xi}, P^*_N).$$

(This is Σ^* since it results from the substitution of $\tilde{F}(\vec{\xi}, P^*_N)$ in the formula $\nu = \nu$.) Hence in M' we have:

$$\bigvee y\, y = F'(\vec{\xi}, P^*_N),$$

where F' has the same good definition over M'. Thus $\mathrm{rng}(\pi) \subset \mathrm{rng}\, \pi'^{-1}$ and $\pi'^{-1}\pi$ is a Σ^*-preserving map of M to M'. A repeat of this argument then shows that $\mathrm{rng}(\pi') \subset \mathrm{rng}(\pi^{-1})$ and $\pi'^{-1}\pi$ is an isomorphism of M onto M'. But M, M' are transitive. Hence $M = M'$ and $\pi = \pi'$.

QED

This prove uniqueness. We now prove existence. Let $a = p^*_N$. Let $\rho^{n+1} \leq \mu < \rho^n$. Set $\overline{N} = N^{n,a}$. Let $b = a \cap \rho^n_N$ and set:

$$X = h_{\overline{N}}(\mu \cup b) = \text{ the closure of } \mu \cup b \text{ under } \Sigma_1(\overline{N}) \text{ functions.}$$

Let $\overline{\sigma} : \overline{M} \overset{\sim}{\longleftrightarrow} \overline{N}|X$ be the transitivazation of $\overline{N}|X$. By the downward extension lemma, there are unique $M, \sigma \supset \overline{\sigma}, \overline{a}$ such that:

$$\overline{M} = M^{n,\overline{a}},\ \sigma : M \longrightarrow_{\Sigma_1^{(n)}} N,\ \sigma(\overline{a}) = a.$$

Clearly, $\sigma \upharpoonright \mu = \mathrm{id}$. Moreover, $\overline{a} \in R^{(\mu)}_M$. It suffices to prove:

Claim. σ is Σ^*-preserving and $\overline{a} = p^*_M$.

If $\sigma = \mathrm{id}$ and $M = N$, there is nothing to prove, so suppose not. Let $\lambda = \mathrm{crit}(\sigma)$. (Hence $\mu \leq \lambda$.) There is then a $h \leq n$ such that $\rho^{h+1} \leq \lambda < \rho^h$ in N. λ is a regular cardinal in M, since $\sigma(\lambda) > \lambda$. It follows easily that σ witnesses the phalanx $\langle N, M, \lambda\rangle$. Note that $\rho^\omega_M \leq \mu \leq \lambda$, since $\overline{a} \in R^{(\mu)}_M$. We now apply the simplicity lemma, coiterating $N, \langle N, M\lambda\rangle$ with:

$$I^N = \langle\langle N_i\rangle, \langle\nu^N_i\rangle, \langle\pi^N_{i,j}\rangle, T^N\rangle$$

$$I^M = \langle\langle M_i\rangle, \langle\nu^M_i\rangle, \langle\pi^M_{i,j}\rangle, T^M\rangle$$

being the iteration of $N, \langle N, M, \lambda\rangle$ respectively. We assume that the iteration terminates at an $\eta < \omega_1$ and that $\langle \nu_i : 1 \leq i < \eta\rangle$ is the sequence of coindices.

It is now time to mention that some of the steps in the proof of solidity go through with a much weaker assumption on the phalanx $\langle N, M, \lambda\rangle$ and its witness σ. In particular:

Lemma 4.3.5. *Let σ witness $\langle N, M, \lambda\rangle$, where $R_M^{(\lambda)} \neq \varnothing$. If cases 2 or 3 hold, then $M \in N$.*

The reader can convince himself of this by an examination of the solidity proof. But the premiss of Lemma 4.3.5 is given. Hence:

(1) Case 1 applies.

Proof. Suppose not. Let A be $\Sigma_1^{(h)}(N)$ in a such that $A \cap \rho_N^{h+1} \notin N$. Let $\overline{A}$ be $\Sigma_1^{(h)}(M)$ in $\overline{a}$ by the same definition. Then $A \cap \rho_N^{h+1} = \overline{A} \cap \rho_N^{h+1} \in N$, since $\overline{A} \in \underline{\Sigma}_\omega(M) \subset N$. Contradiction!

QED(1)

Then $M_\eta = N_\eta$ and there is no truncation on the main branch of I^N. Then $\pi_{1,\eta}^M : M \longrightarrow_{\Sigma^*} M_\eta$. Hence, by a copying argument, M is a mouse, hence is solid. Since $\mathrm{crit}(\pi_{1,\eta}^M) \geq \lambda$, we have:

(2) $\mathbb{P}(\lambda) \cap M = \mathbb{P}(\lambda) \cap M_\eta$ and $\rho_M^i = \rho_{M_\eta}^i$ for $i > h$.

But:

(3) $\mathrm{crit}(\pi_{1,\eta}^N) \geq \rho^{h+1}$.

Proof. Suppose not. then there is $j + 1 \leq_{T^N} \eta$ such that $\kappa_j < \rho^{h+1}$. Let j be the least such. Let $t = T^N(j+1)$. Then:

$$\kappa_j < \sup\ \pi_{t,j+1}\text{''}\rho_N^{h+1} \leq \rho_{N_{j+1}}^{h+1} \leq \rho_{N_\eta}^{h+1} = \rho_M^{h+1} > \kappa_j.$$

Contradiction!

QED(3)

Hence:

(4) $\rho_N^i = \rho_M^i$ for $i > h$. Moreover if $\rho^i = \rho_N^i$, then $\mathbb{P}(\rho^i) \cap N = \mathbb{P}(\rho^i) \cap M$ for $i > h$.

Using this we get:

(5) $\sigma : M \longrightarrow_{\Sigma^*} N$.

We first show that σ is Σ^*-preserving. By induction on $i \geq h$ we show:

Claim. σ is $\Sigma_1^{(i)}$-preserving.

For $i = h$, this is given. Now let $i = k + 1 \geq h$ and let it hold for k. Let A be $\Sigma_1^{(i)}(M)$. then:

$$Ax \longleftrightarrow \langle H^i, B_x^1, \ldots, B_x^r \rangle \models \varphi$$

where φ is a Σ_1-sentence and:

$$B_x^i \{z \in H^i : \langle z, x \rangle \in B^l\},$$

where B^l is $\Sigma_1^{(k)}(M)$ for $l = 1, \ldots, r$. Let A' be $\Sigma_1^{(k)}(M)$ by the same definition. Then:

$$B_{zx}^l \longleftrightarrow B_{z\sigma(x)}^{l'} \text{ for } z \in H_M^i = H_N^i.$$

Hence $Ax \longleftrightarrow A'\sigma(x)$.

QED(5)

But

(6) σ is strongly Σ^*-preserving.

Proof. Let $\rho^m = \rho^\omega$ in M and N. Let A be $\Sigma_1^{(m)}(M)$ in x such that $A \cap \rho^m \notin M$. Let A' be $\Sigma_1^{(m)}(M)$ in $\sigma(x)$ by the same definition. Then $A \cap \rho^n = A' \cap \rho^m \notin N$, since $\mathbb{P}(\rho^m) \cap M = \mathbb{P}(\rho^m) \cap N$.

QED(6)

But then $\sigma(P_M^*) = P_N^*$. Hence $P_M^* = \overline{a} = \overline{\sigma}'(P_N^*)$. We know that $\overline{a} \in R_M^{(\mu)}$. Hence M is solid above μ.

QED(Lemma 4.3.5)

4.3.3 Condensation

The condensation lemma for L says that if M is transitive and $\pi : M \longrightarrow J_\alpha$ is a reasonable embedding, then $M \triangleleft J_\alpha$. It is natural to ask whether the dame holds when we replace J_α by an arbitrary sound mouse. In order to have any hope of doing this, we must employ a more restrictive notion of reasonable. Let us call $\sigma : M \longrightarrow N$ reasonable iff either $\sigma = \text{id}$ or σ witnesses the phalanx $\langle N, M, \lambda \rangle$ and $\rho_M^\omega \leq \lambda$. We then get:

Lemma 4.3.6. *If N, M are sound mice and $\sigma : M \longrightarrow N$ is reasonable in the above sense, then $M \triangleleft N$.*

It ifs not too hard to prove this directly from the solidity lemma and the simplicity lemma. We shall, however, derive it from a deeper structural lemma:

Lemma 4.3.7. *Let N be a mouse. Let σ witness the phalanx $\langle N, M, \lambda\rangle$. Then M is a mouse. Moreover, if M is sound above λ, then one of the following hold:*

(a) $M = \mathrm{core}_\lambda(N)$ *and* $\sigma = \sigma^N_\lambda$.

(b) M *is a proper segment of* N.

(c) $\pi : N||\gamma \longrightarrow^*_F M$, *where* $F = F^N_\mu$ *such that:*

  (i) $\lambda < \gamma \in N$ *such that* $\rho^\omega_{N||\gamma} < \lambda$.

  (ii) $\lambda = \kappa^{+N||\gamma}$ *where* $\kappa = \mathrm{crit}(F)$.

  (iii) F *is generated by* $\{\kappa\}$.

Remark. In case (c) we say that M is one measure away from N. Then γ is maximal such that λ is a cardinal in $N||\gamma$. Hence $\rho_{N||\gamma} \leq \kappa$. But κ is a cardinal in N and $N||\gamma \in N$. Hence $\rho_{N||\gamma} = \kappa$. But $\pi \restriction \kappa = \mathrm{id}$ and $\pi(p^*_{N|\gamma}) = p^*_M$. Hence $N||\gamma = \mathrm{core}(M)$ and π is the core map. Clearly, μ is least such that $E^M_\mu \neq E^N_\mu$.

Remark. Lemma 4.3.6 follows easily, since the possibilities (a) and (c) can be excluded. (a) cannot hold, since otherwise $M = \mathrm{core}_\lambda(N) = N$ by the soundness of N. Hence $\sigma^\lambda_N = \mathrm{id}$. Contradiction, since $\mathrm{crit}(\sigma^\lambda_N) = \lambda$. If (c) held, then $N^* = \mathrm{core}(M)$ where $N^* = N||\gamma$, and π is the core map. But M is sound. Hence $M = N^* = \mathrm{core}(M)$ and $\pi = \mathrm{id}$. Contradiction!

Remark. Lemma 4.3.7 has many applications, through mainly in setting where the awkward possibility (c) can be excluded (e.g. when λ is a limit cardinal in M). We have given a detailed description of (c) in order to facilitate such exclusions.

We now prove Lemma 4.3.7. We can again assume N to be countable by Löwenheim-Skolem argument. We again coiterate against $\langle N, M, \lambda\rangle$ getting the iterations:

$$I^N = \langle\langle N_i\rangle, \ldots, T^N\rangle,\ I^M = \langle\langle M_i\rangle, \ldots, T^M\rangle$$

with coiteration indices $\langle \nu_i : i < \eta\rangle$, where the coiteration terminates at $\eta < \omega_1$. Then $\pi_{1,\eta} : M \longrightarrow_{\Sigma^*} M_\eta$ and M is a mouse by a copying argument. Now let M be sound above λ. We again consider three cases:

Case 1. $M_\eta = N_\eta$ and I^N has no truncation on the main branch.

We can literally repeat the proof in cases of Lemma 4.3.5, getting:

$$\sigma \text{ is strongly } \Sigma^*\text{-preserving.}$$

Hence $\sigma(p^*_M) = p^*_N$ where M is sound above λ and $\sigma = \sigma^N_\lambda$.

QED(Case 1)

Case 2. M_η is a proper segment of N_η.

We can literally repeat the proof in Case 2 of the solidity Lemma, getting: M is a proper segment of N.

Case 3. The above cases fail.

Then $M_\eta = N_\eta$ and I^N has a truncation on the main branch. Let $j+1$ be the last truncation point on the main branch. Then M is a mouse and $\pi^M_{1,\eta}$ is strongly Σ^*-preserving. Hence $\pi^M_{1,\eta}(p^*_M) = p^*_{M_\xi}$. But $\kappa_i \geq \lambda$ for all $i \leq_{T^M} \eta$. Hence $\mathrm{crit}(\pi_{1,\eta}) \geq \lambda$. Hence:

$$M = \mathrm{core}_\lambda(M_\eta) \text{ and } \pi_{1,\eta} = \sigma^{M_\xi}_\lambda,$$

since M is sound above λ. We also know:

$$\kappa_i \geq \lambda_j \geq \lambda \text{ for } j+1 <_{T^N} i+1 <_{T^N} \eta.$$

Hence $\mathrm{crit}(\pi^N_{j+1,\eta}) \geq \lambda$ and $\pi^N_{j+1,\eta}(p^*_{N_{j+1}}) = p^*_{N_\eta} = p^*_{M_\eta}$. Hence:

$$M = \mathrm{core}_\lambda(N_{j+1}) \text{ and } \sigma^{N_{j+1}}_\lambda = (\pi^N_{j+1,\eta})^{-1} \circ \pi^M_{1,\eta}.$$

We consider two cases:

Case 3.1. $\kappa_j \geq \lambda$.

Then N^*_j is a proper initial segment of N_j, hence is sound. Since $\kappa_j \geq \lambda$, it follows as before that $M = \mathrm{core}_\lambda(N^*)$. Hence $M = N^*_j$ by the soundness of N^*_j. But this means that M was not moved in the iteration I^M up to $t = T^N(j+1)$, since if $h < t$ in the least point active in I^*, then $E^M_{\nu_h} \neq \varnothing$ and hence $E^{N_t}_{\nu_h} = E^{N^*_j}_{\nu_h} = \varnothing$. Hence $N^*_j \neq M$. Contradiction!

Thus $M_t = M = N^*_j$ is a proper segment of N_t. Hence the coiteration terminates at $t < \eta$. Contradiction!

QED(Case 3.1)

Case 3.2. Case 3.1 fails.

Then $\kappa_j < \lambda$. But $\tau_j \geq \lambda$, since otherwise τ_j is a cardinal in N and $N^*_j = N$. Hence $j+1$ is not a truncation point in I^N. Contradiction!

Thus $\tau_j = \lambda$. λ is regular in M, since $\sigma(\lambda) > \lambda$. But then $\lambda = \kappa_i^+$ in M and $\sigma(\lambda) = \kappa_j^+$ in N. Hence λ is not a cardinal in N. $E_\lambda^M = \emptyset$, since λ is a cardinal in M. But $E_\lambda^N = \emptyset$, since otherwise κ_j, being a cardinal in N, would be a cardinal in $N||\lambda$. Hence $N||\lambda$ would be an active premouse of type 3. Contradiction!

But 0 is inactive in I^N and $\nu_1 =$ the least ν such that $E_\nu^M \neq E_\nu^N$. Hence $\nu_i \geq \nu_1 > \lambda$ for all i which are active in I^N. Hence no $i < t$ is active in I^N, since otherwise $\kappa_j < \lambda_i$. But $t = T(j+1)$ is the least t such that t is active in I^N and $\kappa_j < \lambda_t$. Contradiction!

But then $N = N_t$ and $N_j^* = N^* = N||\gamma$, where γ is maximal such that $\tau = \lambda$ is a cardinal in $N||\gamma$. Hence $\kappa_j = \kappa =$ the cardinal predecesor of τ in N^*. $\kappa = \rho_{N^*}^\omega$, since κ is a cardinal in N and $N^* \in N$. We have:

$$\kappa_i \geq \lambda \text{ for } 1 \leq_{T^M} i+1 \leq_{T^M} \eta$$

Hence $\mathrm{crit}(\pi_{1,\eta}^M) \geq \lambda$. But:

$$\kappa_i \geq \lambda_t \geq \lambda \text{ for } j+1 <_{T^N} i+1 <_{T^N} \eta$$

Hence $\mathrm{crit}(\pi_{j+1,\eta}^N) \geq \lambda$. Hence:

$$M = \mathrm{core}_\lambda(N_{j+1}),\ (\pi_{j+1,\eta}^N)^{-1} \circ \pi_{1,\eta}^M = \sigma_\lambda^{N_{j+1}},$$

$\rho_{N^*}^\omega \leq \kappa$. But then $\rho_{N^*}^\omega = \kappa$ since κ is a cardinal in N and $N^* \in N$. Set $\langle \tilde{N}, \tilde{F} \rangle = N_j||\nu_j$. Then:

$$\pi_{t,j+1} : N_j^* \longrightarrow_{\tilde{F}}^* N_{j+1}$$

By closeness we have: $\tilde{F}_\kappa \in \underline{\Sigma}_1(N^*)$. Hence $\tilde{F}_\kappa \in \underline{\Sigma}_1(N^*) \subset N||\sigma(\tau)$, where $\sigma(\tau)$ is regular in N and $\gamma < \sigma(\tau)$. Set: $\bar{Q} = N||\tau$. By a standard construction there is a unique triple $\langle Q, F, \bar{\pi} \rangle$ such that F is a full extender at κ with base $\bar{Q}$, $\bar{\pi} \colon \bar{Q} \longrightarrow_F Q$ is the extension of $\langle \bar{Q}, F \rangle$, F is generated by $\{\kappa\}$ and $F_\kappa = \tilde{F}_\kappa$. (To see this we note that $\tilde{F}_\kappa$ is a normal ultrafilter on $\bar{Q}$ at κ. Hence we can form the ultraproduct $\bar{\pi} \colon \bar{Q} \longrightarrow_{\tilde{F}_\kappa} Q$. Q is well-founded , since each element of Q has the form $\bar{\pi}(f)(\kappa)$ where $f \in \bar{Q}$, $f \colon \kappa \longrightarrow \bar{Q}$ and:

$$\begin{aligned} \bar{\pi}(f)(\kappa) \in \tilde{\pi}(g)(\kappa) &\iff \{\xi \colon f(\xi) \in g(\xi)\} \in \tilde{F}_\kappa \\ &\iff \pi_{t,i+1}^N(f)(\kappa) \in \pi_{t,i+1}^N(g)(\kappa). \end{aligned}$$

Set: $F = \bar{\pi} \upharpoonright \mathbb{P}(\kappa)$. Then Q, F, π have the above properties.) The construction of Q, F, $\bar{\pi}$ can be carried out in the ZFC^- model $N||\sigma(\tau)$, since

$\bar{Q}, \tilde{F}_\kappa \in N||\sigma(\tau)$. Then $Q, F, \tilde{\pi} \in N$. It is easily seen that F is close to N^*. Hence we can form the Σ^* ultrapower:

$$\pi \colon N^* \longrightarrow_F^* M'.$$

M' is transitive, since each of its element has the form $\pi(f)(\kappa)$, where $f \in \Gamma^*(\kappa, N^*)$ and as before:

$$\pi(f)(\kappa) \in \pi(g)(\kappa) \iff \pi_{t,i+1}^N(f)(\kappa) \in \pi_{t,i+1}^N(g)(\kappa).$$

There is a $\Sigma_0^{(n-1)}$ preserving map $\sigma \colon M' \longrightarrow N_{i+1}$ defined by:

$$\sigma(\pi(f)(\kappa)) = \pi_{t,i+1}(f)(\kappa)$$

for $f \in \Gamma^*(\kappa, N^*)$. Since π takes $\rho_{N^*}^{n-1}$ cofinally to $\rho_{M'}^{n-1}$ and $\pi t, i+1$ takes $\rho_{N^*}^{n-1}$ cofinally to $\rho_{N_{j+1}}^{n-1}$, we know that σ' takes $\rho_{N^*}^{n-1}$ cofinally to $\rho_{N'}^{n-1}$. Hence why σ is $\Sigma_1^{(n-1)}$-preserving. Since $\sigma \restriction \kappa = \mathrm{id}$ and $\kappa \geq \rho_{N^*}^n$, it follows easily that σ' is Σ^* preserving.

Claim 1. M' is sound above τ. Hence $M = M' = \mathrm{core}_\tau(N_{j+1})$.

Proof. Let $\rho^n \leq \kappa < p^{n-1}$ in N^*. Hence $\kappa = \rho^n = \rho^\omega$ in N^*. Let $x \in M'$. Then $x = \pi(f)(\kappa)$, where $f \in \Gamma^*(\kappa, N^*)$.

By the soundness of N^* we may assume:

$$f(\xi) = F(\xi, a, \vec{\zeta})$$

where F is a good $\Sigma_1^{(n-1)}(N^*)$ function, $a = p_{N^*}^n$ and $\zeta_1, \ldots, \zeta_r < \kappa$. Hence:

$$\pi(f)(\kappa) = F'(\kappa, \pi(a), \vec{\zeta})$$

where F' is $\Sigma_1^{(n-1)}(M')$ by the same good definition, $\pi(a) = p_{M'}^n$, and $\vec{\zeta} < \tau$. But $\kappa < \tau$, where $\rho^n < \tau < \rho^{n-1}$ in M'.

QED(Claim 1)

All that remains is to show:

Claim 2. $\langle Q, F\rangle = N||\mu$ for a $\mu \leq \gamma$.

Proof. We note that if $\langle Q, F\rangle = N||\mu$, then we automatically have $\mu \leq \gamma$, since τ is then a cardinal in $N||\mu$ and γ is maximal s.t. τ is a cardinal in $N||\gamma$.

(1) $\langle Q, F\rangle \in N$.

Proof.$(E^{N_\gamma}_{\nu_j})_\kappa = F_\kappa \in N||\sigma(\tau)$, where $N||\sigma(\tau)$ is a ZFC^- model. Hence $\langle Q, F\rangle \in N||\sigma(\tau)$ since the construction of $\langle Q, F\rangle$ can be carried out in $N||\sigma(\tau)$ by absoluteness.

(2) $\rho^1_{\langle Q,F\rangle} \leq \tau$.

Proof. As above, let $\overline{\pi} : N||\sigma(\tau) \longrightarrow Q$ be the extension map given by F. By §3.2 we know that $\overline{\pi}$ is $\underline{\Sigma}_1(\langle Q, F\rangle)$ and that $\langle Q, F\rangle$ is amenable. But then there is a $\underline{\Sigma}_1(\langle a, \pi\rangle)$ partial map G of $N||\tau$ onto Q defined by: $G(f) = \overline{\pi}(f)(\kappa)$ for $f \in N||\tau, : f : \kappa \longrightarrow N||\tau$.

QED(2)

Define a map $\tilde{\sigma} : \langle Q, F\rangle \longrightarrow N_j||\nu_j$ by:

$$\tilde{\sigma}(\overline{\pi}(f)(\kappa)) := \tilde{\pi}(f)(\kappa) \text{ for } f \in N|\tau, f : \kappa \longrightarrow N||\tau,$$

where $\tilde{\pi} = \pi^N_{t,i}\restriction(N||\tau)$ is the extension of $\langle N_j||\tau, F\rangle$.

Then:

(3) $\tilde{\sigma} : \langle Q, F\rangle \longrightarrow_{\Sigma_0} N_j||\nu_j$. In fact, it is also cofinal.

(4) $\tilde{\sigma}\restriction\tau + 1 = \text{id}$.

Proof. Set:

$$i^+ =: \text{ the least } \eta > i \text{ such that } \eta = \overline{\overline{\eta}} \geq \omega \text{ in } Q$$
$$pl := \langle i^+ : i < \kappa\rangle.$$

Then $\overline{\pi}(pl)(\kappa) = \kappa^{+Q} = \kappa^{+N_j||\nu_j} = \tilde{\pi}(pl)(\kappa)$.

Set:

$$\Gamma =: \{f \in N||\tau : f : \kappa \longrightarrow \kappa \wedge f(i) < i^+ \text{ for } i < \kappa\}$$
$$\dot{<} = \{\langle f, g\rangle \in \Gamma : \{i : f(i) \in g(i)\} \in F_\kappa\}$$

Then every $\xi < \tau$ has the form $\overline{\pi}(f)(\kappa)$ fo an $f \in \Gamma$. Clearly, $f\dot{<}g \longleftrightarrow \overline{\pi}(f)(a) < \pi(g)(a)$ for $f, g \in \Gamma$. Hence by $\dot{<}$-induction on $g \in \Gamma$:

$$\pi(g)(\kappa) = \{\overline{\pi}(\kappa) : f\dot{<}g\}.$$

But $F_\kappa = (E^{N_j}_{\nu_j})_\kappa$. Hence the same holds for $\tilde{\pi}$ in place of $\overline{\pi}$. Thus, by $\dot{<}$-induction on $g \in \Gamma$:

$$\tilde{\pi}(g)(\kappa) = \{\tilde{\pi}(\kappa) : f\dot{<}g\} = \{\pi(\kappa) : f\dot{<}g\} = \overline{\pi}(f)(\kappa).$$

Hence $\tilde{\sigma}\restriction\tau = \text{id}$. But:

$$\tilde{\sigma}(\tau) = \tilde{\sigma}(\overline{\pi}(pl)(\kappa)) = \overline{\pi}(pl)(\kappa) = \tau$$

QED(4)

Redoing the proof of (2) with more care, we get:

(5) $\varnothing \in R^{(\tau)}_{\langle Q,F\rangle}$.

Proof. $X \subset \kappa$ and $X = \kappa$ are both $\Sigma_1(\langle Q,F\rangle)$, since:

$$X \subset \kappa \longleftrightarrow X \in \mathrm{dom}(F),\ X = \kappa \longleftrightarrow X \in \mathrm{On} \cap \mathrm{dom}(F).$$

Thus this suffices to show that $\overline{\pi}$ is $\Sigma_1(\langle Q,F\rangle)$. We note that if $f : X \overset{\text{onto}}{\longrightarrow} u$ and u is transitive, then $\overline{\pi}(f) : \overline{\pi}(X) \overset{\text{onto}}{\longrightarrow} \overline{\pi}(u)$ and $\overline{\pi}(u)$ is transitive. But $\overline{\pi}(X) = F(X)$ for $X \subset \kappa$. Hence $y = \overline{\pi}(x)$ can be expressed by saying that there are:

$$X, Y, f, u, X', Y', f', u'$$

such that:

$$\begin{aligned}
&\bigvee u \bigwedge X, Y \in \mathrm{dom}(F) \wedge f : X \overset{\text{onto}}{\longrightarrow} u \wedge x = f(0)\\
&\wedge \bigwedge \xi, \zeta \in X (f(\xi) \in f(\zeta) \longleftrightarrow \prec \xi, \zeta \succ \in Y)\\
&\wedge X' = F(X) \wedge Y' = F(Y) \wedge f' : X' \overset{\text{onto}}{\longrightarrow} u' \wedge y = f'(0)\\
&\wedge \bigwedge \xi, \zeta \in X' (f'(\xi) \in f'(\zeta) \longleftrightarrow \prec \xi, \zeta \succ \in Y')
\end{aligned}$$

QED(5)

We then prove:

(6) One of the following holds:

(a) $\langle Q,F\rangle = \mathrm{core}_\tau(N_j||\nu_j)$ and $\tilde{\sigma}$ is the core map.

(b) $\langle Q,F\rangle$ is a proper segment of $N_j||\nu_j$

(c) $\rho^\omega > \tau$ in $\langle Q,F\rangle$.

Proof. If $\tilde{\sigma} = \mathrm{id}, \langle Q,F\rangle = N_j||\nu_j$, then (a) holds. Now let $\tilde{\sigma} \neq \mathrm{id}$. Let $\tilde{\lambda} = \mathrm{crit}(\tilde{\sigma})$. Then $\tilde{\lambda} > \tau$ by (4). We know $\rho^1 \leq \tau < \tilde{\lambda}$ in $\langle Q,F\rangle$. Moreover $\tilde{\sigma}$ is Σ_0-preserving. It follows easily that $\tilde{\sigma}$ verifies the phalanx $\langle N_j||\nu_j, \langle Q,F\rangle, \tilde{\lambda}\rangle$. $\langle Q,F\rangle$ is then a mouse. Moreover, it is sound above τ since $\varnothing \notin R^{(\sigma)}_{\langle Q,F\rangle}$. Hence it is sound above $\tilde{\lambda}$ since $\tau < \tilde{\lambda}$. We then coiterate $N_j||\nu_j$ against $\langle N_j||\nu_j, \langle Q,F\rangle, \tilde{\lambda}\rangle$, using all that we have learned up until now. We consider the same three cases. In case 1, (a) holds. In case 2, (b) holds. We now consider case 3, using what we have learned up to now. We know that $\tilde{\lambda}$ is a successor cardinal in $\langle Q,F\rangle$ and that its predecessor $\tilde{\kappa}$ is a limit cardinal in $\langle Q,F\rangle$. Since $\tau < \tilde{\lambda}$ is a successor cardinal in $\langle Q,F\rangle$, we conclude: $\tau < \tilde{\kappa} = \rho^\omega$.

(7) $\langle Q,F\rangle$ is a proper segment of N.

Proof. Suppose not. We derive a contradiction. (c) cannot hold, since $\rho^\omega \leq \tau$ in $\langle Q,F\rangle$. We now show that (b) cannot occur. In fact, we show:

Claim. There is no $i \leq \eta$ such that $\langle Q, F\rangle$ is a proper segment of N_i.

Proof. Suppose not, Then $N_i \neq N$. Hence there is a least $h < i$ which is active in I^N. Then $J^{E^{N_i}}_{\nu_h} = J^{E^N}_{\nu_h}$, where $\nu_h > \tau$ is regular in N_i. But $\rho^\omega_{\langle Q,F\rangle} \leq \tau$. Hence $\langle Q, F\rangle$ is a proper segment of $J^{E^N}_{\nu_h}$, hence of N. Contradiction! QED(Claim)

We now show that (a) cannot occur. If $\nu_j \in N_j$ then $N||\nu_j$ is sound, hence sound above τ. Hence:

$$\langle Q, F\rangle = \mathrm{core}_\tau(N_j||\nu_j) = N_j||\nu_j$$

is a proper segment of N_j. Contradiction! Thus $N_j = N_j||\nu_j$. If there is no truncation on the main branch on $I^M|j+1$, then $N = N_j$. But τ then a cardinal in N_j and not in N. Contradiction! Hence there is a least truncation point $(i+1) \leq_T j$. Let $h = T(i+1)$ and $\pi = \pi_{h,j}$. Then:

$$\pi\colon N_i^* \longrightarrow_{\Sigma^*} N_j, \kappa_i = \mathrm{crit}(\pi),$$

N_j has the form $\langle J^E_\gamma, F'\rangle$. Hence N_i has the form $\langle J^{E^*}_{\nu^*}, F^*\rangle$ where $\kappa_i = \mathrm{crit}(F^*)$, $\tau_i = \tau(F^*)$. But then $\pi(\tau_i) = \tau = \tau(F')$. Hence $\tau \in \mathrm{rng}(\pi)$. Hence $\kappa_i > \tau$, since $(\kappa_i, \lambda_i) \cap \mathrm{rng}(\pi) = \emptyset$. Since N_i^* is sound, being a proper segment of N_h. Hence it is sound above τ. Since $\pi(p^*_{N_i}) = p^*_{N_j}$ and $\pi\restriction\tau = \mathrm{id}$, we conclude:

$$N_i^* = \mathrm{core}(N_j) = \langle Q, F\rangle.$$

But then $\langle Q, F\rangle$ is a proper segment of N_h. Contradiction!

QED(7)

QED(Lemma 4.3.7)

Using the condensation lemma, we prove a sharper version of the initial segment condition for mice:

Lemma 4.3.8. *Let $N = \langle J^E_\nu, F\rangle$ be an active mouse. Let $\overline{\lambda} \in N$. Let $\overline{F} = F|\lambda$ be a full extender. Set:*

$$M = \langle J^E_{\overline{\nu}}, \overline{F}\rangle \text{ where } \overline{\pi} : J^E_\tau \longrightarrow J^E \text{ is the extension of } \vec{F}$$

. Then M is a a proper segment of N.

Proof. Let $\kappa = \mathrm{crit}(F)$. Define $\tau = \tau_F, \lambda = \lambda_F, \nu = \nu_F$ as usual. Hence: $\tau = \kappa^{+N}, \lambda = F(\lambda)$. Then $\overline{\tau} = \tau_{\overline{F}}, \overline{\lambda} = \lambda_{\overline{F}}, \overline{\nu} = \nu_{\overline{F}}$. Let $\pi : J^E_\tau : J^E_\nu$ be the extension of F. Define: $\sigma : J^E_{\overline{\tau}} \longrightarrow J^E_\tau$ by:

$$\sigma(\overline{\pi}(f)(\alpha)) = \pi(f)(\alpha) \text{ for } \alpha < \overline{\lambda}, f \in J^E_\tau, \mathrm{dom}(f) = u.$$

Then $\overline{\lambda} = \text{crit}(\lambda), \sigma(\overline{\lambda})$ and σ is Σ_0-preserving, where:

$$\rho_M^\omega \leq \overline{\lambda} \text{ and } \varnothing \notin R_M^{(\overline{\lambda})}.$$

This is because $\overline{\pi}$ is $\Sigma_1(M)$ and each element of M has the form $\overline{\pi}(f)(\alpha)$ where $f \in J_\tau^E$ and $\alpha < \overline{\lambda}$. It follows easily that σ witnesses the phalanx $\langle N, M, \overline{\lambda}\rangle$. Applying the condensation lemma, we see that one of the possibilities (a), (b), (c) holds. (c) cannot hold since $\overline{\lambda}$ is a limit cardinal in M. (a) cannot hold, since $M \in N$ by the initial segment condition. If (a) holds, we would have: $\sigma(p_M^*) = p_N^*, \sigma \restriction \overline{\lambda} = \text{id}$, where σ is Σ^*-preserving. But then $\rho_M^\omega = \rho_N^\omega$. Let $\rho = \rho_N^\omega$. Let A be $\Sigma^*(N)$ in p_N^* such that $A \cap \rho \notin N$. Let $\overline{A}$ be $\Sigma^*(M)$ in p_M^* by the same defition. Then:

$$A \cap \rho = \overline{A} \cap \rho \in \underline{\Sigma}^*(M) \subset N.$$

Contradiction! Thus, only the possibility (b) remains.

QED(Lemma 4.3.8)

As a corollary of the proof of Lemma 4.3.7, we obtain a lemma which will be very useful in the next chapter. We first define:

Definition 4.3.4. Let M be a premouse. Set:

$$\rho = \rho_M =: \rho_M^\omega, \mu = \mu_M = \{\xi \in M \mid \text{card}(\xi) \leq \rho \text{ in } M\}.$$

Lemma 4.3.9. *Let N be a fully iterable premouse. Let $M = \text{core}(N)$. Let $\mu = \mu_M$. Then $\mu = \mu_N$ and $M||\mu = N||\mu$.*

Proof. If $N = M$ there is nothing to prove, so assume $N \neq M$. Let $\sigma : M \longrightarrow N$ be the core map. Since $\sigma \neq \text{id}$, it has a critical point λ. Clearly $\lambda \geq \rho = \rho_M = \rho_N$, since $\sigma|\rho = \text{id}$. It is easily seen that σ verifies the phalanx $\langle N, M, \lambda\rangle$. Note that the two possibilities (b), (c) in the condensation lemma(4.3.7) cannot hold, since (b) would require: $M \in N$ and (c) would imply that M is unsound. Coiterate $\langle N, M, \lambda\rangle$, N to get I^M, I^N as in the proof of lemma 4.3.7. Then the cases 2 and 3 cannot hold, since then either (b) or (c) would follow. Hence case 1 holds-i.e. $M_\zeta = N_\zeta$ and I^N has no truncation on its main branch. We know that I^M has no truncation on its main branch, where $\kappa_i \geq \lambda \geq \rho$ for i on the main branch. Thus $\rho = \rho_{N_\zeta}$ and $\kappa_i > \rho$ for all i.

Then $\mu = \mu_M = \rho^{+M} = \rho^{+N_\zeta}$ and $M||\mu = N_\zeta||\mu$. Now suppose $\kappa_i = \rho$, where $i+1$ is the first point above 1 on the main branch. Then $\pi_{1,i+1} : M \longrightarrow_{E_{\nu_i}^{M_i}} M_{i+1}$ where $\rho = \rho_{M_{i+1}}$ and $\mu = \tau_i = \rho^{+M}$. But then $\tau_i = \rho^{+M_{i+1}}$ and $M||\tau_i = M_{i+1}||\tau_i$. Since $\kappa_j \geq \lambda_i$ for $j+1$ on the main branch with

$j+1 >_T i+1$, we conclude: $\tau_i = \rho^{+M_\zeta} = \mu_{N_\zeta}$ and $M||\tau_i = N_\zeta||\tau_i$, since $\pi^M_{i+1,\zeta}|\lambda_i = \mathrm{id}$. We have shown:

Claim 1. $\mu = \mu_{N_\zeta}$ and $M||\mu = N_\zeta||\mu$.

But since $\rho = \rho^\omega_{N_\zeta}$ we must have $\kappa_i \geq \rho$ for all $i+1$ on the main branch of I^N, since otherwise $\pi^N_{0\zeta}(\rho) = \rho^\omega_{N_\zeta} > \rho$. Hence we can respect the above proof on the N-side to get:

Claim 2. Let $\mu = \mu_N$. Then $\mu = \mu_{N_\zeta}$ and $N||\mu = N_\zeta||\mu$.

QED(Lemma 4.3.9)

We have defined $\mu = \mu_M$ in such a way that $\mu \notin M$ is possible.In fact we could have: $\rho = \mu = \mathrm{ht}(M)$. However, by the above proof:

Lemma 4.3.10. *Let N be fully iterable and $N \neq M = \mathrm{core}(N)$. Then for all fully iterable N' with $M = \mathrm{core}(N')$ we have:*

Let $\mu' = \mu_{N'}$. Then $\mu' \in N'$ and $\mu = \rho^{+N'}$.

We also note:

Lemma 4.3.11. *Let J^A_α be a constructible extension of J^A_β(i.e. $\beta \leq \alpha$ and $A \subset J^A_\beta$). Assume: $\rho = \rho^{J^A_\alpha} \geq \beta$. Then $J^A_\alpha = \mathrm{core}(J^A_\alpha)$ and $\sigma = \mathrm{id}$ is the core map.*

4.4 Mouselikeness

In §3 we showed that every normally iterable premouse which has the unique branch property is fully iterable. In the present chapter we have derived several deep structural properties of fully iterable premice. We shall call a premouse which has these properites *mouselike*, be it iterable or not. We define:

Definition 4.4.1. Let N be a premouse. N is *condensable* if and only if

(i) N is solid

(ii) Let $M = \mathrm{core}(N)$, $\rho = \rho^\omega_M = \rho^\omega_N$ and $\mu = \rho^{+N}$. Then $\mu = \rho^{+M}$ and $M||\mu = N||\mu$.

(iii) Let σ witness the phalanx $\langle N, M, \lambda\rangle$, where M is sound above λ. Then one of the alternatives (a), (b), (c) in lemma 4.3.7 hold.

Definition 4.4.2. N is *mouselike* if and only if every initial segment $N' \triangleleft N$ is condensible.

Definition 4.4.3. N is *precondensible*(or *pre-mouselike*) if and only if every *proper* initial segment $N' \triangleleft N$ is condensible.

We have seen that every fully iterable premouse W is condensible. Since every $N' \triangleleft N$ is then also fully iterable, we conclude that N is mouselike.

The definition of "condensible" becomes simpler if we assume N to be sound and solid. The conditions (i), (ii) are then vacuously true. (iii) then says that, if σ witnesses $\langle N, M, \lambda\rangle$ and M is sound above λ, then either (b) or (c) hold. (If (a) holds, then $M = \text{core}_\lambda(M)$ and $\sigma = \sigma_M^\lambda$. But by soundness, $M = \text{core}(M)$ and $\sigma_M^\lambda = \sigma_M = \text{id}$, contradicting the fact that $\lambda = \text{crit}(\sigma)$.)

In §4.1 we defined a premouse to be *presolid* if and only if all of it's proper initial segments are solid. Lemma 4.1.13 said that the property of being presolid is uniformly $\Pi_1(M)$ for premice M. Hence:

Lemma 4.4.1. *Let M, N be premice. Then*

- *If M is presolid and $\pi : M \longrightarrow_{\Sigma_1} N$, then N is presolid.*
- *If N is presolid and $\pi : M \longrightarrow_{\Sigma_0} N$, then M is presolid.*

We shall prove:

Lemma 4.4.2. *The property of being pre-mouselike is uniformly $\Pi_1(M)$ for premice M.*

Hence:

Lemma 4.4.3. *Let M, N be premice. Then:*

- *If M is pre-mouselike and $\pi : M \longrightarrow_{\Sigma_1} N$, then N is pre-mouselike.*
- *If N is pre-mouselike and $\pi : M \longrightarrow_{\Sigma_0} N$, then M is pre-mouselike.*

As preparation for the proof of lemma 4.4.2, we list a series of facts which are implicit in what we have done this far, but may not always have been made explicit.

Definition 4.4.4. $M = \langle |M|, E, F\rangle$ is a set *model* if and only if $|M|$ is transitive and $E, F \subset |M|$.

(*Note* we can, of course, generalize this to models with more than two predicates.)

In the following let U be any set which is transitive and closed under rudimentary functions.

Fact 1. The set $\{M \in U : M \text{ is a model}\}$ is uniformly $\Delta_1(U)$.

Models have a first order language $\mathbb{L}$ with predicate symbols $\dot{\in}, \dot{=}, \dot{E}, \dot{F}$. $\dot{\in}, \dot{=}$ are interpreted by $\in, =$ respectively and $\dot{E}, \dot{F}$ by E, F. We assume an "arithmetization" of $\mathbb{L}$, whereby the formulae of $\mathbb{L}$ are identified with objects in ω or V_ω in such a way that the normal syntactic relation and operation become recursive. (In §1.4.1 we proposed an arithmetization of languages over an admissible set. If we take the admissible set as V_ω, we get a suitable arithmetization of $\mathbb{L}$.)

Definition 4.4.5. The *satisfaction relation* is defined as follows: $M \models \varphi[f]$ means:

- M is a model
- φ is a formula of $\mathbb{L}$.
- f is a *variable interpretation* -i.e. f is function such that $\mathrm{dom}(f)$ is a finite set of variables and $\mathrm{ran}(f) \subset M$
- All variables occurring free in φ lie in $\mathrm{dom}(f)$
- φ becomes a true statement in M if each $v \in \mathrm{dom}(f)$ is interpreted by $f(v)$.

(*Note* informally we write: $M \models \varphi[a_1, \ldots, a_m/v_1, \ldots, v_m]$ for $M \models \varphi[f]$ where $\mathrm{dom}(f) = \{v_1, \ldots, v_m\}$ and $a_i = f(v_i)$ for $i = 1, \ldots, n$. When the context permits, it is customary to omit the list of variables and write: $M \models \varphi[a_1, \ldots, a_m]$.)

Fact 2. $\{\langle M, \varphi, f\rangle \mid M \in U \wedge M \models \varphi[f]\}$ is uniformly $\Delta_1(U)$.

Definition 4.4.6. A model M is *amenable* if and only if $\bigwedge x \in M(E \cap x, F \cap x \in M)$.

Definition 4.4.7. M is a J-model if and only if M is amenable and $|M| = J_\alpha[E]$ where $\alpha = \mathrm{On} \cap |M|$.

(Note: we write $\mathrm{ht}(M)$ for $\mathrm{On} \cap |M|$.)

Fact 3. There is a Π_2 sentence φ such that

$$M \text{ is a } J\text{-model} \longleftrightarrow M \models \varphi.$$

(Hence $\{M \in U \mid M \text{ is a } J\text{-model}\}$ is uniformly $\Delta_1(U)$.)

Definition 4.4.8. M is acceptable if and only if it is a J-model and, whenever $\eta \geq \omega$ is a cardinal in M(i.e. $\eta < \mathrm{ht}(M)$ and for all $\xi < \eta$ there is no $f \in M$ mapping ξ onto η.), then:

$$\bigwedge \xi < \eta \ \mathbb{P}(\xi) \cap M \subset J^E_\eta.$$

Fact 4. There is a Π_2 sentence φ such that for J-model M:

$$M \text{ is acceptable} \longleftrightarrow M \models \varphi$$

(Hence $\{M \in U \mid M \text{ is acceptable}\}$ is uniformly $\Delta_1(U)$.)

In §1.6 we expanded the language $\mathbb{L}$ to a *many sorted* language $\mathbb{L}^*$ which is more suitable for analyzing acceptable structures N. $\mathbb{L}^*$ contains variables of type n for $n < \omega$, two original variables of $\mathbb{L}$ being of type 0. Variables of type i range over $N^i = J^E_{\rho^i_N}$, where $\rho^i \leq \mathrm{ht}(N)$ and $\rho^0 = \mathrm{ht}(N)$. We then defined an appropriate satisfaction relation for $\mathbb{L}^*$ formulae. $R(x_1^{i_1}, \dots, x_n^{i_n})$ is an $\mathbb{L}^*$-*definable relation* on N(with arity $\langle i_1, \dots, i_n \rangle$) if and only if there is an $\mathbb{L}^*$-formula $\varphi(v_1^{i_1}, \dots, v_n^{i_n})$ with:

$$R(\vec{x})) \longleftrightarrow N \models \varphi[\vec{x}].$$

We defined a hierarchy $\Sigma_n^{(m)}(n = 0, 1)$ of $\mathbb{L}^*$-formulas and defined a $\Sigma_n^{(m)}(N)$ relation to be a relation which is N-definable by a $\Sigma_n^{(m)}$-formula. This hierarchy is better suited to acceptable structures than the Levy hierarchy.

The following fact is implicit in §2.6. As far as we can tell, however, we have hitherto not stated it explicitly, although we have made tacit use of it(for instance in the proof of Lemma 4.1.13).

Fact 5. Let N be acceptable. Let $\varphi(v_1^{i_1}, \dots, v_m^{i_m})$ be any formula in the many sorted language $\mathbb{L}^*$ developed in §2.6. There is a formula $\tilde{\varphi}$ in the first order language $\mathbb{L}$ of N such that

$$N \models \varphi[x_1, \dots, x_m] \longleftrightarrow N \models \tilde{\varphi}[x_1, \dots, x_m]$$

for $x_j \in H_N^{i_j}(j = 1, \dots, m)$. Moreover the function $\varphi \mapsto \tilde{\varphi}$ is recursive.

Proof(sketch). Let $\mathbb{L}^m$ consist of formulas with variables of type $i \leq m$. By induction on m, we construct the function $\varphi \mapsto \tilde{\varphi}$ for $\varphi \in \mathbb{L}^m$. It clearly suffices to have $\tilde{\rho}^i, \tilde{H}^i$ $(i \leq m)$, since we can then form $\tilde{\varphi}$ by replacing $\bigwedge v^i \dots$ by $\bigwedge v(\tilde{H}^i v \to \dots)$ everywhere. We proceed by induction on m. The case $m = 0$ is trivial, since $\mathbb{L}^0$ is the set of non sorted formulas in the language of N. Moreover we have: $\rho^0 = \mathrm{ht}(N)$, $H^0 = |N|$. Now let it hold at m.

Let $T^m(x_m, \ldots, x_0)$ be the predicate defined in §2.6 preceding the proof of lemma 2.6.17. Set:

$$T'(i, z, \vec{x}) \longleftrightarrow \langle J^E_{\rho^m}, T^{m,x_{m-1},\ldots,x_0} \rangle \models \varphi_i[z, x_m]$$

where $T^{m,x_{m-1},\ldots,x_0} = \{y \mid T^m(y, x_{m-1}, \ldots, x_0)\}$ and $\langle \varphi_i \mid i < \omega \rangle$ is a fixed enumeration of Σ_1 formulae with two free variables. Thus T' is $\Sigma_1^{(m)}(N)$. Moreover, it is *universal* in the sense that, if D is any $\underline{\Sigma}_1^{(m)}(N)$ subset of H^m, then there are $i < \omega$, $\vec{x}$ such that

$$D(z) \longleftrightarrow T'(i, z, \vec{x}).$$

But then:

$$\xi < \rho^{m+1} \longleftrightarrow \bigwedge i < \omega \bigwedge \vec{x} (T_i^{\vec{x}} \cap \xi) \cap \xi \in N$$

and:

$$x \in H^{m+1} \longleftrightarrow \bigvee \xi < \rho^m x \in J^E_\xi.$$

These definitions of ρ^n, H^n are by formulae lying in $\mathbb{L}^m$. That gives us $\tilde{\rho}^{m+1}, \tilde{H}^{m+1}$.

QED(Fact 5)

In §2.6.3 we introduced the class of *m-sound* acceptable models. N is *sound* if and only if it is m-sound for every $m < \omega$.

Fact 6. For $m < \omega$ there is an $\mathbb{L}$-sentence φ_m such that,

$$N \text{ is } n\text{-sound} \longleftrightarrow N \models \varphi_m.$$

Moreover $m \mapsto \varphi_m$ is a recursive function. Hence $\{N \in U \mid N \text{ is sound}\}$ is uniformly $\Pi_1(U)$.

In §3.3 we introduced the class of *premice* and proved:

Fact 7. There is an $\mathbb{L}$-sentence φ such that

$$N \text{ is a premouse } \longleftrightarrow N \models \varphi.$$

(Hence $\{N \in U \mid N \text{ is a premouse}\}$ is uniformly $\Delta_1(U)$.)

In §4.1 we defined the class of *m-solid* premice. We call N *solid* if and only if it is m-solid for all $m < \omega$. Using Fact 5:

Fact 8. For $m < \omega$ there is an $\mathbb{L}$-sentence φ_m such that

$$N \text{ is } m\text{-solid } \longleftrightarrow N \models \varphi_m.$$

Moreover $m \mapsto \varphi_m$ recursive. (Thus $\{N \in U \mid N \text{ is solid}\}$ is uniformly $\Pi_1(U)$.)

In §4.3.2 we defined what it means for a premouse N to be *sound above* λ, where $\lambda \in N$. The definition was equivalent to:

Definition 4.4.9. Let $\lambda \in N$. N is *m-sound above* λ if and only if

- $\rho^m \leq \lambda < \rho^{m-1}$ and N is i-sound for $i < m$.
- Let $a \in P_N^m$. Set $b = a \cap \rho_N^{m-1}$, $\bar{N} = N^{n,a \setminus \rho^{m-1}}$. Then every $x \in \bar{N}$ has the form $h(i, \langle \xi, b \rangle)$ where $i < \omega$, $\xi < \lambda$ and h is the canonical Σ_1-Skolem function for $\bar{N}$.

Definition 4.4.10. N is *sound above* λ if and only if it is m-sound above λ for some m.

By Fact 5 it follows that:

Fact 9. Let $\lambda \in N$. For each $m < \omega$ there is a formula $\varphi_m \in \mathbb{L}$ such that

$$N \text{ is } m\text{-sound above } \lambda \text{ if and only if } N \models \varphi_m[\lambda].$$

Moreover, the function $m \mapsto \varphi_m$ is recursive. Hence:

Fact 10.

- $\{\langle N, \lambda \rangle \in U \mid N \text{ is } m\text{-sound above } \lambda\}$ is $\Delta_1(U)$
- $\{\langle N, \lambda \rangle \in U \mid N \text{ is sound above } \lambda\}$ is $\Sigma_1(U)$

In §4.2 we defined what it means to say that σ witnesses the phalanx $\langle N, M, \lambda \rangle$. We aim to prove the following lemma, which in turn, implies lemma 4.4.2:

Lemma 4.4.4. *Let N be sound and solid. Let $N \in U$, where U is transitive and rudimentarily closed. 'N is condensible' is uniformly $\Pi_1(U)$ in the parameter N.*

The proof will stretch over several sublemmas. U could be quite small- e.g. it could be the closure of $|N| \cup \{N\}$ under rudimentary functions. We call $\langle \sigma, M, \lambda \rangle$ a *counterexample* to the condensibility of N if σ witnesses $\langle N, M, \lambda \rangle$, M is sound above λ, and (b), (c) both fail. At first glance it might seem that there could be a counterexample in V which is not in U. But this is not so:

Lemma 4.4.5. *Let σ witness $\langle N, M, \lambda\rangle$, where M is sound above λ. Then $M \in N$ and $\sigma \in U$.*

Proof. Let $\rho^n \le \lambda < \rho^m$ in M, where $n = m+1$. Let $\bar{a} \in [\mathrm{ht}(M)]^{<\omega}$ such that, letting $\bar{a}^{(i)} = \bar{a} \cap \rho^i$ for $i = 0, \dots, m$, we have:

- Every $x \in M^{m,\bar{a}}$ is $\Sigma_1(M^{m,a})$ in parameters $\bar{a}^{(m)}$, ξ such that $\xi < \lambda$
- $\bar{a}^{(i)} \in R^i_M$ for $i < m$.

Set: $a = \sigma(\bar{a})$, $a^{(i)} = \sigma(\bar{a}^{(i)}) = a \cap \rho^i_N$. Then $\sigma|M^{m,\bar{a}} : M^{n,\bar{a}} \longrightarrow_{\Sigma_0} N^{n,a}$ and $\bar{a}$, M is the unique pair b, Q such that $b \in R^m_Q$ and $Q^{m,b} = M^{m,\bar{a}}$. Moreover σ is the unique $\sigma \supset \sigma|M^{m,\bar{a}}$ such that $\sigma(\bar{a}) = a$ and $\sigma : M \longrightarrow_{\Sigma^{(n)}_0} N$ strictly. We consider two cases:

Case 1. $m = 0$(Hence $N = N^{m,a}$, $M = M^{m,\bar{a}}$)

We consider two subcases:

case 1.1. $\sup \sigma\text{"}\rho^0_M < \rho^0_N$. Set:

$$\tilde{\rho} = \sup \sigma\text{"}\rho^0_M ; \tilde{N} = N|\tilde{\rho} = \langle J^{E^N}_{\tilde{\rho}}, E^N_\nu \cap J^{E^N}_{\tilde{\rho}}\rangle$$

where $\nu = \rho^0_N = \mathrm{ht}(N)$. Then $\tilde{N}$ is amenable and $\tilde{N} \in N$, since N is amenable. We have: $\sigma : M \longrightarrow_{\Sigma_1} \tilde{N}$ cofinally. Let $\tilde{h} = h_{\tilde{N}}$, $h = h_M$. Clearly $a = \sigma(\bar{a}) \in \tilde{N}$. Set:

$$\tilde{h}^a(\xi) \simeq \tilde{h}((\xi)_0, \langle(\xi)_1, a\rangle) \text{ for } \xi < \lambda,$$

where $\xi =: \prec (\xi)_0, (\xi)_1 \succ$. Set:

$$h^{\bar{a}}(\xi) \simeq h_M((\xi)_0, \langle(\xi)_1, \bar{a}\rangle) \text{ for } \xi < \lambda.$$

Then $\sigma(h^{\bar{a}}(\xi)) \simeq \tilde{h}^a(\xi)$. Set: $\tilde{M} = \langle|\tilde{M}|, \tilde{\in}, \tilde{=}, \tilde{E}, \tilde{F}\rangle$, where:

- $|\tilde{M}| =: \mathrm{dom}(\tilde{h}^a)$
- $\xi \tilde{\in} \zeta \longleftrightarrow: \tilde{h}^a(\xi) \in \tilde{h}^a(\zeta)$
- $\xi \tilde{=} \zeta \longleftrightarrow: \tilde{h}^a(\xi) = \tilde{h}^a(\zeta)$
- $\tilde{E}\xi \longleftrightarrow: \tilde{h}^a(\xi) \in E^N$
- $\tilde{F}\xi \longleftrightarrow: \tilde{h}^a(\xi) \in E^N_\nu$

Then:

(1) $\tilde{M} \in N$, since $\tilde{N} \in N$. $h^{\bar{a}}$(hence M) is recoverable from $\tilde{M}$ by the recursion:

$$h^{\bar{a}}(\xi) = \{h^{bara}(\zeta) \mid \zeta \tilde{\in} \xi\} \text{ for } \xi \in \tilde{M}.$$

λ is easily seen to be a regular cardinal in M, since $\sigma(\lambda) > \lambda$. Hence $\sigma(\lambda)$ is a regular cardinal in N. Hence:

$$|\tilde{M}| \in \mathbb{P}(\lambda)_N \subset J^{E^N}_{\sigma(\lambda)}$$

by acceptability. Hence M can be recovered from $\tilde{M}$ in the ZFC^- model $J^{E^N}_{\sigma(\lambda)}$. Hence:

(2) $M \in N$

But then:

$$\sigma = \{\langle \tilde{h}^a(\xi), h^{\bar{a}}(\xi)\rangle \mid \xi \in |M|\}$$

where $\tilde{h}^a, h^{\bar{a}} \in N$. Thus:

(3) $\sigma \in \underline{\Sigma}_\omega(N) \subset U$.

QED(Case 1.1)

Case 1.2. Case 1.1 fails.

Then $\tilde{N} = N, \tilde{h}^a = h^a$, where $h^a(\xi) \simeq h_N((\xi)_0, \langle(\xi)_1, a\rangle)$ for $\xi < \lambda$. We have $\sigma : M \longrightarrow_{\Sigma_1} N$ cofinally.

Case 1.2.1. $\lambda < \rho^\omega_N$.

Then $\tilde{M} \in N$, since $\langle J^{E^N}_{\rho^\omega_N}, B\rangle$ is amenable whenever $B \subset J^{E^N}_{\rho^\omega_N}$ is $\underline{\sigma}^*(N)$. The rest of the proof is exactly like Case 1.1.

QED(Case 1.2.1)

Case 1.2.2. The above cases fail.

Then $\rho^\omega \leq \lambda$ in N. We conclude that:

(4) $p^* \setminus \lambda \not\subset a$, where $p^* = p^*_N$.

Proof. If not, $\rho^\omega \cup p^* \subset \operatorname{ran}(\sigma) \prec_{\Sigma_1} N$. But then $M = N$, $\sigma = \operatorname{id}$ by the soundness of N. Contradiction! Since $\lambda = \operatorname{crit}(\sigma)$.

QED(4)

Let $\eta \in (p^* \setminus \lambda) \setminus a$ be maximal.(Hence $\eta \geq \lambda$) Then $a \setminus \eta = p^* \setminus (\eta+1)$. Let $\rho^{i+1} \leq \eta < \rho^i$ in N.(Since we core in Case 1, we know that $i = 0$, but we preserve the more general formulation for later use.) Let $X = h_{N^{i,a\setminus\eta}}(\eta \cup (a \setminus \eta))$. Let $\bar{\pi} : Q' \stackrel{\sim}{\longleftrightarrow} X$ be the transitivation of X. Then $\bar{\pi} : Q' \prec_{\Sigma_1} N^{i,a\setminus\eta}$ and by solidity we have: $\bar{N} \in N$, where $\bar{N}, b$ are the unique objects such that $\bar{N}^{i,b} = Q'$. Moreover; there is unique $\pi \supset \bar{\pi}$ such that

$$\pi : \bar{N} \longrightarrow_{\Sigma_1^{(i)}} N \text{ and } \pi(b) = a \setminus \eta.$$

In the present case we know that $i = 0$ and $\eta \geq \lambda$. Let $\pi^{-1}(a) = b' = b \cup (a \cap (\lambda, \eta))$.

$$\pi^{-1}(a) = b' = b \cup (a \cap (\lambda, \eta)).$$

Set: $h^{b'}(\xi) \simeq h_{\bar{N}}((\xi)_0, \langle (\xi)_1, b' \rangle)$ for $\xi < \lambda$. Then $|\tilde{M}| = \text{dom}(h^{b'})$ and:

$$\xi \tilde{\in} \zeta \longleftrightarrow h^{b'}(\xi) \in h^{b'}(\zeta) \text{ for } \xi, \zeta \in \lambda,$$

etc. Thus $\tilde{M} \in N$, since $\bar{N} \in N$. The rest of the proof is exactly as in Case 1.1.

QED(Case 1)

Case 2. $m > 0$. Let $m = r + 1$.

There is a good $\Sigma_1^{(m)}(M)$ function $\bar{G}$ such that each $x \in M$ has the form $\bar{G}(\zeta, \bar{a})$ for an $\zeta < \rho_M^m$. Let G be a good $\Sigma_1^{(m)}(N)$ function by the same good definition. Then:

$$\sigma(\bar{G}(\zeta, \bar{a})) \simeq G(\sigma(\zeta), a) \text{ for } \zeta < \rho_M^m.$$

Set: $\bar{Q} = M^{m,\bar{a}}$, $Q = N^{m,a}$. Then $\sigma|\bar{Q} : \bar{Q} \longrightarrow_{\Sigma_0^{(m)}} Q$. Let $\tilde{\rho} = \sup \sigma" \rho_M^m$. Set:

$$\tilde{Q} = Q|\tilde{\rho} =: \langle J_{\tilde{\rho}}^{E^N}, T_N^{m,a} \cap J_{\tilde{\rho}}^{E^N} \rangle.$$

Then $\sigma : \bar{Q} \longrightarrow_{\Sigma_1^{(m)}} \tilde{Q}$ cofinally. We now set:

- $h^{\bar{a}}(\xi) \simeq h_{\bar{Q}}((\xi)_0, \langle (\xi)_1, \bar{a} \rangle)$.
- $\tilde{h}^a(\xi) \simeq h_{\tilde{Q}}((\xi)_0, \langle (\xi)_1, a \rangle)$.
- $\bar{G}^{\bar{a}}(\xi) \simeq \bar{G}(\bar{h}^{\bar{a}}(\xi), \bar{a})$.
- $\tilde{G}^a(\xi) \simeq G(\tilde{h}^a(\xi), a)$.

Then $\sigma(\bar{G}^{\bar{a}}(\xi)) = \tilde{G}^a(\xi)$ for $\xi < \lambda$. Moreover, each $x \in M$ has the form $\bar{G}^{\bar{a}}(\xi)$ for an $\xi < \lambda$. Set:

- $\tilde{M} = \mathrm{dom}(\tilde{G}^a)$
- $\xi \tilde{\in} \zeta \longleftrightarrow \tilde{G}^a(\xi) \in \tilde{G}^a(\zeta)$ for $\xi, \zeta < \lambda$
- $\xi \tilde{=} \zeta \longleftrightarrow \tilde{G}^a(\xi) = \tilde{G}^a(\zeta)$ for $\xi, \zeta < \lambda$
- $\tilde{E}\xi \longleftrightarrow \tilde{G}^a(\xi) \in E^N$ for $\xi < \lambda$
- $\tilde{F}\xi \longleftrightarrow \tilde{G}^a(\xi) \in E^N_\nu$ for $\xi < \lambda$, where $\nu = \mathrm{ht}(N)$.

Then $\tilde{M}/\tilde{=}$ is isomorphic to M and the function $\tilde{G}^a$ is obtainable from $\tilde{M}$ by the recursion:

$$\bar{G}^a(\xi) = \{\tilde{G}^a(\zeta) \mid \zeta \tilde{\in} \xi\}.$$

Hence it suffices to prove:

Claim. $\tilde{M} \in N$.

Since just as before we will then have:

$$|\tilde{M}| \in \mathbb{P}(\lambda) \cap N \subset J^{E^N}_{\sigma(\lambda)}$$

and we can recover M from $\tilde{M}$ in the ZFC^- model $J^{E^N}_{\sigma(\lambda)}$ by the above recursion. But then: $\sigma = \{\langle \tilde{G}^a(\xi), \bar{G}^{\bar{a}}(\xi)\rangle \mid \xi \in |\tilde{M}|\}$. Hence $\sigma \in \underline{\Sigma}_\omega(N) \subset U$ by the above Fact. We prove the Claim by cases as before:

Case 2.1. $\tilde{\rho} < \rho^m_N$.

Then $\tilde{M} \in N$, since $N^{m,a}$ is amenable.

Case 2.2. Case 2.1 fails.

Case 2.2.1. $\lambda < \rho^\omega_N$.

Then $\tilde{M} \in N$ for the same reason as before.

Case 2.2.2. The above cases fail.

Just as before we conclude:

(5) $p^* \setminus \lambda \not\subset a$.

We again let η be maximal. Let $\rho^{i+1} \leq \eta < \eta^i$ in N. Hence $i \leq m$. As before let:

$$X = h_{N^{i,a \setminus \eta}}(\eta \cup (a \setminus \eta)).$$

Let $\bar{\pi}' : Q' \overset{\sim}{\longleftrightarrow} X$ be the transitivation of X. Then $\bar{\pi}' : Q' \prec_{\Sigma_1} N^{i,a\setminus\eta}$. But then as before, $N' \in N$, where N', b are the unique objects such that $N'^{i,b} = Q'$. Moreover, there is a unique $\pi' \supset \bar{\pi}'$ such that

$$\pi' : N' \longrightarrow_{\Sigma_1^{(i)}} N \text{ and } \pi'(b) = a \setminus \eta.$$

Let $\pi'^{-1}(a) = a' = b \cup (a \cap (\lambda, \eta))$. Now let $Q = N'^{m,a'}$. Let $G'(\zeta, a')$ be $\Sigma_1^{(m)}(N')$ by the same good definition as $G(\zeta, a)$. Then:

$$\pi'(G'(\zeta, a')) = G(\pi'(\zeta), a)$$

for $\zeta < \rho^m_{N'}$. Let $\rho' = \sup \pi'^{-1} \circ \sigma" \rho^m_M$. Set:

$$Q' = Q|\rho' =: \langle J^{E^{N'}}_{\rho'}, T^{m,a'}_{N'} \cap J^{E^{N'}}_{\rho'} \rangle.$$

$$h'^{a'}(\xi) \simeq h_{Q'}((\xi)_0, \langle (\xi)_1, a' \rangle)$$

for $\xi < \lambda$. Set:

$$G'^{a'}(\xi) \simeq G'(h'^{a'}(\xi), a') \text{ for } \xi < \lambda.$$

Then: $|\tilde{M}| = \mathrm{dom}(G'^{a'})$, $\xi \tilde{\in} \zeta \longleftrightarrow G'^{a'}(\xi) \in G'^{a'}(\zeta)$ for $\xi, \zeta < \lambda$, etc. But since $N' \in N$, we conclude $\tilde{M} \in N$.

QED(Lemma 4.4.5)

Tweaking this proof a bit, we get:

Lemma 4.4.6. *For each $n < \omega$ there is a formula $\varphi_n \in \mathbb{L}$ such that for all sound and solid N, $N \models \varphi_n[M, \lambda, \tilde{\lambda}]$ if and only if there is σ witnessing $\langle N, M, \lambda \rangle$ such that the following hold:*

- $\rho^{n+1} \leq \lambda < \rho^n$ *in* M
- M *is sound above* λ
- $\tilde{\lambda} = \sigma(\lambda)$

Proof. $N \models \varphi_n[M, \lambda, \tilde{\lambda}]$ says that there are $a, \bar{a}, b, \bar{b}$ such that

- $a \in [\rho^0_N]^{<\omega}, \bar{a} \in [\rho^0_M]^{<\omega}$
- $b = a \cap \rho^n_N, \bar{b} = \bar{a} \cap \rho^n_M$
- $\bar{a} \in P^{n+1}_{\bar{M}}$ and $\rho^{n+1} \leq \lambda < \rho^n$ in M

- M is sound above λ
- $M^{n,\bar{a}} \models \varphi[\vec{\xi}, \bar{b}] \rightarrow N^{n,a} \models \varphi[\vec{\xi}, \tilde{\lambda}, b]$ for all Σ_0 formulas φ and all $\xi_0, \ldots, \xi_{n-1} < \lambda$.
- $\tilde{\lambda} > \lambda$
- For $m = 0$: Let $h, \bar{h}$ be the Skolem function for N, M respectively. If $\bar{h}(i, \langle \xi, \bar{a} \rangle)$ is a cardinal in M, then $h(i, \langle \xi, a \rangle)$ is a cardinal in N(where $\xi < \lambda$).

We see that this can be expressed by an $\mathbb{L}$-formula φ_n using Fact 5 and the facts:

- M-satisfaction relation is uniformly $\Delta_1(N)$ in M
- $N^{n,a}$ satisfaction relation for Σ_0-formulae is uniformly $\Sigma_1(N^{n,a})$.

The direction $(\leftarrow)$ of an equivalence then follows easily by lemma 4.4.5. To prove the other direction we note that if h is the canonical Skolem function for $N^{n,a}$ and $\bar{h}$ is the Skolem function for $M^{n,\bar{a}}$, then for all $\xi < \lambda$:

$$\langle i, \langle \xi, \bar{b}, \lambda \rangle \rangle \in \mathrm{dom}(\bar{h}) \longrightarrow \langle i, \langle \xi, b, \tilde{\lambda} \rangle \rangle \in \mathrm{dom}(h).$$

Hence we can define $\bar{\sigma} : M^{n,\bar{a}} \longrightarrow_{\Sigma_0} N^{n,a}$ by:

$$\bar{\sigma}(\bar{h}(i, \langle \xi, \bar{b}, \lambda \rangle)) = \begin{cases} h(i, \langle \xi, b, \tilde{\lambda} \rangle), & \text{if } \bar{h}(i, \langle \xi, \bar{b}, \lambda \rangle) \text{ is defined;} \\ \text{otherwise undefined.} \end{cases}$$

Applying the downward extension lemma, we get:

There are unique M', a' with $M'^{n,a'} = M^{n,\bar{a}}$ and $a' \in R^n_{M'}$.

By uniqueness we conclude: $M' = M, a' = \bar{a}$. But then there is a unique $\sigma' \supset \bar{\sigma}$ such that $\sigma' : M \longrightarrow_{\Sigma_0^{(n)}} N$ and $\sigma'(\bar{a}) = a$. Thus, by uniqueness, $\sigma' = \sigma$.

QED(Lemma 4.4.6)

Condensability for N says that if σ,$\langle N, M, \lambda \rangle$ are as in lemma 4.4.3, then one of the conclusions (b), (c) hold.

Lemma 4.4.7. *Let σ,$\langle N, M, \lambda\rangle$ be as in lemma 4.4.6. Then there is a formula $\chi \in \mathbb{L}$ such that*

$$N \models \chi[M, \lambda, \sigma(\lambda)] \longleftrightarrow \text{ (b) or (c) hold.}$$

Proof. χ says that either $\bigvee \alpha \in N(M = N||\alpha)$, or that there are $\kappa, \gamma, \mu \in N$ such that

- λ is the cardinal successor of κ in M.
- $\rho^1_{N||\lambda} = \kappa$.
- $\mu \leq \gamma$, $E^N_\mu \neq \emptyset$ and $\text{crit}(E^N_\mu) = \kappa$, E^N_μ is generated by $\{\kappa\}$.
- $(N||\tilde{\lambda}) \models$ There is π such that $\pi : N||\gamma \longrightarrow_{E^N_\mu} M$.

This can be written as an $\mathbb{L}$-formula by Fact 5 and the fact that for $Q \in N$, Q-satisfaction is uniformly $\Delta_1(N)$ in Q. The asserted equivalences then hold because statements of the form:

$$\bigvee \pi \quad \pi : Q \longrightarrow^*_F Q'$$

are absolute in transitive ZFC^- models.

QED(Lemma 4.4.7)

Set:

$$\psi_n =: \bigwedge u \bigwedge v \bigwedge w(\varphi_n(u, v, w) \longrightarrow \chi(u, v, w)).$$

Then obviously:

Lemma 4.4.8. *Let N be sound and solid. Then*

$$N \models \psi_n \longleftrightarrow N \text{ is condensable.}$$

It is apparent from the above proofs that the function $n \mapsto \psi_n$ is recursive. Hence, if N is sound and solid, then:

$$\bigwedge n \; N \models \psi_n \longleftrightarrow N \text{ is condensable.}$$

But $\bigwedge n \; N \models \psi_n$ is uniformly $\Pi_1(U)$ in N, since N-satisfaction is uniformly $\Delta_1(U)$ in N. This proves lemma 4.4.4.

Lemma 4.4.2 then follows, since it says:

$$\bigwedge \alpha \in M(\text{Lim}(\alpha) \longrightarrow \bigwedge n \; (N||\alpha) \models \psi_n).$$

QED(Lemma 4.4.2)

4.4.1 Σ_1-acceptability

Definition 4.4.11. Let $N = \langle J^A_\alpha, B\rangle$ be a J-model. N is Σ_1-acceptable if and only if it is acceptable and whenever $\gamma > \omega$ is a limit cardinal in N, then $J^A_\gamma \prec_{\Sigma_1} J^A_\alpha$.

Lemma 4.4.9. *Every pre-mouselike premouse is Σ_1-acceptable.*

Proof. We proceed by induction on $\alpha = \text{ht}(N)$. If $\alpha = \omega$, the assertion is vacuously true. If α is a limit of limit ordinals, then the assertion is trivial, since any cardinal γ in N is a cardinal in $N||\beta$ for $\beta > \gamma$. There remains the case: $\alpha = \beta + \omega$. Let $M = \langle J^E_\beta, F\rangle$, where $F = E_\beta$. Then $N = \langle J^{E'}_\alpha, \emptyset\rangle$, where

$$E' = E * F = E \cup (\{\beta\} \times F).$$

Let $\rho = \rho^\omega_M$. Then ρ is the largest cardinal in N. Let $\gamma > \omega$ be a limit cardinal in N. Then $\gamma \leq \rho$. If $\rho < \beta$, then γ, ρ are cardinals in M. Now let ψ be a Σ_1 formula such that

$$J^{E'}_\alpha \models \psi[x] \text{ where } x \in J^{E'}_\gamma.$$

We must prove:

Claim. $J^{E'}_\gamma \models \psi[x]$.

We first note that:

$$|N| = \text{rud}(|M| \cup \{M\}) = \text{rud}(|M| \cup \{E\} \cup \{F\}),$$

where $\text{rud}(Y)$ is the closure of Y under rudimentary functions. Let $\psi = \bigvee v\psi'$, where ψ' is Σ_0 in the language of N. Then:

(1) $N \models \psi'[t, x]$ for a $t \in N$

Since $N = J^{E'}_\alpha$ and $E' = E * F$, (1) can be equivalently written as:

(2) $N \models \varphi[t, x, |M|, E, F]$, where φ is a Σ_0 formula containing only the predicate $\in$.

Let $t = f(x, z, |M|, E, F)$ where f is rudimentary and $z \in M$. Recall that rudimentary functions are *simple* in the sense of §2.2. This means that, given the function f: (2) reduces uniformly to:

(3) $N \models \varphi'[x, z, |M|, E, F]$, where φ' is a Σ_0 formula containing only the predicate $\in$.

But this can easily be converted into an equivalent statement of the form:

(4) $M \models \chi'[x, z]$, where χ' is a first order formula in the language of M. Set $\chi = \bigvee v \chi'$. Then:

(5) $M \models \chi[x]$.

In order to derive Claim 1, we show:

Claim 2. There is $\bar{\beta} < \gamma$ such that, letting $\bar{M} = M||\bar{\beta}$, $\bar{N} = M||\bar{\alpha}$, $\bar{\alpha} = \bar{\beta} + \omega$, we have: $\bar{M} \models \chi[x]$.

But then $\bar{M} \models \chi'[x, z]$ for a $z \in \bar{M}$. We then reverse the above chain of equivalent reductions to get: $\bar{N} \models \psi'[\bar{t}, x]$, where $\bar{t} = f(x, z, |\bar{M}|, \bar{E}, \bar{F})$ and f is the above mentioned rudimentary function. Thus: $\bar{N} \models \psi[x]$ and $J^E_\gamma \models \psi[x]$, since $\bar{N} \triangleleft J^E_\gamma$, proving Claim 1.

Our procedure will be to first define $\bar{M}$ and then, using the condensability of M, show that $\bar{M}$ is a proper segment of J^E_γ. We can assume that w.l.o.g. that the formula χ is a Σ_m-formula for some $m < \omega$. Choose $n < \omega$ such that $n \geq m$ and $\rho^\omega_M = \rho^n_M$. Since M is sound, it has a standard parameter a. Hence $a \in P^n_M$. Hence $a \in R^n_m$ by soundness. Now let δ' be the least cardinal in M such that $x \in J^E_{\delta'}$. Then δ' is a successor cardinal in M(hence in N). Let δ be the immediate successor cardinal of δ' in M (and N). Then $\delta < \gamma$. Let X be the smallest $X \prec_{\Sigma_1} M^{n,a}$ such that $(\delta' + 1) \cup a \subset X$. Then $X = \tilde{h}"\delta'$, where

$$\tilde{h}(\prec i, \xi \succ) \simeq h(i, \langle \xi, \delta', a \rangle)$$

and h is the Skolem function for $M^{n,a}$. Let $\bar{\pi} : \bar{Q} \overset{\sim}{\longleftrightarrow} X$ be the transitivation of X. Then $\bar{\pi} : \bar{Q} \longrightarrow_{\Sigma_1} M^{n,a}$. By the downward extension of embeddings lemma(Lemma 2.6.32) we conclude:

(a) There are unique $\bar{M}, \bar{a}$ such that $\bar{a} \in R^n_{\bar{M}}$ and $\bar{M}^{n,\bar{a}} = \bar{Q}$.

(b) There is a unique $\pi \supset \bar{\pi}$ such that $\bar{\pi} : \bar{M} \longrightarrow_{\Sigma_1^{(n)}} M$ and $\pi(\bar{a}) = a$.

But M is sound and a is its standard parameter. Hence $\bar{M}, \bar{a}, \pi$ are the unique objects given by our earlier downward extension lemma and we have:

(6) $\pi : \bar{M} \longrightarrow_{\Sigma_{n+1}} M$.

We now show:

(7) $\bar{M} \in J^E_\delta$.

Proof. $\tilde{h}$ is $\Sigma_1^{(n)}(M)$ in $a \cup \{\delta'\}$ and is a partial map of δ' unto X. Thus $\bar{h} = \bar{\pi}^{-1}\tilde{h}$ is $\Sigma_1^{(n)}(M)$ in $\bar{a} \cup \{\delta'\}$ and is a partial map of δ' onto $\bar{M}^{n,\bar{a}}$. Since $\bar{a} \in R^n_{\bar{M}}$, there is a partial map $\bar{g}$ of $\bar{M}^{n,\bar{a}}$ onto $\bar{M}$ which is $\Sigma_1^{(n)}(\bar{M})$ in $\bar{a}$. Let g be $\Sigma_1^{(n)}(M)$ in a by the same definition. Then $\bar{k} = \bar{g}\bar{h}$ is a $\underline{\Sigma}^*(\bar{M})$ map of δ' onto $\mathrm{ran}(\pi)$, since $g\bar{\pi} = \pi\bar{g}$. Set:

- $|\tilde{M}| =: \mathrm{dom}(k) \subset \delta'$.
- $x \tilde{\in} y \longleftrightarrow: k(x) \in k(y)$ for $x, y \in |\tilde{M}|$.
- $x \tilde{=} y \longleftrightarrow: k(x) = k(y)$ for $x, y \in |\tilde{M}|$.
- $\tilde{E}x \longleftrightarrow: k(x) \in E$, $\tilde{F}x \longleftrightarrow: k(x) \in F$ for $x \in |\tilde{M}|$.

Set: $\tilde{M} =: \langle |\tilde{M}|, \tilde{\in}, \tilde{=}, \tilde{E}, \tilde{F} \rangle$. Then $\tilde{M} \in J^E_\gamma$, since $\langle J^E_\rho, D \rangle$ is amenable for all $\underline{\Sigma}^*(M)$ sets D, and δ is a cardinal in J^E_ρ. But J^E_δ is a ZFC^- model, since δ is a successor cardinal in J^E_ρ. $\tilde{E}$ is well founded. Hence $j \in J^E_\delta$, where $j : \tilde{M} \longrightarrow \bar{M}$ is defined by the recursion: $j(x) = j\text{"}\tilde{\in}\text{"}\{x\}$ for $x \in |\tilde{M}|$. Hence $\bar{M} \in J^E_\delta$.

QED(7)

Set: $\bar{\delta} = \pi^{-1}(\delta)$. It follows easily that $\pi \restriction \delta = \mathrm{id}$. But $\pi(\bar{\delta}) = \delta > \bar{\delta}$, sicne $\bar{\delta} \in J^E_\delta$. Thus $\bar{\delta} = \mathrm{crit}(\pi)$. Using this, we show:

(8) π verifies the phalanx $\langle M, \bar{M}, \bar{\delta} \rangle$.

Proof.

- $\pi : \bar{M} \longrightarrow M$.
- π is $\Sigma_1^{(n)}$-preserving, where $\bar{\delta} < \rho^n_{\bar{M}}$.
- $\rho^{n+1}_{\bar{M}} < \bar{\delta}$, since $\bar{h}$ is a $\Sigma_1^{(n)}(\bar{M})$ partial map of $\delta' < \bar{\delta}$ onto $\bar{M}^{n,a}$.
- ξ is a cardinal in $\bar{M}$ if and only if $\pi(\xi)$ is a cardinal in M, by (6).

QED(8)

But M is condensable. Hence $\bar{M}$ satisfies one of the three conditions (a), (b), (c) in the condensation lemma. But:

(9) (a) does not hold, since otherwise:

$$\rho^n_{\bar{M}} = \mathrm{ht}(\bar{M}^{n,\bar{a}}) < \delta < \rho.$$

But we can also show:

(10) (c) does not hold.

Proof. Suppose not. Then there is $\eta \in M$ such that $\rho^{\omega}_{J^E_\eta} = \kappa < \delta$, where κ is the largest cardinal in J^E_δ. Moreover, there is $\mu \leq \eta$ such that $\sigma : J^E_\eta \longrightarrow_F \bar{M}$, where $F = E_\mu$ and $\kappa = \text{crit}(F)$. But then $\kappa = \delta'$ would be a limit cardinal in $\bar{M}$. Contradiction!, since δ' is a successor cardinal.

QED(10)

Thus (b) holds, and $\bar{M} \lhd M$. Since $\bar{\beta} = \text{ht}(\bar{M}) < \delta$, we have:

(11) $\bar{M} = M||\bar{\beta} = \langle J^{\bar{E}}_{\bar{\beta}}, \bar{F}\rangle$ where $\bar{\beta} < \delta$.

Moreover, if $\bar{\alpha} = \bar{\beta} + \omega$ and $\bar{N} = M||\bar{\alpha}$, we have:

(12) $\bar{N} = M||\bar{\alpha} = J^{\bar{E}*\bar{F}}_{\bar{\alpha}}$.

By (6) we know: $\bar{M} \models \chi[x]$, hence:

(13) $\bar{M} \models \chi'[x, z]$ for a $z \in \bar{M}$.

Reversing our earlier chain of equivalences, we see that (13) is equivalent to:

(14) $\bar{N} \models \varphi'[x, z, |\bar{M}|, \bar{E}, \bar{F}]$.

Set $\bar{t} = f(x, z, |\bar{M}|, \bar{E}, \bar{F})$ where f is the rudimentary function used above. Then (14) is equivalent to:

(15) $\bar{N} \models \varphi[\bar{t}, x, |\bar{M}|, \bar{E}, \bar{F}]$,

which is, in turn, equivalent to:

(16) $\bar{N} \models \psi'[\bar{t}, x]$.

Hence $\bar{N} \models \psi[x]$, where $\bar{N} \lhd J^E_\delta$.

QED(Lemma 4.4.9)

Call a premouse N *fully preiterable.* If every proper $M \lhd N$ is fully iterable. By lemma 4.4.9 we of course have:

Corollary 4.4.10. *Every fully preiterable premouse is Σ_1-acceptable.*

(Hence of course, every fully iterable premouse is Σ_1-acceptable.)

4.4.2 Mouselikeness in 1-small premice

The reader may wonder why we develop theory of mouselikeness and pre-mouselikeness in such detail, when we already know that these properties hold for all fully iterable mice. The reason is that we may encounter iterations where we can verify the mouselikeness of a structure without yet knowing it to be fully iterable. We give an example involving *1-small premice*, which were introduced in §3.8 and will be our main object of investigation in the ensuing chapters. We call a 1-small premouse N *unrestrained* if and only if

- $N = J_\alpha^E$ is a constructible extension of J_β^E, where $\beta \leq \rho_N^\omega$.
- β is Woodin in $J_{\alpha+\omega}^{E^N}$, where $\alpha = \mathrm{ht}(N)$.

Otherwise we call N *restrained.* Restrained premice have the unique branch property-i.e. any normal iteration of limit length has at most one cofinal well founded branch. Hence, by Theorem 3.6.1 and Theorem 3.6.2 we know that N is fully iterable if it is normally iterable. Happily, however, it turns out that if N is unrestrained and pre-mouselike, then it is mouselike. We, in fact, prove:

Lemma 4.4.11. *Let $N = J_\alpha^E$ be 1-small, where $\beta \leq \alpha$ is Woodin in $J_{\alpha+\omega}^E$. If J_β^E is pre-mouselike; then N is mouselike.*

Proof. Since β is Woodin in $J_{\alpha+\omega}^E$. We have $\beta \leq \rho_N^\omega$, N is then a constructible extension of J_β^E by 1-smallness,

(1) N is sound, by Lemma 2.5.22.

(2) N is solid, by Lemma 4.1.16.

Now let σ witness $\langle N, M, \lambda\rangle$ where M is sound above λ. By Lemma 4.4.5:

(3) $M \in N$, $\sigma \in \underline{\Sigma}_\omega(N)$.

Claim. One of the conditions (b), (c) holds.

(4) If $\lambda \geq \beta$, the (b) holds.

Proof. $\lambda \neq \beta$, since otherwise $\sigma(\lambda) > \beta$ is Woodin in N. Contradiction! But then $\sigma(\beta) = \beta$. Hence M is a constructible extension of J_β^E, since $\sigma : M \longrightarrow_{\Sigma_0} N$. But then $M \triangleleft N$ is a proper segment of N and (b) holds.

QED(4)

From now on assume: $\lambda < \beta$. Thus:

(5) $M \in J^E_\beta$.

Proof. Let $\gamma = \mathrm{ht}(M)$. There is $f \in N$ such that $f : \lambda \overset{\text{onto}}{\longrightarrow} \gamma$, since M is sound above λ. Moreover M is coded by a $b \subset \lambda$. Hence $b \in J^E_\beta$, since β is a cardinal in N. But β is a regular limit cardinal in N. Hence J^E_β is a transitive model of ZFC. Hence b can be decoded in J^E_β. Hence $M \in J^E_\beta$.

QED(5)

(6) $\sigma(\lambda) \leq \beta$

Proof. Otherwise $\beta < \sigma(\lambda)$ is the unique Woodin cardinal in N. Hence some $\bar{\beta} < \lambda$ is the unique Woodin cardinal in M. Hence $\beta = \sigma(\bar{\beta}) = \bar{\beta} < \beta$, and $\bar{\beta} < \lambda$. Contradiction!

QED(6)

Let $\varphi_m \in \mathbb{L}$ be the formula in Lemma 4.4.6, where $\rho^{m+1} \leq \lambda < \rho^m$ in M. Without loss of generality, suppose φ_m to be Σ_r in the Levy hierarchy. Pick $n \geq r$ such that $\rho^n = \rho^\omega$ in N. Let $a \in P^n_N$. Let $Q = N^{n,a}$. Let h be the canonical Σ_1 Skolem function for Q. Working in $J^E_{\alpha+\omega}$, we define sequences $X_i \prec_{\Sigma_1} Q$, $\alpha_i < \alpha$ for $i < \omega$ as follows: let $\beta_0 < \beta$ such that $M \in J^E_{\beta_0}$ and $\sigma(\lambda) < \beta_0$ if $\sigma(\lambda) < \beta$. Set: $X_i = h(\beta_i) =: \{h(i,\xi) \mid \xi < \beta_i\}$, $\beta_{i+1} = \mathrm{lub}\, \beta \cap X_i$.

Since β is a regular limit cardinal in $J^E_{\alpha+\omega}$, it follows that $\beta_i < \beta$ for $i < \omega$, where the sequence $\langle \beta_i \mid i < \omega \rangle$ is defined from φ. Hence $\langle \beta_i \mid i < \omega \rangle$ is N-definable by Fact 5. Hence $\langle \beta_i \mid i < \omega \rangle \in J^E_{\alpha+\omega}$ and

$$\bar{\beta} =: \sup_{i<\omega} \beta_i < \beta.$$

Set $X = h(\bar{\beta}) = \bigcup_{i<\omega} X_i$. Then $X \in J^E_{\alpha+\omega}$. Let $\bar{\pi} : \bar{Q} \overset{\sim}{\longleftrightarrow} X$. Thus $\bar{\pi} : \bar{Q} \prec_{\Sigma_1} Q$ and by the downward extension Lemma there are unique $\bar{N}, \bar{a}$ such that $\bar{a} \in R^n_{\bar{N}}$ and $\bar{N}^{n,\bar{a}} = \bar{Q}$. Moreover there is a unique $\pi \supset \bar{\pi}$ such that $\pi(\bar{a}) = a$ and $\pi : \bar{N} \longrightarrow_{\Sigma_1} N$. Since $a \in R^n_N$, we then get: $\pi : \bar{N} \longrightarrow_{\Sigma_n} N$. But then $\bar{N} \models \varphi_m[M, \lambda, \tilde{\lambda}]$, where $\tilde{\lambda} = \sigma(\lambda)$ if $\sigma(\lambda) < \beta$ and $\tilde{\lambda} = \bar{\beta}$ is $\sigma(\lambda) = \bar{\beta}$. Hence:

(7) There is $\bar{\sigma}$ witnessing $\langle \bar{N}, M, \lambda \rangle$ where $\bar{\sigma}(\lambda) = \sigma(\lambda)$ if $\sigma(\lambda) < \beta$ and $\bar{\sigma}(\lambda) = \bar{\beta}$ if $\sigma(\lambda) = \beta$.

Clearly $\bar{N}$ is a constructible extension of $J^E_{\bar{\beta}}$ and $\bar{\beta}$ is Woodin in $\bar{N}$ if $\beta < \alpha$. Using this, we get:

(8) $\bar{N} \triangleleft J^E_{\beta}$, where $\mathrm{ht}(N) < \beta$.

Proof. Since $\bar{\beta} < \beta$, there is a least $\nu < \beta$ such that $E_\nu \neq \emptyset$. But then J^E_ν is a constructible extension of $J^E_{\bar{\beta}}$ and $\bar{\beta}$ is not Woodin in J^E_ν by 1-smallness. Hence $\bar{\alpha} < \nu$, where $\bar{\alpha} = \mathrm{ht}(\bar{N})$ and $\bar{N} = J^E_{\beta} || \bar{\alpha}$.

QED(8)

Since J^E_{β} is pre-mouselike, we conclude that $\bar{N} \models \chi[M, \lambda, \bar{\sigma}(\lambda)]$. We can w.l.o.g. assume n to be chosen so that χ is Σ_n in the Levy hierarchy. But then:

$$N \models \chi[M, \lambda, \sigma(\lambda)], \text{ since } \pi(\bar{\sigma}(\lambda)) = \sigma(\lambda).$$

Hence (b) or (c) hold.

QED(Lemma 4.4.11)