Problem Set 9, due Jan 21, 50 points

Algebraic Geometry I, Winter 18/19

Please email any comments or corrections to mdawes@math.uni-bonn.de

$\mathcal{O}_X$-modules II

Problem 1. Let $\mathcal{F}, \mathcal{G}, \mathcal{H}$ be $\mathcal{O}_X$-modules.

(a) Define a functorial homomorphism of $\mathcal{O}_X$-modules

(1) $$\mathscr{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \otimes_{\mathcal{O}_X} \mathcal{H} \to \mathscr{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{H})$$

(Hint: Define a homomorphism from the tensor product presheaf $U \mapsto \mathscr{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})(U) \otimes_{\mathcal{O}_X(U)} \mathcal{H}(U)$ to the right hand side, and take the induced map from the sheafification.)

Proof. For each open set $U \subset X$, define the map

(2)
$$\Gamma(U, \mathscr{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})) \times \Gamma(U, \mathcal{H}) \to \mathrm{Hom}(\mathcal{F}_{|U}, (\mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{H})_{|U}) = \mathrm{Hom}(\mathcal{F}_{|U}, \mathcal{G}_{|U} \otimes_{\mathcal{O}_U} \mathcal{H}_{|U})$$

by sending a section $s \in \Gamma(V, \mathcal{F})$ (for $V \subset U$), to the image of $w_V(s) \otimes t_{|V} \in \Gamma(V, \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{H})$. □

(b) If $\mathcal{F}$ is locally free, show that (1) is an isomorphism.

Proof. As morphisms of sheaves are bijective if and only if they are bijective on stalks, the statement is a local one: we simply need to check that for each $x \in X$ there is an open neighbourhood U so that the restriction of the map (2) to U is an isomorphism. Therefore, without loss of generality, assume that $\mathcal{F} = \mathcal{O}_X^n$ for some $n \geq 0$. The map (2) decomposes as the sequence of isomorphisms

$$\begin{aligned}\mathscr{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \otimes_{\mathcal{O}_X} \mathcal{H} &\xrightarrow{\sim} \mathscr{H}om_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{G})^n \otimes_{\mathcal{O}_X} \mathcal{H} \xrightarrow{\sim} \mathcal{G}^n \otimes_{\mathcal{O}_X} \mathcal{H} \\ &\xrightarrow{\sim} (\mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{H})^n \xrightarrow{\sim} \mathscr{H}om_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{H})^n \\ &\xrightarrow{\sim} \mathscr{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{H}),\end{aligned}$$

and is therefore an isomorphism. □

(c) If $\mathcal{L}$ is an invertible sheaf, conclude that

$$\mathcal{L}^\vee \otimes \mathcal{L} \cong \mathscr{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{L}) \cong \mathcal{O}_X.$$

(Hint: The first isomorphism follows from (a). For the second construct a natural morphism $\mathcal{O}_X \to \mathscr{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{L})$ and check it is an isomorphism on stalks.)

Proof. If $\mathcal{L}$ is a locally free $\mathcal{O}_X$-module of rank 1, then its dual $\mathcal{L}^\vee$ is locally free of rank 1. By part (b), there is a functorial isomorphism

$$\mathcal{L}^\vee \otimes_{\mathcal{O}_X} \mathcal{L} \xrightarrow{\sim} \mathscr{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{L}).$$

As $\mathcal{L}|_U \cong \mathcal{O}_U$ and $\mathcal{L}^\vee|_U \cong \mathscr{H}om_{\mathcal{O}_U}(\mathcal{O}_U, \mathcal{O}_U) \cong \mathcal{O}_U$, then each of the maps on stalks $(\mathcal{L} \otimes_X \mathcal{L}^\vee)_p \to \mathcal{O}_{X,p}$ are isomorphisms (where $\mathcal{L}^\vee \otimes \mathcal{L} \to \mathcal{O}_X$ is defined by $f \otimes g \mapsto f(g)$), and we conclude that $\mathscr{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{L}) \cong \mathcal{O}_X$. □

Problem 2. Let X be a Noetherian scheme and let $\mathcal{F}$ be a coherent sheaf. For any $x \in X$ the stalk $\mathcal{F}_x$ is naturally an $\mathcal{O}_{X,x}$ module, and the quotient

$$\mathcal{F}_x/\mathfrak{m}_x\mathcal{F}_x = \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} k(x)$$

is a vector space over the residue field $k(x)$. Let $\varphi(x)$ be the dimension of this vector space.

a) Show that $\varphi(x)$ is finite for all $x \in X$. (Hint: If $\mathcal{F}$ is coherent, then there exist an open neighbourhood U of x and a surjection $\mathcal{O}_X|_U^{\oplus r} \twoheadrightarrow \mathcal{F}|_U \to 0$ for some $r \geq 1$. Then use that the tensor product is right exact.)

Proof. By using the hint, as $\mathcal{F}$ is coherent, there exists an open neighbourhood $U \ni x$ and a surjection

$$\mathcal{O}_X|_U^{\oplus r} \twoheadrightarrow \mathcal{F}|_U \to 0$$

for some $r \geq 1$, which induces a surjection of stalks

$$(\mathcal{O}_X^{\oplus r})_x \twoheadrightarrow \mathcal{F}_x \to 0.$$

As $\otimes_{X,x} k(x)$ defines a right exact functor, we obtain

$$(\mathcal{O}_X^{\oplus r})_x \otimes_{\mathcal{O}_{X,x}} k(x) \twoheadrightarrow (\mathcal{F}_x) \otimes_{\mathcal{O}_{X,x}} k(x) \to 0.$$

As coherence implies that $(\mathcal{O}_X^{\oplus r})_x \otimes_{\mathcal{O}_{X,x}} k(x)$ is a finite vector space, the result follows. □

b) If $\mathcal{F}$ is locally free, show that φ is locally constant.

Proof. For any $n \in \mathbb{N}$, define $U_n := \varphi^{-1}(n) \subset X$. For all $x \in U_n$, there exist an open neighbourhood U of x so that $\mathcal{F}|U \cong \mathcal{O}_X|_U^n$ is free. Therefore, for every $y \in U$, $\varphi(y) = n$ and so U_n is open, and so $\varphi(x)$ is locally constant. □

c) Prove the converse: If X is integral and φ is locally constant, then $\mathcal{F}$ is locally free. (Let K and k be the fraction field and the residue field of $\mathcal{O}_{X,x}$ respectively. Show that if M is an $\mathcal{O}_{X,x}$-module such that $\dim_K M \otimes K = \dim_k M \otimes k$, then M is free. Conclude that $\mathcal{F}_x$ is free for all $x \in X$. Show this implies the claim by 'spreading out'.)

Proof. Let $x \in X$ and let $U = \operatorname{Spec} A$ be an affine neighbourhood of x. Without loss of generality, we can assume that $\mathcal{F}|_U = \tilde{M}$ for a finitely generated A-module M. Select representatives $\{m_1, \dots, m_n\} \subset M$ for a $k(x)$-basis of $M_\mathfrak{p}/\mathfrak{p}M_\mathfrak{p}$. By the Nakayama lemma, $\{m_1, \dots, m_n\}$ generate $M_\mathfrak{p}$ as an $A_\mathfrak{p}$-module. Therefore, they generate $M_\mathfrak{q}$ as an $A_\mathfrak{q}$-module for $\mathfrak{q} \subset \mathfrak{p}$. As φ is constant, then the image of $\{m_1, \dots, m_n\}$ in $M_\mathfrak{q}/\mathfrak{q}M_\mathfrak{q}$ are linearly independent. Therefore, if $\sum a_i m_i = 0$ in $M_\mathfrak{p}$, then $a_i = 0$ for all $A_\mathfrak{q}/\mathfrak{q}A_\mathfrak{q}$ for all $\mathfrak{q} \subset \mathfrak{p}$. Because X is reduced, then $a_i = 0$ in $A_\mathfrak{q}$. Therefore, m_i are linearly independent over $A_\mathfrak{p}$ and so $M_\mathfrak{p}$ is a free $A_\mathfrak{p}$-module. Therefore, $\mathcal{F}$ is locally free. □

d) Apply the criterion from c) to decide for the following A-modules M which of the quasi-coherent sheaves $\tilde{M}$ is locally free:
   - $A = \mathbb{C}[x]$, $M = (x)$
   - $A = \mathbb{C}[x_1, x_2], M = (x_1, x_2)$
   - $A = \mathbb{C}[x_1, x_2]$, $M = A/I$ for some ideal I
   - $A = \mathbb{C}[x, y]/(y^2 - x^2(x-1))$ and M the ideal sheaf of the origin
   - the ideal sheaf of a closed point on a normal curve.

e) Show that in (c) it is enough to assume that X is reduced. Show by example that the condition that X is reduced is necessary.

Quasicoherent modules on Proj A

Problem 3. Let $X = \mathbb{P}^n_k$ be n-dimensional projective space over a field k. Determine $\Gamma(X, \mathcal{O}_X(d))$ for all $d \in \mathbb{Z}$. space of homogeneous polynomials of degree d.

Proof. We shall actually prove a slightly stronger result. If S is a graded ring, let $X = \mathrm{Proj}(S)$. If $\mathcal{F}$ is a sheaf of $\mathcal{O}_X$-modules, then there is a graded S-module $\Gamma_*(\mathcal{F})$ associated to $\mathcal{F}$ whose group is given by $\Gamma_*(\mathcal{F}) = \bigoplus_{n\in\mathbb{Z}} \Gamma(X, \mathcal{F}(n))$ and if $s \in S_d$ and $t \in S_n$, the product $s.t \in S_{n+d}$ is given by the tensor product $s \otimes t$, using the natural map $\mathcal{F}(n) \otimes \mathcal{O}_x(d) \cong \mathcal{F}(n+d)$.

Now let $S = k[x_0, \dots, x_n]$. We show that $\Gamma_*(\mathcal{O}_X) \cong S$. Cover X with the standard open sets $D(x_i)$. Then each section $t \in \Gamma(X, \mathcal{O}_X(n))$ is defined uniquely by sections $t_i \in \mathcal{O}_X(n)(D(x_i))$ (for $i = 0, \dots, n$) that agree on the intersections $D(x_i x_j)$ for all i, j. The section t_i is a homogeneous element of degree n in the localisation S_{x_i} and its restriction to $D(x_i x_j)$ is the image of the element in $S_{x_i x_j}$. Therefore, by summing over all n, we conclude that $\Gamma_*(\mathcal{O}_X)$ can be identified with the set of $(n+1)$-tuples $(t_0, \dots, t_n)$ so that for all i, $t_i \in S_{x_i}$ and for each i, j, the images of t_i and t_j in $S_{x_i x_j}$ agree.

As none of the x_i are zero divisors in S, the localisations $S \to S_{x_i}$ and $S_{x_i} \to S_{x_i x_j}$ are all injective. Moreoever, each ring is a subring of $S' = S_{x_0,\dots,x_n}$. Therefore, $\Gamma_*(\mathcal{O}_X)$ is the intersection $\bigcap S_{x_i}$ taken in S'. As any homogeneous element of S' can be written uniquely as a product $x_0^{i_0} \dots x_n^{i_n} f(x_0, \dots, x_n)$ where $i_j \in \mathbb{Z}$ and f is a homogeneous polynomial not divisible by any x_i, then the element belongs to S_{x_i} if and only if $i_j \geq 0$ for $j \neq i$. Therefore, the intersection of all the S_{x_i} is precisely S.

□

Problem 4. Let $A = k[x, y]$ where x, y are of degree $a, b \geq 1$ respectively with a, b coprime. Consider the weighted projective space $X = \mathbb{P}(a, b)$.

(a) Show that $X \cong \mathbb{P}^1_k$. (Hint: Consider $D(x)$ and $D(y)$.)

Proof. We can actually prove a much stronger result. Suppose that $a_0, \dots, a_n$ are positive integers without a common factor and let $c := gcd(a_1, \dots, a_n)$. I claim that $\mathbb{P}(a_0, \dots, a_n) \cong \mathbb{P}(a_0, a_1/c, \dots, a_n/c)$. Let $R = k[x_0, \dots, x_n]$ be the graded ring where x_i has weight a_i and let $S = \bigoplus_{i \geq 0} R_{ci}$. As $R = S[c]$ then, using the Veronese embedding from problem sheet 8, we conclude that $\mathbb{P}(a_0, \dots, a_n) = \text{Proj } R = \text{Proj } S$.

Now take a monomial $x_0^{p_0} \dots x_n^{p_n}$ of degree mc for some m. As $\sum p_i a_i = cm$, then $c | p_0 a_0$. However, as the a_i have no common factor, then $c | p_0$. Therefore, x_0 only occurs in R as x_0^c. Therefore, $R = k[x_0^c, x_1, \dots, x_n]$ and so $\mathbb{P}(a_0, \dots, a_n) = \text{Proj } R = \mathbb{P}(a_0, a_1/c, \dots, a_n/c)$.

From which we conclude that $\mathbb{P}(a, b) \cong \mathbb{P}^1$. □

(b) Under the isomorphism of (a) the sheaves $\mathcal{O}_X(n)$ correspond to quasi-coherent sheaves on $\mathbb{P}^1$. Which ones? (You may assume $a = 1$ and $b = 2$ for simplicity.)

Proof. Let $\varphi : X \to \mathbb{P}^1$ be the Veronese embedding from part (a). We have

$$\varphi_* \mathcal{O}_X(n) = \mathcal{O}_{\mathbb{P}^1}(f(n))$$

where the function f is determined by the quasi-periodicity

$$f(n + ab) = f(n) + 1$$

and the initial data

$$f(n) = 0 \iff n \in \{ia + jb | 0 \leq i \leq b - 1, 0 \leq j \leq a - i\}.$$

Sketch of proof: Write $x = t^a, y = u^b$. Then whenever

$$n = ia + jb$$

for some i, j as above, the modules

$$(A(n)_{t^a})_0 \,, (A(n)_{u^b})_0$$

are both generated by the same element $t^{ia} u^{jb}$. Hence the transition functions are trivial which implies $\mathcal{O}_X(n)$ is trivial. The quasi-periodicity is left to you. □

Line bundles on curves

Problem 5. Let C be a normal curve over a field k and let K be its function field. Recall that for an open subset $U \subset C$ we have

$$\mathcal{O}_X(U) = \{f \in K | \nu_x(f) \geq 0 \text{ for all } x \in U\}$$

where $\nu_x : K \to \mathbb{Z} \cup \{\infty\}$ is the valuation corresponding to the point $x \in C$, and we used the convention $\nu_x(0) = \infty$.

Let $D = (n_x)_{x\in C_0} \in \mathbb{Z}^{C_0}$ be a tuple of integers, almost all vanishing, indexed by the set C_0 of closed points of C. We also write

$$D = \sum_{x\in C} n_x \cdot x$$

and call D a *divisor* on C. For every divisor on C define an $\mathcal{O}_X$-module by

$$\mathcal{L}_D(U) := \{s \in K | \nu_x(s) \geq n_x \text{ for all } x \in U\}$$

with the natural restriction morphisms and $\mathcal{O}_X$-module structure.
(a) Describe the restriction morphisms and the $\mathcal{O}_X$-module structure on $\mathcal{L}_D$. Why is $\mathcal{L}_D$ a sheaf?

Proof. For $V \subset U$ define restriction maps by the inclusions $\Gamma(U, \mathcal{L}_D) \hookrightarrow \Gamma(V, \mathcal{L}_D)$. Multiplication by K defines a scalar multiplication $\Gamma(U, \mathcal{O}_X) \times \Gamma(U, CL_D) \to \Gamma(U, \mathcal{L}_D)$, which endows $\mathcal{L}_D$ with an $\mathcal{O}_X$-module structure. □

(b) Show that $\mathcal{L}_D$ is an invertible sheaf. (Hint: You can either use Problem 2, or construct an isomorphism $\mathcal{O}_X|_{U_x} \cong \mathcal{O}_X|_{U_x}$ for an open neighbourhood U_x of any $x \in X$ directly. If $n_x = 0$ this is immediate. If $n_x \neq 0$ find an open subset V and a $t \in \mathcal{O}_X(V)$ such that t_x is the uniformizer in $\mathcal{O}_{X,x}$ and t vanishes nowhere else in V. Then divide or multiply by t appropriately.)

Proof. Let $\mathrm{Supp}(D)$ be the set of $x \in X_0$ (where X_0 denotes the set of closed points) where $n_x \neq 0$. For each $x \in \mathrm{Supp}(D)$ let U_x be an open affine neighbourhood of x containing no other point of $\mathrm{Supp}(D)$ and so that there exists a section $s_x \in \Gamma(U_x, \mathcal{O}_X)$ whose germ in x is the chosen uniformising element $\pi_x \in \mathcal{O}_{X,x}$. By shrinking U_x (if neccessary) we can assume that $\nu_y(s_x) = 0$ for all $y \in U_x \cap X_0$. Define $V := X\setminus \mathrm{Supp}(D)$. Then V and U_x for $x \in \mathrm{Supp}(D)$ define an open cover for X. By definition, $\mathcal{L}_{D|V} \cong \mathcal{O}_{X|V}$ and, over U_x, multiplication by $s_x^{n_x} \in K$ defines an isomorphism

$$\mathcal{O}_{X|U_x} \xrightarrow{\sim} \mathcal{L}_{|U_x}.$$

□

(c) Calculate $\mathcal{L}_D \otimes_{\mathcal{O}_C} \mathcal{L}_{D'}$ for two divisors D, D'. Conclude that we have a group homomorphism from the group of divisors of C into the Picard group,

$$\mathrm{Div}(C) := \mathbb{Z}^{C_0} \to \mathrm{Pic}(C).$$

Proof. For divisors D and D', $\mathcal{L}_D \otimes_{\mathcal{O}_X} \mathcal{L}_{D'} = \mathcal{L}_{D+D'}$ is immediate from the definition. Therefore, there exists a homomorphism from $\mathrm{Div}(C) \to \mathrm{Pic}(C)$. □

(d*) (**Bonus**, +5 points) If $f \in K$ and $D = (\nu_x(f))_{x\in C_0}$ show that $\mathcal{L}_D \cong \mathcal{O}_C$.
(e*) (**Bonus**, +5 points) Conclude that $\mathrm{Pic}(\mathbb{A}^1_k) = 0$.

(Hint: Show the image of $\mathrm{Div}(\mathbb{A}^1_k) \to \mathrm{Pic}(\mathbb{A}^1_k)$ vanishes using (d). Now let $\mathcal{L}$ be any invertible sheaf, consider a trivialization $\varphi : \mathcal{O}|_U \to \mathcal{L}|_U$ over an open set U. Extend $\varphi(1)$ to a global section of $\mathcal{L}$.)

(f*) (**Bonus**, +5 points) Conclude that $\mathrm{Pic}(\mathbb{P}^1_k) = \mathbb{Z}$.