# Problem Set 9, due Jan 21, 50 points

Algebraic Geometry I, Winter 18/19

## $\mathcal{O}_X$-modules II

**Problem 1.** Let $\mathcal{F}, \mathcal{G}, \mathcal{H}$ be $\mathcal{O}_X$-modules.

(a) Define a functorial homomorphism of $\mathcal{O}_X$-modules

$$\mathscr{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \otimes_{\mathcal{O}_X} \mathcal{H} \to \mathscr{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{H}) \tag{1}$$

(Hint: Define a homomorphism from the tensor product presheaf $U \mapsto \mathscr{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})(U) \otimes_{\mathcal{O}_X(U)} \mathcal{H}(U)$ to the right hand side, and take the induced map from the sheafification.)

(b) If $\mathcal{F}$ is locally free of finite rank, show that (1) is an isomorphism.

(c) If $\mathcal{L}$ is an invertible sheaf, conclude that

$$\mathcal{L}^\vee \otimes \mathcal{L} \cong \mathscr{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{L}) \cong \mathcal{O}_X.$$

(Hint: The first isomorphism follows from (a). For the second construct a natural morphism $\mathcal{O}_X \to \mathscr{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{L})$ and check it is an isomorphism on stalks.)

**Problem 2.** Let $X$ be a Noetherian scheme and let $\mathcal{F}$ be a coherent sheaf. For any $x \in X$ the stalk $\mathcal{F}_x$ is naturally an $\mathcal{O}_{X,x}$ module, and the quotient

$$\mathcal{F}_x/\mathfrak{m}_x\mathcal{F}_x = \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} k(x)$$

is a vector space over the residue field $k(x)$. Let $\varphi(x)$ be the dimension of this vector space.

a) Show that $\varphi(x)$ is finite for all $x \in X$. (Hint: If $\mathcal{F}$ is coherent, then there exist an open neighbourhood $U$ of $x$ and a surjection $\mathcal{O}_X|_U^{\oplus r} \twoheadrightarrow \mathcal{F}|_U \to 0$ for some $r \geq 1$. Then use that the tensor product is right exact.)

b) If $\mathcal{F}$ is locally free, show that $\varphi$ is locally constant.

c) Prove the converse: If $X$ is integral and $\varphi$ is locally constant, then $\mathcal{F}$ is locally free. (Let $K$ and $k$ be the fraction field and the residue field of $\mathcal{O}_{X,x}$ respectively. Show that if $M$ is an $\mathcal{O}_{X,x}$-module such that $\dim_K M \otimes K = \dim_k M \otimes k$, then $M$ is free. Conclude that $\mathcal{F}_x$ is free for all $x \in X$. Show this implies the claim by 'spreading out'.)

d) Apply the criterion from c) to decide for the following $A$-modules $M$ which of the quasi-coherent sheaves $\tilde{M}$ is locally free:

- $A = \mathbb{C}[x]$, $M = (x)$
- $A = \mathbb{C}[x_1, x_2], M = (x_1, x_2)$
- $A = \mathbb{C}[x_1, x_2]$, $M = A/I$ for some ideal $I$
- $A = \mathbb{C}[x, y]/(y^2 - x^2(x-1))$ and $M$ the ideal sheaf of the origin
- the ideal sheaf of a closed point on a normal curve.

e) Show that in (c) it is enough to assume that $X$ is reduced. Show by example that the condition that $X$ is reduced is necessary.

## Quasicoherent modules on Proj $A$

**Problem 3.** Let $X = \mathbb{P}^n_k$ be $n$-dimensional projective space over a field $k$. Determine $\Gamma(X, \mathcal{O}_X(d))$ for all $d \in \mathbb{Z}$.

**Problem 4.** Let $A = k[x, y]$ where $x, y$ are of degree $a, b \geq 1$ respectively with $a, b$ coprime. Consider the weighted projective space $X = \mathbb{P}(a, b)$.

(a) Show that $X \cong \mathbb{P}^1_k$. (Hint: Consider $D(x)$ and $D(y)$.)
(b) Under the isomorphism of (a) the sheaves $\mathcal{O}_X(n)$ correspond to quasi-coherent sheaves on $\mathbb{P}^1$. Which ones? (You may assume $a = 1$ and $b = 2$ for simplicity.)

## Line bundles on curves

**Problem 5.** Let $C$ be a normal curve over a field $k$ and let $K$ be its function field. Recall that for an open subset $U \subset C$ we have

$$\mathcal{O}_X(U) = \{f \in K | \nu_x(f) \geq 0 \text{ for all } x \in U\}$$

where $\nu_x : K \to \mathbb{Z} \cup \{\infty\}$ is the valuation corresponding to the point $x \in C$, and we used the convention $\nu_x(0) = \infty$.

Let $D = (n_x)_{x \in C_0} \in \mathbb{Z}^{C_0}$ be a tuple of integers, almost all vanishing, indexed by the set $C_0$ of closed points of $C$. We also write

$$D = \sum_{x \in C} n_x \cdot x$$

and call $D$ a *divisor* on $C$. For every divisor on $C$ define an $\mathcal{O}_X$-module by

$$\mathcal{L}_D(U) := \{s \in K | \nu_x(s) \geq n_x \text{ for all } x \in U\}$$

with the natural restriction morphisms and $\mathcal{O}_X$-module structure.
(a) Describe the restriction morphisms and the $\mathcal{O}_X$-module structure on $\mathcal{L}_D$. Why is $\mathcal{L}_D$ a sheaf?
(b) Show that $\mathcal{L}_D$ is an invertible sheaf. (Hint: You can either use Problem 2, or construct an isomorphism $\mathcal{O}_X|_{U_x} \cong \mathcal{O}_X|_{U_x}$ for an open neighbourhood $U_x$ of any $x \in X$ directly. If $n_x = 0$ this is immediate. If $n_x \neq 0$ find an open subset $V$ and a $t \in \mathcal{O}_X(V)$ such that $t_x$ is the uniformizer in $\mathcal{O}_{X,x}$ and $t$ vanishes nowhere else in $V$. Then divide or multiply by $t$ appropriately.)
(c) Calculate $\mathcal{L}_D \otimes_{\mathcal{O}_C} \mathcal{L}_{D'}$ for two divisors $D, D'$. Conclude that we have a group homomorphism from the group of divisors of $C$ into the Picard group,

$$\mathrm{Div}(C) := \mathbb{Z}^{C_0} \to \mathrm{Pic}(C).$$

(d*) (**Bonus**, +5 points) If $f \in K$ and $D = (\nu_x(f))_{x \in C_0}$ show that $\mathcal{L}_D \cong \mathcal{O}_C$.
(e*) (**Bonus**, +5 points) Conclude that $\mathrm{Pic}(\mathbb{A}^1_k) = 0$.
(Hint: Show the image of $\mathrm{Div}(\mathbb{A}^1_k) \to \mathrm{Pic}(\mathbb{A}^1_k)$ vanishes using (d). Now let $\mathcal{L}$ be any invertible sheaf, consider a trivialization $\varphi : \mathcal{O}|_U \to \mathcal{L}|_U$ over an open set $U$. Extend $\varphi(1)$ to a global section of $\mathcal{L}$. )
(f*) (**Bonus**, +5 points) Conclude that $\mathrm{Pic}(\mathbb{P}^1_k) = \mathbb{Z}$.