Problem Set 8, due Jan 14, 50 points

Algebraic Geometry I, Winter 18/19

Projective spectrum

Problem 1. (a) Let A, B be graded rings and let $\varphi : A \to B$ be a graded ring homomorphism (i.e. a ring homomorphism that preserves the grading). Let

$$U = \{\mathfrak{p} \in \text{Proj } B | \varphi(A_+) \nsubseteq \mathfrak{p}\}.$$

Show that U is an open subset of Proj B and that φ determines a natural morphism $f : U \to \text{Proj } A$.

Proof. We show that $U = \text{Proj } B \backslash V(\varphi(A_+))$. If $\mathfrak{p} \in U$, then $\mathfrak{p} \notin V(\varphi(A_+))$ $\Rightarrow U \subset \text{Proj } B \backslash V(\varphi(A_+))$. Similarly, if $\mathfrak{p} \notin V(\varphi(A_+))$ then $\mathfrak{p} \not\supset \varphi(A_+) \Rightarrow$ $\mathfrak{p} \in U$, and so $\text{Proj } B \backslash V(\varphi(A_+)) \subset U$. Therefore, $\text{Proj } B \backslash V(\varphi(A_+)) = U$, which is open.

Define $f : U \to \text{Proj } A$ by $f : \mathfrak{p} \mapsto \varphi^{-1}\mathfrak{p}$. Then, as $f^{-1}(V(\mathfrak{p})) = V(\varphi^{\#}(\mathfrak{p}))$ then f^{-1} maps closed sets to closed sets, and so f is continuous. Moreover, the maps $f|_{\text{Spec } B_{\varphi(\mathfrak{p})}}$ induced by $\varphi : (A_\mathfrak{p})_0 \to (B_{\varphi(\mathfrak{p})})_0$ glue naturally (see G-W rmk 13.7) and so define a morphism of ringed spaces. $\square$

(b) Let $A = \mathbb{C}[x, y, z]$ where the generators x, y, z are of degree $1, 1, n$ respectively for some $n \geq 1$. Consider the ring homomorphisms

$$\varphi_1 : \mathbb{C}[x, y, z] \to \mathbb{C}[x, y, w], \quad x \mapsto x, y \mapsto y, z \mapsto w^n,$$

where w is of degree 1, and

$$\varphi_2 : \mathbb{C}[x_0, x_1, \dots , x_n, z] \to \mathbb{C}[x, y, z],$$
$$x_0 \mapsto x^n,\ x_1 \mapsto x^{n-1}y,\ x_2 \mapsto x^{n-2}y^2,\ \dots ,x_n \mapsto y^n,\ z \mapsto z$$

where $x_0, \dots, x_n$ are all of degree n. Find the corresponding open subset U and describe the associated morphisms f_i. Which of the f_i is a closed immersion? (For the closed immersion part see also Problem 2)

Proof. We start with φ_1 and f_1. Let $A = \mathbb{C}[x, y, z]$ and let $B = \mathbb{C}[x, y, w]$. Recall that $U = \{\mathfrak{p} \in \text{Proj } B \mid \varphi_1(A_+) \not\subset \mathfrak{p}\}$. Therefore,

$$\begin{aligned} U^c &= V(\varphi_1(A_+)) \\ &= \{\mathfrak{p} \mid x \in \mathfrak{p}, y \in \mathfrak{p}, w^n \in \mathfrak{p}\} \\ \Rightarrow w \in \mathfrak{p} & \\ &= \{\mathfrak{p} \mid (x, y, w) \subset \mathfrak{p}\} = \emptyset \end{aligned}$$

Therefore, $U = \text{Proj } B = \mathbb{P}^2$.

The variety Proj $A =: \mathbb{P}(1,1,n)$ is known as a weighted projective space and the morphism f_1 corresponds to taking the quotient of $\mathbb{P}^2$ by the cyclic group C_n. For more details, you might like to consult

`https://homepages.warwick.ac.uk/~masda/surf/more/grad.pdf`.

For the case of φ_2, we note that if C is a graded ring and $C[n] := \bigoplus_{d\geq 0} C_{nd}$ then

$$\text{Proj } C[n] \cong \text{Proj } C.$$

This (rather surprising) isomorphism is known as the *Veronese embedding*, and more details can be found in the above link.

Let $A = \mathbb{B}[x_0, x_1, \ldots, x_n, z]$ and $B = \mathbb{C}[x, y, z]$. As $\operatorname{Im} \varphi_2 = \bigoplus_d B_{nd}$ then, by Question 2),

$$f_2 : \text{Proj } B[n] \to \text{Proj } A = \mathbb{P}(1, \ldots, n)$$

is a closed immersion. On the other hand Proj $B[n] = \mathbb{P}(1,1,n)$ because of the Veronese embedding. □

(c) (**Bonus**, +5 points) Let $A = k[x_0, \ldots, x_n]$ where x_i is of some degree $a_i \geq 1$ for all i. Follow the ideas of (b) to show that the weighted projective space

$$\mathbb{P}(a_0, \ldots, a_n) = \text{Proj } k[x_0, \ldots, x_n]$$

admits a closed embedding into some (unweighted) projective space $\mathbb{P}^N_k$.

Problem 2. Let $\varphi : A \to B$ be a *surjective* graded ring homomorphism.

Show that the open set U defined in Problem 1(a) is equal to Proj B, and that the induced morphism

$$f : \text{Proj } B \to \text{Proj } A$$

is a closed immersion. (Hint: Describe the map f on the preimage of the open affine subsets $D(g)$, $g \in A$.)

Proof. We first show that $U =$ Proj B. As φ is surjective and a homomorphism of graded rings, then $\varphi(A_+) = B_+$. Therefore $U = \{\mathfrak{p} \in \text{Proj } B \mid B_+ \not\subset \mathfrak{p}\} = \text{Proj } B$ by definition.

We now show that the induced map f : Proj $B \to$ Proj A is a closed immersion. As $B = A/\operatorname{Ker}\varphi$, then there exists a 1-1 correspondence between homogeneous prime ideals of A containing $\operatorname{Ker}\varphi$ and homogeneous prime ideals of B. Therefore, f : Proj $B \to$ Proj A is a homeomorphism onto the closed subset $V(\operatorname{Ker}(\varphi))$.

For all $g \in A$, the map $\mathcal{O}_{\text{Proj } A}(D(g)) = (A_g)_0 \to (B_{\varphi(g)})_0 = \mathcal{O}_{\text{Proj } B}(f^{-1}D(g))$ is surjective. Therefore, $f^{\#}$ is surjective, and so f is a closed immersion. □

$\mathcal{O}_X$-modules

Problem 3. Let $X = \operatorname{Spec} A$ be an affine scheme. Show that the functors $\sim$ and Γ are adjoint in the following sense: for any A-module M and for any sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ there is a natural isomorphism

$$\operatorname{Hom}_A(M, \Gamma(X, \mathcal{F})) \cong \operatorname{Hom}_{\mathcal{O}_X}(\widetilde{M}, \mathcal{F}).$$

Proof. We define the two morphisms explicitly. For all $\psi : M \to \Gamma(X, \mathcal{F})$ let $\psi : \tilde{M} \to \mathcal{F}$ be defined on a basic open subset by $M_f \to \mathcal{F}(D(f))$, $mf^{-n} \mapsto \frac{\psi(m)|D(f)}{f^n}$ on all $D(f)$ for $f \in A$. This induces a morphism $\tilde{M} \to \tilde{\mathcal{F}}$ on all open subsets. For $\phi : \tilde{M} \to \mathcal{F}$, let $\psi = \phi(X)$. One checks that the two morphisms are inverse to one another. □

Let $\mathcal{F}, \mathcal{G}$ be $\mathcal{O}_X$ modules on a scheme X. We define their tensor product $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ to be the sheaf associated to the presheaf

$$U \mapsto \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U). \tag{1}$$

The right hand side of (1) is naturally an $\mathcal{O}_X(U)$-module, and we give $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ the induced $\mathcal{O}_X$-module structure.

Problem 4.

a) Find an example where the presheaf (1) is not a sheaf. Hence sheafification is necessary when taking tensor products of $\mathcal{O}_X$-modules.

Proof. Let $X = \operatorname{Spec} \mathbb{C}[x] \cong \mathbb{A}^1$ and define the $\mathcal{O}_X$-module $\mathcal{F}$ by

$$\mathcal{F}(U) = \begin{cases} 0 & \text{if } 0 \in U \\ \mathcal{O}_X(U) & \text{if } 0 \notin U. \end{cases}$$

Let $\mathcal{G}$ be the skyscraper sheaf at a point $p \in U$, $p \neq 0$

$$\mathcal{G}(U) = \begin{cases} \mathbb{C} & \text{if } p \in U \\ 0 & \text{if } p \notin U. \end{cases}$$

Then $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} = \mathcal{G}$ but $\mathcal{F}(X) = 0$. Therefore, $\mathcal{F}(X) \otimes_{\mathcal{O}_X(X)} \mathcal{G}(X) = 0 \neq \mathcal{G}(X) = \mathbb{C}$. □

b) Assume that $\mathcal{F}, \mathcal{G}$ are quasi-coherent. Show that the tensor product $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ is again quasi-coherent, and that for every open affine $U \subset X$ we have

$$\Gamma(U, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}) = \Gamma(U, \mathcal{F}) \otimes_{\Gamma(U, \mathcal{O}_X)} \Gamma(U, \mathcal{G}).$$

Proof. Without loss of generality, assume $X = \operatorname{Spec} A$ is affine. Then $\mathcal{F} \cong \tilde{M}$ and $\mathcal{G} \cong \tilde{N}$ where M and N are A-modules. It suffices to show that there exists an isomorphism $\tilde{M} \otimes_{\tilde{A}} \tilde{N} \cong (\widetilde{M \otimes_A N})$ that is functorial in M and N. The sheaf $\tilde{M} \otimes_{\tilde{A}} \tilde{N}$ is attached to the presheaf $U \mapsto \mathcal{H}(U) := \Gamma(U, \tilde{M}) \otimes_{\Gamma(U, \tilde{A})} \Gamma(U, \tilde{N})$ defined for principal open subsets $U = D(f)$ for $f \in A$. Moreover, there are functorial isomorphisms $\mathcal{H}(D(f)) \cong M_f \otimes_{A_f} N_f \cong (M \otimes_A N)_f \cong$

$\Gamma(D(f), (\widetilde{M \otimes_A N}))$ that are compatible with the restriction from $D(f)$ to $D(g) \subset D(f)$. Hence, the presheaf $\mathcal{H}$ is already a sheaf and $\tilde{M} \otimes_{\tilde{A}} \tilde{N} = \widetilde{M \otimes N}$, as desired. □

c) Are the sheaves in your example in a) quasi-coherent? If $\mathcal{F}$ and $\mathcal{G}$ are quasi-coherent, is the pre-sheaf (1) a sheaf?

Proof. No, $\mathcal{F}$ is not quasi-coherent. However (1) is still not a sheaf in general. For example, take the line with two origins L and the skyscraper sheaves $\mathcal{F}_{0_1}$ and $\mathcal{F}_{0_2}$ at the first and second origins. Then $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} = 0$ but $\mathcal{F}(L) = \mathbb{C}$, $\mathcal{G}(L) = \mathbb{C}$, and $\mathcal{F}(L) \otimes_{\mathbb{C}[x]} \mathcal{G}(L) \neq 0$. □

Recall that a morphism of $\mathcal{O}_X$-modules $\mathcal{F} \to \mathcal{G}$ is a morphism of sheaves of abelian groups $\mathcal{F} \to \mathcal{G}$ such that $\mathcal{F}(U) \to \mathcal{G}(U)$ is a $\mathcal{O}_X(U)$-module homomorphism for all $U \subset X$. We write

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$$

for the set of all $\mathcal{O}_X$-module homomorphisms $\mathcal{F} \to \mathcal{G}$. For any two $\mathcal{O}_X$-modules define a sheaf $\mathscr{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ by the assignment

$$U \mapsto \mathrm{Hom}_{\mathcal{O}_X|U}(\mathcal{F}|_U, \mathcal{G}|_U) \tag{2}$$

with restriction morphisms given by the restriction of sheaf homomorphisms, endowed with the natural $\mathcal{O}_X$-module structure.

Problem 5.

a) Show that (2) indeed defines a sheaf and not only a presheaf.

Proof. Let $U = \bigcup_{i \in I} U_i$ be an open covering of an open subset $U \subset X$. We first show that if $x \in \mathscr{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ restricts to zero on an open cover, then x is the zero element. Let $x \in \mathscr{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ be such that $x|_{U_i} = 0$ for all $i \in I$. Let $y \in \mathcal{F}(U)$ be arbitrary. As $x|_{U_i}(y|U_i) = 0$ then $x(U)(y) = 0$. As y is arbitrary, then $x(U) = 0$ and so $x = 0$.

We next show that if $y_i \in \mathscr{H}om(\mathcal{F}, \mathcal{G})(U_i)$ are such that $y_i|U_i \cap U_j = y_j|U_i \cap U_j$ then there exists $y \in \mathscr{H}om(\mathcal{F}, \mathcal{G})(U)$ so that $y|U_i = y_i$ for each i. Let $x_i \in \mathscr{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ be such that $x_i = x_j$ on $U_i \cap U_j$. If $V \subset U$, then $W_i := U_i \cap V$ cover V. Fix a section $x \in \mathcal{F}(V)$ and let $x_i := x|W_i$ and let z_i denote the image of x_i under $y_i(W_i)$. As the sections y_i are compatible, then each of the z_i are compatible. As $\mathcal{G}$ is a sheaf, then there exists a global section $z \in \mathcal{G}(V)$ so that $z|W_i = z_i$. Define $y \in \mathscr{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ by $y(V) : \mathcal{F}(V) \to \mathcal{G}(V)$ by $x \mapsto z$ for all $V \subset U$. Therefore, $y|U_i = y_i$. □

b) Define a natural homomorphism

$$\mathscr{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})_x \to \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{G}_x).$$

Is it injective/surjective/bijective in general? If not give counter-examples.

Proof. Let $X = \text{Spec } \mathbb{C}[x] \cong \mathbb{A}^1$ and let $\mathcal{F}$ be ths skyscraper sheaf at 0, defined by

$$\mathcal{F}(U) = \begin{cases} 0 & \text{if } 0 \in U \\ \mathcal{O}_X(U) & \text{if } 0 \notin U. \end{cases}$$

For the failure of injectivity, note that $\mathscr{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F})_0 \neq 0$ but $\mathcal{F}_0 = 0$. For the failure of surjectivity, let $\mathcal{Q}$ be the quotient of $\mathcal{F} \to \mathcal{O}_X$. As $\mathcal{Q}_0 = \mathcal{O}_{X,0}$, then $\mathscr{H}om(\mathcal{Q}_0, \mathcal{O}_{X,0}) \neq 0$. However, the stalk $\mathscr{H}om(Q, \mathcal{O}_X)_0 = 0$. ☐

Projective spaces are proper!

Problem 6.(Bonus: +10pts) The goal of this problem is to show that the structure morphism from projective space over a ring R,

$$\mathbb{P}^n_R = \text{Proj } R[x_0, \dots, x_n] \to \text{Spec } R,$$

is proper. We first discuss some motivation and then try to turn that motivation into a proof. For simplicity lets take $R = \mathbb{C}$ and $n = 1$. The idea is to use the valuative criterion. For that we need to understand how to 'take limits' in projective space. As explained in the lecture, points in $\mathbb{P}^1_{\mathbb{C}}$ correspond bijectively to $\mathbb{C}^*$-orbits in $\mathbb{C}^2$ excluding the zero orbit. So let us consider curves in $\mathbb{C}^2$ and try to take their limits, keeping in mind that we have the extra freedom of scaling by $\mathbb{C}^*$. For example consider the curve given by

$$t \mapsto (a/t, b/t^2)$$

for some $(a, b) \neq 0$. If $t \to 0$ a limit does not exist in $\mathbb{C}^2$ since the point moves off to infinity. However, under the $\mathbb{C}^*$-scaling

$$(a/t, b/t^2) \sim (at, b),$$

so assuming $b \neq 0$ the rescaled curve $t \mapsto (at, b)$ has the unique limit $(0, b)$ for $t \to 0$. In another direction, consider the curve

$$t \mapsto (at, bt^2).$$

If we take the limit for $t \to 0$ we get the point $(0, 0)$, which does not define a point in projective space. So to obtain a limit in $\mathbb{P}^1$ we need to rescale, this time by dividing by t (assuming $a \neq 0$) to get the limit $(a, 0)$.

(a) Consider the curve in $\mathbb{C}^2$ defined by $t \mapsto (p(t), q(t))$ for some non-constant rational functions $p(t), q(t) \in \mathbb{C}(t)$ Find a $k \in \mathbb{Z}$ such that the rescaled curve $t \mapsto t^k(p(t), q(t))$ has a unique non-zero limit as $t \to 0$. Is k unique?
(Hint: Let $\nu(p)$ denote the order of vanishing (the valuation) of $p(t)$ at the origin. Can you express k in terms of $\nu(p)$ and $\nu(q)$?)

We want to interpret taking this limit in terms of the valuation criterion. So let $A = \mathbb{C}[t]_{(t)}$ and let $K = \mathbb{C}(t)$ be the fraction field, let

$$T = \text{Spec } A, \quad U = \text{Spec } K.$$

Consider an 'infinitesimal path' $\gamma : U \to \mathbb{P}^1$. Finding a limit of γ corresponds to finding a morphism $\tilde{\gamma} : T \to \mathbb{P}^1$ that extends γ, i.e. for which $\tilde{\gamma}|_U = \gamma$. We may assume that γ is non-constant, so has dense image and induces an inclusion $\gamma^* \colon k(\mathbb{P}^1) \hookrightarrow K$. Let

$$f_{01} = \gamma^*(x_0/x_1), \quad f_{10} = \gamma^*(x_1/x_0)$$

In the interpretation of (a) we have $f_{01} = p(t)/q(t)$ and $f_{10} = q(t)/p(t)$.

(b) Show that either $f_{01} \in A$ or $f_{10} \in A$.

Without loss of generality assume $f_{01} \in A$. Consider the ring homomorphism $\mathbb{C}[x_0/x_1] \to A$ that sends x_0/x_1 to f_{01}. Let $T \to D(x_1)$ be the induced map, and $\tilde{\gamma}$ be the composition $T \to D(x_1) \to \mathbb{P}^1$.

(c) Show that $\tilde{\gamma} : T \to \mathbb{P}^1$ satisfies $\tilde{\gamma}|_U = \gamma$ and hence is the desired limit.

(d) Show that in the above argument we may replace $\mathbb{C}[t]_{(t)}$ by any valuation ring over $\mathbb{C}$. Conclude that $\mathbb{P}^1_{\mathbb{C}}$ is proper over Spec $\mathbb{C}$.

(e) Generalize the above argument to the case $\mathbb{P}^n$ and R arbitrary. (Hint: You may assume $R = \mathbb{Z}$ by a base change argument. In the valuation criterion assume that the generic point of T maps into $\cap_{i=0}^n D(x_i)$. Then consider the elements $f_{ij} = \gamma^*(x_i/x_j)$ and their valuations.)