Problem Set 8, due Jan 14, 50 points

Algebraic Geometry I, Winter 18/19

Projective spectrum

Problem 1. (a) Let A, be graded rings and let $\varphi : A \to B$ be a graded ring homomorphism (i.e. a ring homomorphism that preserves the grading). Let

$$U = \{\mathfrak{p} \in \text{Proj } B | \varphi(A_+) \nsubseteq \mathfrak{p}\}.$$

Show that U is an open subset of Proj B and that φ determines a natural morphism $f : U \to \text{Proj } A$.

(b) Let $A = \mathbb{C}[x, y, z]$ where the generators x, y, z are of degree $1, 1, n$ respectively for some $n \geq 1$. Consider the ring homomorphisms

$$\varphi_1 : \mathbb{C}[x, y, z] \to \mathbb{C}[x, y, w], \quad x \mapsto x, y \mapsto y, z \mapsto w^n,$$

where w is of degree 1, and

$$\varphi_2 : \mathbb{C}[x_0, x_1, \dots , x_n, z] \to \mathbb{C}[x, y, z],$$
$$x_0 \mapsto x^n,\ x_1 \mapsto x^{n-1}y,\ x_2 \mapsto x^{n-2}y^2,\ \dots ,x_n \mapsto y^n,\ z \mapsto z$$

where $x_0, \dots , x_n$ are all of degree n. Find the corresponding open subset U and describe the associated morphisms f_i. Which of the f_i is a closed immersion? (For the closed immersion part see also Problem 2)

(c) (**Bonus**, +5 points) Let $A = k[x_0, \dots , x_n]$ where x_i is of some degree $a_i \geq 1$ for all i. Follow the ideas of (b) to show that the weighted projective space

$$\mathbb{P}(a_0, \dots , a_n) = \text{Proj } k[x_0, \dots , x_n]$$

admits a closed embedding into some (unweighted) projective space $\mathbb{P}^N_k$.

Problem 2. Let $\varphi : A \to B$ be a *surjective* graded ring homomorphism. Show that the open set U defined in Problem 1(a) is equal to Proj B, and that the induced morphism

$$f : \text{Proj } B \to \text{Proj } A$$

is a closed immersion. (Hint: Describe the map f on the preimage of the open affine subsets $D(g)$, $g \in A$.)

$\mathcal{O}_X$-modules

Problem 3. Let $X = \text{Spec } A$ be an affine scheme. Show that the functors $\sim$ and Γ are adjoint in the following sense: for any A-module M and for any sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ there is a natural isomorphism

$$\text{Hom}_A(M, \Gamma(X, \mathcal{F})) \cong \text{Hom}_{\mathcal{O}_X}(\widetilde{M}, \mathcal{F}).$$

Let $\mathcal{F}, \mathcal{G}$ be $\mathcal{O}_X$ modules on a scheme X. We define their tensor product $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ to be the sheaf associated to the presheaf

$$U \mapsto \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U). \tag{1}$$

The right hand side of (1) is naturally an $\mathcal{O}_X(U)$-module, and we give $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ the induced $\mathcal{O}_X$-module structure.

Problem 4.

a) Find an example where the presheaf (1) is not a sheaf. Hence sheafification is necessary when taking tensor products of $\mathcal{O}_X$-modules.

b) Assume that $\mathcal{F}, \mathcal{G}$ are quasi-coherent. Show that the tensor product $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ is again quasi-coherent, and that for every open affine $U \subset X$ we have

$$\Gamma(U, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}) = \Gamma(U, \mathcal{F}) \otimes_{\Gamma(U, \mathcal{O}_X)} \Gamma(U, \mathcal{G}).$$

c) Are the sheaves in your example in a) quasi-coherent? If $\mathcal{F}$ and $\mathcal{G}$ are quasi-coherent, is the pre-sheaf (1) a sheaf?

Recall that a morphism of $\mathcal{O}_X$-modules $\mathcal{F} \to \mathcal{G}$ is a morphism of sheaves of abelian groups $\mathcal{F} \to \mathcal{G}$ such that $\mathcal{F}(U) \to \mathcal{G}(U)$ is a $\mathcal{O}_X(U)$-module homomorphism for all $U \subset X$. We write

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$$

for the set of all $\mathcal{O}_X$-module homomorphisms $\mathcal{F} \to \mathcal{G}$. For any two $\mathcal{O}_X$-modules define a sheaf $\mathscr{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ by the assignment

$$U \mapsto \mathrm{Hom}_{\mathcal{O}_X|U}(\mathcal{F}|_U, \mathcal{G}|_U) \tag{2}$$

with restriction morphisms given by the restriction of sheaf homomorphisms, endowed with the natural $\mathcal{O}_X$-module structure.

Problem 5.

a) Show that (2) indeed defines a sheaf and not only a presheaf.

b) Define a natural homomorphism

$$\mathscr{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})_x \to \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{G}_x).$$

Is it injective/surjective/bijective in general? If not give counter-examples.

Projective spaces are proper!

Problem 6.(Bonus: +10pts) The goal of this problem is to show that the structure morphism from projective space over a ring R,

$$\mathbb{P}^n_R = \mathrm{Proj}\ R[x_0, \ldots, x_n] \to \mathrm{Spec}\ R,$$

is proper. We first discuss some motivation and then try to turn that motivation into a proof. For simplicity lets take $R = \mathbb{C}$ and $n = 1$. The idea is to use the valuative criterion. For that we need to understand how to 'take limits' in projective space. As explained in the lecture, points in $\mathbb{P}^1_{\mathbb{C}}$

correspond bijectively to $\mathbb{C}^*$-orbits in $\mathbb{C}^2$ excluding the zero orbit. So let us consider curves in $\mathbb{C}^2$ and try to take their limits, keeping in mind that we have the extra freedom of scaling by $\mathbb{C}^*$. For example consider the curve given by

$$t \mapsto (a/t, b/t^2)$$

for some $(a, b) \neq 0$. If $t \to 0$ a limit does not exist in $\mathbb{C}^2$ since the point moves off to infinity. However, under the $\mathbb{C}^*$-scaling

$$(a/t, b/t^2) \sim (at, b),$$

so assuming $b \neq 0$ the rescaled curve $t \mapsto (at, b)$ has the unique limit $(0, b)$ for $t \to 0$. In another direction, consider the curve

$$t \mapsto (at, bt^2).$$

If we take the limit for $t \to 0$ we get the point $(0, 0)$, which does not define a point in projective space. So to obtain a limit in $\mathbb{P}^1$ we need to rescale, this time by dividing by t (assuming $a \neq 0$) to get the limit $(a, 0)$.

(a) Consider the curve in $\mathbb{C}^2$ defined by $t \mapsto (p(t), q(t))$ for some non-constant rational functions $p(t), q(t) \in \mathbb{C}(t)$ Find a $k \in \mathbb{Z}$ such that the rescaled curve $t \mapsto t^k(p(t), q(t))$ has a unique non-zero limit as $t \to 0$. Is k unique?
(Hint: Let $\nu(p)$ denote the order of vanishing (the valuation) of $p(t)$ at the origin. Can you express k in terms of $\nu(p)$ and $\nu(q)$?)

We want to interpret taking this limit in terms of the valuation criterion. So let $A = \mathbb{C}[t]_{(t)}$ and let $K = \mathbb{C}(t)$ be the fraction field, let

$$T = \text{Spec } A, \quad U = \text{Spec } K.$$

Consider an 'infinitesimal path' $\gamma : U \to \mathbb{P}^1$. Finding a limit of γ corresponds to finding a morphism $\tilde{\gamma} : T \to \mathbb{P}^1$ that extends γ, i.e. for which $\tilde{\gamma}|_U = \gamma$. We may assume that γ is non-constant, so has dense image and induces an inclusion $\gamma^* \colon k(\mathbb{P}^1) \hookrightarrow K$. Let

$$f_{01} = \gamma^*(x_0/x_1), \quad f_{10} = \gamma^*(x_1/x_0)$$

In the interpretation of (a) we have $f_{01} = p(t)/q(t)$ and $f_{10} = q(t)/p(t)$.

(b) Show that either $f_{01} \in A$ or $f_{10} \in A$.

Without loss of generality assume $f_{01} \in A$. Consider the ring homomorphism $\mathbb{C}[x_0/x_1] \to A$ that sends x_0/x_1 to f_{01}. Let $T \to D(x_1)$ be the induced map, and $\tilde{\gamma}$ be the composition $T \to D(x_1) \to \mathbb{P}^1$.

(c) Show that $\tilde{\gamma} : T \to \mathbb{P}^1$ satisfies $\tilde{\gamma}|_U = \gamma$ and hence is the desired limit.

(d) Show that in the above argument we may replace $\mathbb{C}[t]_{(t)}$ by any valuation ring over $\mathbb{C}$. Conclude that $\mathbb{P}^1_{\mathbb{C}}$ is proper over Spec $\mathbb{C}$.

(e) Generalize the above argument to the case $\mathbb{P}^n$ and R arbitrary. (Hint: You may assume $R = \mathbb{Z}$ by a base change argument. In the valuation criterion assume that the generic point of T maps into $\cap_{i=0}^n D(x_i)$. Then consider the elements $f_{ij} = \gamma^*(x_i/x_j)$ and their valuations.)