Algebraic Geometry I. Problem Sheet 7.

This problem sheet is to be submitted on Monday 10/12/2018 before the lecture.

Please email any comments or corrections to `mdawes@math.uni-bonn.de`.

It is possible to score a total of 50 points by answering the non-bonus problems. Additional points can be scored by answering the bonus problems. The total score (which may exceed 50) will count towards the final score for the semester and is given by the sum of the points scored for bonus and non-bonus problems.

(1) (10 points.) Prove the following.

(a) Show that an open subset U of a scheme X is quasi-compact if and only if U can be covered by finitely many open affine subschemes.

Proof. ($\Leftarrow$) Suppose $\{U_i\}_{i\in I}$ is an open cover of X with U_i affine and $|I| < \infty$. As affine schemes are quasi-compact, then each U_i admits a finite cover $\{V_{ij}\}_j$. Therefore, $\cup_j\{V_{ij}\}_j$ is an open affine cover for X.
($\Rightarrow$) As X is a scheme, X admits an open affine cover. As X is quasi-compact, then X admits a finite open affine subcover. □

(b) Recall that a morphism $f : X \to Y$ is called quasi-compact if the pre-image of every open quasi-compact is quasi-compact. Show that a morphism f is quasi-compact if and only if there exists a cover $Y = \cup_i U_i$ by affine open subschemes U_i such that $f^{-1}(U_i)$ is quasi-compact.

Proof. Let U_i be an open affine cover for Y.
($\Rightarrow$) As affine schemes are quasi-compact then, by assumption, $f^{-1}(U_i)$ is quasi-compact.
($\Leftarrow$) Let $V \subset Y$ be open and quasi-compact. Then $f^{-1}(V) = \bigcup_i f^{-1}(V \cap U_i)$. As $\{V \cap U_j\}$ covers V, then there exists a finite subcover $\{V \cap U_j\}_{j=1}^n$ for V. As $f^{-1}(U_j)$ is quasi-compact, it admits a finite cover. Therefore $f^{-1}(V) \cap f^{-1}(U_j)$ admits a finite subcover As $\bigcup_{j=1}^n f^{-1}(V \cap U_j) = \bigcup_{j=1}^n (f^{-1}(V) \cap f^{-1}(U_j))$, then $f^{-1}(V)$ admits a finite cover and so $f^{-1}(V)$ is quasi-compact. □

(2) (16 points.) Prove the following.

(a) A closed immersion is a morphism of finite type.

Proof. Suppose that $f : X \to Y$ is a closed immersion. If $U \subset Y$ is open, then the restriction $f|_{f^{-1}(U)} : f^{-1}(U) \to U$ is a closed immersion. (Moreover, $f|_{f^{-1}(U)}$ is a homeomorphism onto the closed subset $f(X) \cap U \subset U$ and the map of sheaves $\mathcal{O}_Y|_U \to f_*\mathcal{O}_X|_{f^{-1}(U)}$ is surjective, as it defines a surjection on stalks.) Therefore, it suffices to check locally. Without loss of generality, suppose $Y = \mathrm{Spec}(A)$ and $X \cong \mathrm{Spec}(A/I)$. As A/I is a finitely generated A-algebra, the morphism f is of finite type. □

(b) A quasi-compact open immersion is of finite type.

Proof. It suffices to show that f is locally of finite type. If $f : X \to Y$ is an isomorphism onto $U \subset Y$ and $\mathrm{Spec}(A)$ is any open affine of Y, then $f^{-1}(\mathrm{Spec}(A)) = f^{-1}(U \cap \mathrm{Spec}(A))$. Therefore, $U \cap \mathrm{Spec}\, A$ can be covered by open affine sets $\mathrm{Spec}(A_i)$. As $f : X \to U$ is an isomorphism, then $f^{-1}(\mathrm{Spec}(A)$ is covered by open affine sets of the form $\mathrm{Spec}(A_i)$ and each A_i is a finitely generated A-algebra. □

(c) A composition of two morphisms of finite type is of finite type.

Proof. Suppose $f : X \to Y$, $g : Y \to Z$ are morphisms of finite type, let $h = g \circ f$, and let V be an affine open subset of Z. Then $g^{-1}(V)$ is a finite union of two affine subsets

U_i of Y. Each $f^{-1}(U_i)$ is a finite union of affine open subset W_{ij} of X. The composition $W_{ij} \to U_i \to V$ is of finite type. As $h^{-1}(V) = \bigcup_{i,j} W_{ij}$, then h is of finite type □

(d) Morphisms of finite type are stable under base extension. i.e. for any morphism of finite type $f : X \to S$ and for any morphism $b : S' \to S$ the induced map $f' : X' = X \times_S S' \to S'$ is of finite type.

Proof. As the finite type property is local on the base, we can assume that $S = \mathrm{Spec}(A)$ and $S' = \mathrm{Spec}(B)$ are both affine. If $X = f^{-1}(\mathrm{Spec}(A))$ is covered by $\mathrm{Spec}(C_i)$ where C_i is a finite A-algebra, then $(f')^{-1}(S') = X \times_S S'$ is covered by $\mathrm{Spec}(B \otimes_A C_i)$ and as C_i is a finitely generated A-algebra, then $B \otimes_A C_i$ is a finitely generated B-algebra. □

(e) If X and Y are schemes of finite type over S, then $X \times_S Y$ is of finite type over S.

Proof. Without loss of generality, use a local argument. Suppose $S = \mathrm{Spec}(A)$ is affine and let $\mathrm{Spec}(B_i)$ and $\mathrm{Spec}(B'_j)$ be finite covers of X and Y with B_i, B'_j finite A-algebras. Then $\mathrm{Spec}(B_i \otimes_A B'_j)$ is a finite cover of $X \times_S Y$ and $B_i \otimes_A B_j$ is a finite A-algebra. □

(f) If $X \xrightarrow{f} Y \xrightarrow{g} Z$ are two morphisms such that f is quasi-compact and $g \circ f$ is of finite type, then f is of finite type.

Proof. Without loss of generality, use a local argument. Take an open affine cover $\mathrm{Spec}(C_i)$ of Z where $(g \circ f)^{-1}(\mathrm{Spec}(C_i)$ is covered by finitely many $\mathrm{Spec}(A_{ij}) \subset X$. Let $\mathrm{Spec}(B_{ik})$ be a cover of $g^{-1}(\mathrm{Spec}(A_i))$. As $f^{-1}(\mathrm{Spec}(B_{ij}))$ is covered by a collection of $\mathrm{Spec}(A_{ij})$, there are ring maps $C_i \to B_{ik} \to A_{ij}$ so that A_{ij} is of finite type over C_i, from which we conclude that A_{ij} is of finite type over B_{ij} and so f is locally of finite type. □

(g) If $f : X \to Y$ is a morphism of finite type and Y is noetherian, then X is noetherian.

Proof. Take an open cover $\mathrm{Spec}(A_i)$ of Y with A_i noetherian. Then $f^{-1}(A_i)$ can be covered by finitely many $\mathrm{Spec}(B_{ij})$ where B_{ij} are finite A_i-algebras (where we have used the statement of Ex. 3.1 p. 30 of Hartshorne). As each A_i is noetherian, then each B_{ij} is noetherian. □

(3) (12 points.) *(Integral and irreducible fibres.)* Find examples of the following.

(a) Show that there exist morphisms $X \to Y$ with Y integral such that all fibres X_y are irreducible but X is not irreducible.

Proof. Let $Y = \mathrm{Spec}(R)$ where R is a discrete valuation ring, k is the residue field of R, and $K = \mathrm{Frac}(R)$. Let $X = \mathrm{Spec}(K \times k)$ and let $X \to Y$ be the morphism induced by $R \to K \times k$. □

(b) Show that there exist morphisms $X \to \mathrm{Spec}(\mathbb{C}[x])$ with X integral, so that the generic fibre X_η is non-empty and integral but no closed fibre is integral.

Proof. Let $X = \mathrm{Spec}(\mathbb{C}[x])$ and let $X \to \mathrm{Spec}(\mathbb{C}[x])$ be defined by $x \mapsto x^2$. □

(c) Show that there exist morphisms $X \to \mathrm{Spec}(\mathbb{Q}[x])$ so that X is integral and has infinitely many irreducible and infinitely many reducible closed fibres.

Proof. Let $X = \mathrm{Spec}(\overline{\mathbb{Q}}[x])$ or $\mathrm{Spec}(\mathbb{C}[x])$. □

(4) (12 points.) *(Morphisms into separated schemes.)* Consider schemes X and Y over a base scheme S. Assume that X is reduced (or, even stronger, integral) and that $Y \to S$ is separated. Show that any two morphisms $f, g : X \to Y$ over S that coincide on a dense open subset $U \subset X$ are equal. (Hint: Prove that, for X reduced, $f = g$ if and only if $f \circ i_x = g \circ i_x$ for all $x \in X$ where $i_x : \mathrm{Spec}\, k(x) \to X$.) Show that the conditions are neccessary by providing counterexamples if one of the hypotheses is dropped.

Proof. Let $\Delta := \Delta_{Y/S}$ denote the diagonal morphism $Y \to Y \times_S Y$, and let h denote the morphism $(f, g) : X \to Y \times_S Y$. Because of the universal property of fibre products, $\Delta \circ f = (f, f)$. Therefore, $(\Delta \circ f)|_U = h|_U$, and so $U \subset h^{-1}(\Delta(Y))$. As Y is separated, then $\Delta(Y)$ is closed. Therefore, $X = h^{-1}(\Delta(Y))$, from which we conclude that $f(x) = g(x)$ for all $x \in X$.

Without loss of generality, suppose $X = \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(B)$. Let φ and ψ denote the corresponding ring homomorphisms defined by f and g, let $b \in B$ and let $a := \varphi(b) - \psi(b)$. Then $a|_U = 0$. Therefore, $U \subset V(a)$ and so $V(A) = \mathrm{Spec}(A)$ because U is assumed to be dense. Therefore, a is nilpotent. As A is reduced, then $a = 0$. Therefore $\varphi = \psi$ and so $f = g$. ☐

(5) **(For 10 bonus points.)** Let S be a base scheme and let $p : G \to S$ be an S-scheme so that the functor $h_G : \mathrm{Sch}_S^{\mathrm{opp}} \to (\mathrm{Sets})$ factors through the forgetful functor (Groups) $\to$ (Sets). Show that G is a group scheme over S. i.e. there exist morphisms $m : G \times_S G \to G$ and $i : G \to G$ as well as a section $e : S \to G$ to p so that the following diagrams commute.

(a) (Associativity)

$$\begin{array}{ccc} G \times_S G \times_S G & \xrightarrow{\mathrm{id}_G \times m} & G \times_S G \\ {\scriptstyle m \times \mathrm{id}_G}\downarrow & & \downarrow{\scriptstyle m} \\ G \times_S G & \xrightarrow[m]{} & G \end{array}$$

(b) (Existence of a neutral element.)

$$\begin{array}{ccc} G & \xrightarrow{(e \circ p, \mathrm{id}_G)} & G \times_S G \\ {\scriptstyle (\mathrm{id}_G, e \circ p)}\downarrow & \searrow^{\mathrm{id}_G} & \downarrow{\scriptstyle m} \\ G \times_S G & \xrightarrow[m]{} & G \end{array}$$

(c) (Existence of inverse elements.)

$$\begin{array}{ccc} G & \xrightarrow{(i, \mathrm{id}_G)} & G \times_S G \\ {\scriptstyle (\mathrm{id}_G, i)}\downarrow & \searrow^{e \circ p} & \downarrow{\scriptstyle m} \\ G \times_S G & \xrightarrow[m]{} & G \end{array}$$