Algebraic Geometry I. Problem Sheet 7.

This problem sheet is to be submitted on Monday 10/12/2018 before the lecture.

Please email any comments or corrections to `mdawes@math.uni-bonn.de`.

It is possible to score a total of 50 points by answering the non-bonus problems. Additional points can be scored by answering the bonus problems. The total score (which may exceed 50) will count towards the final score for the semester and is given by the sum of the points scored for bonus and non-bonus problems.

(1) (10 points.) Prove the following.
   (a) Show that an open subset U of a scheme X is quasi-compact if and only if U can be covered by finitely many open affine subschemes.
   (b) Recall that a morphism $f : X \to Y$ is called quasi-compact if the pre-image of every open quasi-compact is quasi-compact. Show that a morphism f is quasi-compact if and only if there exists a cover $Y = \cup_i U_i$ by affine open subschemes U_i such that $f^{-1}(U_i)$ is quasi-compact.

(2) (16 points.) Prove the following.
   (a) A closed immersion is a morphism of finite type.
   (b) A quasi-compact open immersion is of finite type.
   (c) A composition of two morphisms of finite type is of finite type.
   (d) Morphisms of finite type are stable under base extension. i.e. for any morphism of finite type $f : X \to S$ and for any morphism $b : S' \to S$ the induced map $f' : X' = X \times_S S' \to S'$ is of finite type.
   (e) If X and Y are schemes of finite type over S, then $X \times_S Y$ is of finite type over S.
   (f) If $X \xrightarrow{f} Y \xrightarrow{g} Z$ are two morphisms such that f is quasi-compact and $g \circ f$ is of finite type, then f is of finite type.
   (g) If $f : X \to Y$ is a morphism of finite type and Y is noetherian, then X is noetherian.

(3) (12 points.) *(Integral and irreducible fibres.)* Find examples of the following.
   (a) Show that there exist morphisms $X \to Y$ with Y integral such that all fibres X_y are irreducible but X is not irreducible.
   (b) Show that there exist morphisms $X \to \mathrm{Spec}(\mathbb{C}[x])$ with X integral, so that the generic fibre X_η is non-empty and integral but no closed fibre is integral.
   (c) Show that there exist morphisms $X \to \mathrm{Spec}(\mathbb{Q}[x])$ so that X is integral and has infinitely many irreducible and infinitely many reducible closed fibres.

(4) (12 points.) *(Morphisms into separated schemes.)* Consider schemes X and Y over a base scheme S. Assume that X is reduced (or, even stronger, integral) and that $Y \to S$ is separated. Show that any two morphisms $f, g : X \to Y$ over S that coincide on a dense open subset $U \subset X$ are equal. (Hint: Prove that, for X reduced, $f = g$ if and only if $f \circ i_x = g \circ i_x$ for all $x \in X$ where $i_x : \mathrm{Spec}\, k(x) \to X$.) Show that the conditions are neccessary by providing counterexamples if one of the hypotheses is dropped.

(5) **(For 10 bonus points.)** Let S be a base scheme and let $p : G \to S$ be an S-scheme so that the functor $h_G : \mathrm{Sch}_S^{\mathrm{opp}} \to (\mathrm{Sets})$ factors through the forgetful functor $(\mathrm{Groups}) \to (\mathrm{Sets})$. Show that G is a group scheme over S. i.e. there exist morphisms $m : G \times_S G \to G$ and $i : G \to G$ as well as a section $e : S \to G$ to p so that the following diagrams commute.

(a) (Associativity)

$$\begin{array}{ccc} G \times_S G \times_S G & \xrightarrow{\mathrm{id}_G \times m} & G \times_S G \\ {\scriptstyle m \times \mathrm{id}_G}\downarrow & & \downarrow{\scriptstyle m} \\ G \times_S G & \xrightarrow[m]{} & G \end{array}$$

(b) (Existence of a neutral element.)

$$\begin{array}{ccc} G & \xrightarrow{(e \circ p, \mathrm{id}_G)} & G \times_S G \\ {\scriptstyle (\mathrm{id}_G, e \circ p)}\downarrow & \searrow^{\mathrm{id}_G} & \downarrow{\scriptstyle m} \\ G \times_S G & \xrightarrow[m]{} & G \end{array}$$

(c) (Existence of inverse elements.)

$$\begin{array}{ccc} G & \xrightarrow{(i, \mathrm{id}_G)} & G \times_S G \\ {\scriptstyle (\mathrm{id}_G, i)}\downarrow & \searrow^{e \circ p} & \downarrow{\scriptstyle m} \\ G \times_S G & \xrightarrow[m]{} & G \end{array}$$