Algebraic Geometry I. Problem Sheet 6.

This problem sheet is to be submitted on Monday 3/12/2018 before the lecture.

Please email any comments or corrections to `mdawes@math.uni-bonn.de`.

It is possible to score a total of 50 points by answering the non-bonus problems. Additional points can be scored by answering the bonus problems. The total score (which may exceed 50) will count towards the final score for the semester and is given by the sum of the points scored for bonus and non-bonus problems.

(1) (12 points.) A scheme $(X, \mathcal{O}_X)$ is said to be *reduced* if for every open set $U \subset X$, the ring $\mathcal{O}_X(U)$ has no nilpotent elements.

(a) Show that $(X, \mathcal{O}_X)$ is reduced if and only if for every $P \in X$, the local ring $\mathcal{O}_{X,P}$ has no nilpotent elements.

Proof. ($\Rightarrow$) Suppose $\mathcal{O}_P$ is reduced for all $P \in X$ and let $s \in \mathcal{O}_X(U)$ be nilpotent. For each stalk, there exists n so that the image of s^n in $\mathcal{O}_P$ is zero. Therefore, as $\mathcal{O}_P$ is reduced then $s_P = 0$. As $S_P = 0$ for all $P \in X$, then $s = 0$.
($\Leftarrow$) Suppose $\mathcal{O}_X(U)$ is reduced for all U. If (s', U) represents nilpotent $s \in \mathcal{O}_P$ then, for some n, $((s')^n, U) \sim (0, V)$. Therefore, $(s'|_{U\cap V})^n = 0$ and so $s'|_{U\cap V} = 0$ as $\mathcal{O}_X(U \cap V)$ is reduced. Therefore $(s', U) \sim (s', U \cap V) \sim (0, U \cap V)$ and so $s = 0$ in $\mathcal{O}_X(P)$. □

(b) Let $(X, \mathcal{O}_X)$ be a scheme. Let $(\mathcal{O}_X)_{red}$ denote the sheaf associated with the presheaf $U \mapsto \mathcal{O}_X(U)_{red}$ where, for a ring A, A_{red} denotes the quotient of A by the ideal of nilpotent elements. Show that $(X, (\mathcal{O}_X)_{red})$ is a scheme (which is known as the *reduced scheme associated with* X, and denoted by X_{red}). Show that there is a morphism of schemes $X_{red} \to X$, which is a homeomorphism of the underlying topological spaces.

Proof. Let U be an affine neighbourhood of $P \in X$ so that $\phi : (U, \mathcal{O}_X|_U) \xrightarrow{\sim} \mathrm{Spec}(R)$ for some ring R. $(U, (\mathcal{O}_X)_{\mathrm{red}}) \cong \mathrm{Spec}(R_{\mathrm{red}})$. Firstly, the topological spaces $\mathrm{Spec}(R)$ and $\mathrm{Spec}(R_{\mathrm{red}})$ are homeomorphic under the surjection $R \to R/\operatorname{nil}(R)$ as every prime ideal contains the nilradical. As localisation commutes with taking quotients, there is a map $R_f \to (R_f)_{\mathrm{red}} = (R_{\mathrm{red}})_f$, which induces a natural map $\psi : \mathcal{O}_{\mathrm{Spec}(R)} \to \mathcal{O}_{R_{\mathrm{red}}}$ in terms of the open sets $D(f)$. The map $\psi \circ \phi : \mathcal{O}_X(V) \to \mathcal{O}_{\mathrm{Spec}\, R_{\mathrm{red}}}(\phi^{-1}(V))$ is a map to a reduced ring, and therefore factors through $\mathcal{O}_V(V)_{\mathrm{red}}$. By the universal property of sheafification, there is a map $(\mathcal{O}_X)_{\mathrm{red}} \to \phi_* \mathcal{O}_{\mathrm{Spec}\, R_{\mathrm{red}}}$ that is an isomorphism on stalks and therefore, by 1(b) of Problem sheet 2, the map is an isomorphism. □

(c) Let $f : X \to Y$ be a morphism of schemes, where X is assumed to be reduced. Show there is a unique morphism $g : X \to Y_{red}$ so that f is obtained by composing g with the natural map $Y_{red} \to Y$.

Proof. This essentially amounts to noting that any map from a ring R to a reduced ring S factors uniquely through the reduced ring R_{red} as the push-forward of a reduced sheaf is reduced. □

(2) (12 points.) Let k be a field. Describe the fibres at all points of the morphism $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ corresponding to the canonical morphism $A \to B$ in the following cases.

(a) $\mathrm{Spec}(k[T, U]/(TU - 1)) \to \mathrm{Spec}(k[T])$;

Proof. **Remark:** Each example follows Example 3.3.1 in Hartshorne, which you can/should consult for full details.

Identify $\operatorname{Spec}(k[T])$ with $\mathbb{A}^1$. If $a \neq 0$, $a \in \mathbb{A}^1$, $au - 1 = 0$ then $u = a^{-1}$. There is no fibre over $a = 0$. □

(b) $\operatorname{Spec}(k[T,U]/(T^2 - U^2)) \to \operatorname{Spec}(k[T])$;

Proof. For $0 \neq a \in \mathbb{A}^1$, the fibre over a is defined by $a^2 - u^2 \subset \mathbb{A}^1$. For $a = 0$, the fibre is given by $u^2 = 0 \in \mathbb{A}^1$. □

(c) $\operatorname{Spec}(k[T,U]/(T^2 + U^2)) \to \operatorname{Spec}(k[T])$;

Proof. If $a \in \mathbb{A}^1$, $a \neq 0$ $a^2 + u^2 = 0$, the fibre over a is given by $u^2 = -a^2 \in \mathbb{A}^1$. Over $a = 0$, the fibre is given by $u^2 = 0 \in \mathbb{A}^1$. □

(d) $\operatorname{Spec}(k[T,U]/(TU)) \to \operatorname{Spec}(k[T])$.

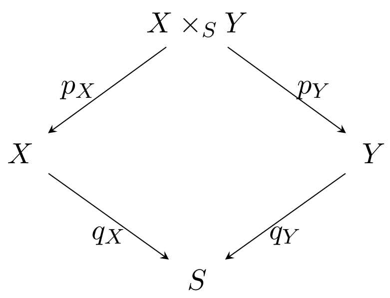
Proof. Let $a \in \mathbb{A}^1$, $a \neq 0$. Then the fibre over 0 is given by $au = 0 \subset \mathbb{A}^1$. If $a = 0$, the fibre is simply $\mathbb{A}^1$. □

(3) (14 points.) Let $|X|$ denote the underlying topological space of a scheme X. Suppose X, Y, Z are schemes and that there are morphisms $f : X \to S$, $g : Y \to S$. Show there exists a natural surjective map of sets

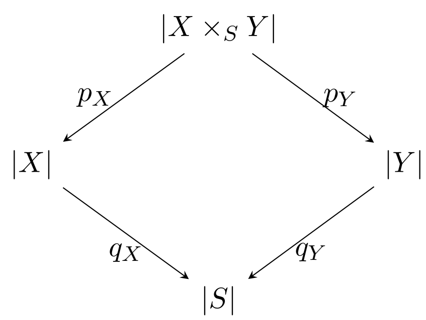
$$\pi : |X \times_S Y| \to |X| \times_{|S|} |Y|.$$

Find examples in which π admits infinite or disconnected fibres.

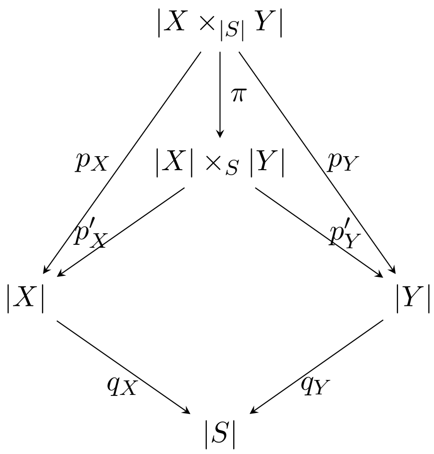
Proof. By assumption,



is universal in the category of varieties. Then, regarding



as a map of sets, there exists a unique product $|X| \times_S |Y|$ in the category of sets and by the universal property, a map π so that



there is a unique product $|X| \times_S |Y|$ in the category of setes □

(4) (12 points.) Let $(X, \mathcal{O}_X)$ be a prevariety over a field k, which is assumed to be algebraically closed. We construct a scheme associated with X as follows. Let $t(X)$ be the set of all irreducible closed subsets of X. For every open subset $U \subset X$ let $U^* \subset t(X)$ be the set of irreducible closed subsets $T \subset X$ such that $T \cap U \neq \varnothing$. Prove the following:

(a) The collection of subsets $\{U^* \mid U \subset X\}$ is a topology on $t(X)$.

Proof. If $\mathcal{J} := \{U^* \mid U \subset X, U \text{ open}\}$, then $t(X) \in \mathcal{J}$ because $t(X) = X^*$ and $\varnothing \in \mathcal{J}$ because $\varnothing^* = \varnothing$. If $U^*, V^* \in \mathcal{J}$, then $(U \cap V)^* = U^* \cap V^*$ and so $U^* \cap V^* \in t(X)$. (By irreducibility, if $T \cap V \neq \varnothing$ and $T \cap U \neq \varnothing$ then $T \cap U$ is irreducible and dense as if $U \cap V \neq \varnothing$, $T \cap (U \cap V) = (T \cap U) \cap V = (T \cap U) \cap (T \cap V) \neq \varnothing$.) If $U_i^* \in \mathcal{J}$ for i in some index set $\mathcal{I}$, then $\cup_{i \in \mathcal{I}} U_i^* = (\cup_{i \in \mathcal{I}} U_i)^*$. Therefore, $\mathcal{J}$ is a topology for $t(X)$. □

(b) With respect to the above topology, a subset $Z \subset t(X)$ is closed if and only if $Z = t(Y)$ for some closed subset $Y \subset X$.

Proof. By part (a), $Z \subset t(X)$ if and only if

$$\begin{aligned} Z &= t(x) \backslash U^* \text{ for some open } U \subset X \\ &= X^* \backslash U^* \\ &= \{\tau \subset X \mid \tau \cap X \neq \varnothing, \tau \cap U = \varnothing, \ \tau \text{ closed}\} \\ &= \{\tau \subset (X \backslash U) \mid \ \tau \text{ closed }, \tau \cap (X \backslash U) \neq \varnothing\}. \end{aligned}$$

□

(c) Show that $\iota_X : x \mapsto \overline{\{x\}}$ for all $x \in X$ induces a homeomorphism of X onto the set of closed points of $t(X)$.

Proof. By definition, using (b). □

We define a sheaf of k-algebras $\mathcal{O}_{t(X)}$ on $t(X)$ by setting, for all open $U \subset X$ with associated open subset $U^* \subset t(X)$,

$$\mathcal{O}_{t(X)}(U^*) := \mathcal{O}_X(U),$$

and restriction maps as for $\mathcal{O}_X$.

(d) Assume that X is affine and $\mathcal{O}_X(X) = A$. Show that $(t(X), \mathcal{O}_{t(X)})$ is a locally ringed space which is isomorphic to $\mathrm{Spec}(A)$.

Proof. See Hartshorne II Proposition 2.6. □

(e) For general X, show that $t(X)$ is a scheme and that $x \in t(X)$ is a closed point if and only if $k(x) = k$.

Proof. For the first part, see Hartshorne II Proposition 2.6. Let $x \in t(X)$ be closed. As $\operatorname{tr.deg}(k(x)) = 0$, then $k(x) = k$.
If $x \in t(X)$ has residue field k, but x is not closed then the irreducible closed subset Z containing x has dimension ≥ 1, therefore $\operatorname{tr.deg}(k(x)) \geq 1$, which is a contradiction. □

(f) Let $f : X \to Y$ be a morphism of prevarieties. Show that f induces a natural map of schemes $t(f) : t(X) \to t(Y)$ and that the induced map

$$\operatorname{Hom}_{\text{pre-vars}}(X, Y) \to \operatorname{Hom}_{\text{schemes}/k}(X, Y)$$

is bijective. (Hint: Treat the affine case first.)

Proof. See Hartshorne II Proposition 2.6 or Mumford II.3 Theorem 2. □

Remark: We will later see that the essential image of the functor t is the category of integral schemes of finite type over k.

(5) **(For 10 bonus points.)** Let $\mathcal{C}$ be a category. For $X \in \mathcal{C}$ let

$$h_X := \operatorname{Hom}_{\mathcal{C}}(-, X) : \mathcal{C}^{op} \to \text{Sets}$$

be the associated functor. Let $F : \mathcal{C}^{op} \to \text{Sets}$ denote an arbitrary functor.

(a) Prove the Yoneda lemma. That is, that the map

$$\operatorname{Hom}(h_X, F) \to F(X)$$

defined by $\eta \mapsto \eta_X(\mathrm{Id}_X)$ is a bijection which is natural in X and F.

(b) Let $X \to S$ and $Y \to S$ be schemes over a base scheme S. Let $\mathcal{C} = (\mathrm{Sch}/S)$ be the category of schemes over S, and let $\mathcal{D} \subset \mathcal{C}$ be the full subcategory consisting of objects $Z \to S \in \mathcal{C}$ with Z affine. If $\operatorname{Hom}_S(X, Y)$ is the set of morphisms $f : X \to Y$ of schemes over S, prove there are bijections

$$\operatorname{Hom}_S(X, Y) \cong \operatorname{Hom}(h_X, h_Y) \cong \operatorname{Hom}(h_{X|\mathcal{D}}, h_{Y|\mathcal{D}})$$

where $F_{|\mathcal{D}}$ denotes the restriction of a functor $F : \mathcal{C}^{op} \to \text{Sets}$ to $\mathcal{D}^{op}$.