

Algebraic Geometry I. Problem Sheet 6.

This problem sheet is to be submitted on Monday 3/12/2018 before the lecture.

Please email any comments or corrections to mdawes@math.uni-bonn.de.

It is possible to score a total of 50 points by answering the non-bonus problems. Additional points can be scored by answering the bonus problems. The total score (which may exceed 50) will count towards the final score for the semester and is given by the sum of the points scored for bonus and non-bonus problems.

- (1) (12 points.) A scheme (X, \mathcal{O}_X) is said to be *reduced* if for every open set $U \subset X$, the ring $\mathcal{O}_X(U)$ has no nilpotent elements.
 - (a) Show that (X, \mathcal{O}_X) is reduced if and only if for every $P \in X$, the local ring $\mathcal{O}_{X,P}$ has no nilpotent elements.
 - (b) Let (X, \mathcal{O}_X) be a scheme. Let $(\mathcal{O}_X)_{red}$ denote the sheaf associated with the presheaf $U \mapsto \mathcal{O}_X(U)_{red}$ where, for a ring A , A_{red} denotes the quotient of A by the ideal of nilpotent elements. Show that $(X, (\mathcal{O}_X)_{red})$ is a scheme (which is known as the *reduced scheme associated with X* , and denoted by X_{red}). Show that there is a morphism of schemes $X_{red} \rightarrow X$, which is a homeomorphism of the underlying topological spaces.
 - (c) Let $f : X \rightarrow Y$ be a morphism of schemes, where X is assumed to be reduced. Show there is a unique morphism $g : X \rightarrow Y_{red}$ so that f is obtained by composing g with the natural map $Y_{red} \rightarrow Y$.
- (2) (12 points.) Let k be a field. Describe the fibres at all points of the morphism $\text{Spec}(B) \rightarrow \text{Spec}(A)$ corresponding to the canonical morphism $A \rightarrow B$ in the following cases.
 - (a) $\text{Spec}(k[T, U]/(TU - 1)) \rightarrow \text{Spec}(k[T])$;
 - (b) $\text{Spec}(k[T, U]/(T^2 - U^2)) \rightarrow \text{Spec}(k[T])$;
 - (c) $\text{Spec}(k[T, U]/(T^2 + U^2)) \rightarrow \text{Spec}(k[T])$;
 - (d) $\text{Spec}(k[T, U]/(TU)) \rightarrow \text{Spec}(k[T])$.
- (3) (14 points.) Let $|X|$ denote the underlying topological space of a scheme X . Suppose X, Y, Z are schemes and that there are morphisms $f : X \rightarrow S, g : Y \rightarrow S$. Show there exists a natural surjective map of sets

$$\pi : |X \times_S Y| \rightarrow |X| \times_{|S|} |Y|.$$

Find examples in which π admits infinite or disconnected fibres.

- (4) (12 points.) Let (X, \mathcal{O}_X) be a prevariety over a field k , which is assumed to be algebraically closed. We construct a scheme associated with X as follows. Let $t(X)$ be the set of all irreducible closed subsets of X . For every open subset $U \subset X$ let $U^* \subset t(X)$ be the set of irreducible closed subsets $T \subset X$ such that $T \cap U \neq \emptyset$. Prove the following:
 - (a) The collection of subsets $\{U^* \mid U \subset X\}$ is a topology on $t(X)$.
 - (b) With respect to the above topology, a subset $Z \subset t(X)$ is closed if and only if $Z = t(Y)$ for some closed subset $Y \subset X$.
 - (c) Show that $\iota_X : x \mapsto \{x\}$ for all $x \in X$ induces a homeomorphism of X onto the set of closed points of $t(X)$.

We define a sheaf of k -algebras $\mathcal{O}_{t(X)}$ on $t(X)$ by setting, for all open $U \subset X$ with associated open subset $U^* \subset t(X)$,

$$\mathcal{O}_{t(X)}(U^*) := \mathcal{O}_X(U),$$

and restriction maps as for \mathcal{O}_X .

- (d) Assume that X is affine and $\mathcal{O}_X(X) = A$. Show that $(t(X), \mathcal{O}_{t(X)})$ is a locally ringed space which is isomorphic to $\text{Spec}(A)$.

- (e) For general X , show that $t(X)$ is a scheme and that $x \in t(X)$ is a closed point if and only if $k(x) = k$.
- (f) Let $f : X \rightarrow Y$ be a morphism of prevarieties. Show that f induces a natural map of schemes $t(f) : t(X) \rightarrow t(Y)$ and that the induced map

$$\mathrm{Hom}_{\mathrm{pre}\text{-vars}}(X, Y) \rightarrow \mathrm{Hom}_{\mathrm{schemes}/k}(X, Y)$$

is bijective. (Hint: Treat the affine case first.)

Remark: We will later see that the essential image of the functor t is the category of integral schemes of finite type over k .

- (5) **(For 10 bonus points.)** Let \mathcal{C} be a category. For $X \in \mathcal{C}$ let

$$h_X := \mathrm{Hom}_{\mathcal{C}}(-, X) : \mathcal{C}^{op} \rightarrow \mathrm{Sets}$$

be the associated functor. Let $F : \mathcal{C}^{op} \rightarrow \mathrm{Sets}$ denote an arbitrary functor.

- (a) Prove the Yoneda lemma. That is, that the map

$$\mathrm{Hom}(h_X, F) \rightarrow F(X)$$

defined by $\eta \mapsto \eta_X(\mathrm{Id}_X)$ is a bijection which is natural in X and F .

- (b) Let $X \rightarrow S$ and $Y \rightarrow S$ be schemes over a base scheme S . Let $\mathcal{C} = (\mathrm{Sch}/S)$ be the category of schemes over S , and let $\mathcal{D} \subset \mathcal{C}$ be the full subcategory consisting of objects $Z \rightarrow S \in \mathcal{C}$ with Z affine. If $\mathrm{Hom}_S(X, Y)$ is the set of morphisms $f : X \rightarrow Y$ of schemes over S , prove there are bijections

$$\mathrm{Hom}_S(X, Y) \cong \mathrm{Hom}(h_X, h_Y) \cong \mathrm{Hom}(h_{X|\mathcal{D}}, h_{Y|\mathcal{D}})$$

where $F|_{\mathcal{D}}$ denotes the restriction of a functor $F : \mathcal{C}^{op} \rightarrow \mathrm{Sets}$ to \mathcal{D}^{op} .