Algebraic Geometry I. Problem Sheet 5.

This problem sheet is to be submitted on Monday 26/11/2018 before the lecture.

Please email any comments or corrections to `mdawes@math.uni-bonn.de`.

It is possible to score a total of 50 points by answering the non-bonus problems. Additional points can be scored by answering the bonus problems. The total score (which may exceed 50) will count towards the final score for the semester and is given by the sum of the points scored for bonus and non-bonus problems.

Assume the field k is algebraically closed, and let R denote a commutative ring with 1.

(1) (10 points.) Prove the following.

(a) A closed subset $V(\mathfrak{a}) \subset \operatorname{Spec}(A)$ is irreducible if and only if $\sqrt{\mathfrak{a}}$ is prime.

Proof. (Irreducible $\Rightarrow$ prime.) Without loss of generality, supose that $\sqrt{\mathfrak{a}} = \mathfrak{a}$. If $\mathfrak{a}$ is not prime, then there exists $a, b \in A \backslash \mathfrak{a}$ so that $ab \in \mathfrak{a}$. Then $V((\mathfrak{a}, a)) \cup V((\mathfrak{a}, b)) = V(\mathfrak{a})$ and as $V((\mathfrak{a}, a))$, $V((\mathfrak{a}, b)) \neq V(\mathfrak{a})$ then $V(\mathfrak{a})$ is not irreducible. Therefore, if $V(\mathfrak{a})$ is irreducible then $\mathfrak{a}$ is prime.
(Irreducible $\Leftarrow$ prime.) Suppose $\mathfrak{a}$ is prime. By (b) and (c), $\overline{\mathfrak{a}} = V(\mathfrak{a})$ and so $\mathfrak{a}$ is irreducible. □

(b) For every $\mathfrak{p} \in \operatorname{Spec} A$, the closure $\overline{\mathfrak{p}}$ is irreducible.

Proof. Suppose $\overline{\mathfrak{p}} = Q_1 \cup Q_2$ with Q_i closed. Then $\mathfrak{p} \in Q_1$ or $\mathfrak{p} \in Q_2$. As Q_1, Q_2 are closed sets containing $\mathfrak{p}$ then $Q_1 = \overline{\mathfrak{p}}$ or $Q_2 = \overline{\mathfrak{p}}$. □

(c) We have $\mathfrak{q} \in \overline{\mathfrak{p}}$ if and only if $\mathfrak{p} \subset \mathfrak{q}$. In particular, $\overline{\mathfrak{p}} = V(\mathfrak{p})$.

Proof. As $\overline{\mathfrak{p}} = \bigcap_{\mathfrak{p} \in V(\mathfrak{a})} V(\mathfrak{a})$ then $\mathfrak{q} \in \overline{\mathfrak{p}}$ if and only if $\mathfrak{q} \in V(\mathfrak{a})$ $\forall$ $\mathfrak{a} \subset \mathfrak{p}$ if and only if $\mathfrak{q} \supset \mathfrak{a}$ $\forall$ $\mathfrak{a} \subset \mathfrak{p}$ if and only if $\mathfrak{q} \supset \mathfrak{p}$.
As $\overline{\mathfrak{p}}$ is irreducible and $V(\mathfrak{p}) = \{\mathfrak{q} \mid \mathfrak{q} \supset \mathfrak{p}\}$, then $\overline{\mathfrak{p}} \in V(\mathfrak{p}) \subset \overline{\mathfrak{p}}$ and by irreducibility $V(\mathfrak{p}) = \overline{\mathfrak{p}}$. □

(d) Every closed irreducible subset $X \subset \operatorname{Spec} A$ has a unique generic point. Which point is it?

Proof. By part (a), $X = V(\mathfrak{p})$ for some prime ideal $\mathfrak{p}$. As $\mathfrak{p} \in V(\mathfrak{p})$ then $\mathfrak{p}$ is a generic point of X.
If $\mathfrak{p}'$ was another generic point of $V(\mathfrak{p})$, then $\mathfrak{p} \subset \mathfrak{p}'$. However, $\mathfrak{p} \in \overline{\mathfrak{p}'}$ and so, by (c), $\mathfrak{p}' \subset \mathfrak{p}$. Therefore, $\mathfrak{p} = \mathfrak{p}'$. □

(2) (10 points.)

(a) Let $f \in R$. Show that $D(f) = \varnothing$ if and only if f is nilpotent.

Proof. ($\Leftarrow$) If f is nilpotent then $f^n = 0$ for some $n > 0$. Therefore, if $\mathfrak{p} \subset R$ is prime then $f \in \mathfrak{p}$ or $f^{n-1} \in \mathfrak{p}$. By induction, $f \in \mathfrak{p}$. Therefore, $\mathfrak{p} \notin D(f)$ and so $D(f) = \emptyset$.
($\Rightarrow$) Suppose $D(f) = \emptyset$ then $f \in \bigcap_{\mathfrak{p} \in X} \mathfrak{p} = \operatorname{Nil}(R)$, therefore f is nilpotent. □

(b) Show that the nilradical of R is equal to the Jacobson radical of R if and only if every non-empty open subset of $\operatorname{Spec} A$ contains a closed point of $\operatorname{Spec} A$.

Proof. ($\Leftarrow$) If $J(R) \neq \operatorname{nil}(R)$ then $\operatorname{Spec}(R) \backslash V(J(R)) \neq \emptyset$ and so there exists an open set containing no closed point.

($\Rightarrow$) Suppose $J(R) = \text{nil}(R)$ and take $\mathfrak{a} \in \operatorname{Spec} R$. If $U(\mathfrak{a}) := \operatorname{Spec}(R) \backslash V(\mathfrak{a}) \neq \emptyset$ I claim there exists a maximal ideal $\mathfrak{m} \in U(\mathfrak{a})$ (and so there exists a maximal ideal $\mathfrak{m}$ so that $\mathfrak{m} \subset \mathfrak{a}$). If, to the contrary, $\mathfrak{a} \subset \mathfrak{m}$ for all maximal ideals $\mathfrak{m}$, then $\mathfrak{a} \subset \text{nil}(R) \Rightarrow \mathfrak{a} \subset \mathfrak{p}$ for all prime ideals $\mathfrak{p}$. Therefore, $V(\mathfrak{a}) = \operatorname{Spec}(R)$ and so $\operatorname{Spec}(R) \backslash V\mathfrak{a}) = \emptyset$, which is a contradiction. □

(3) (10 points.) (Pathological examples.) Find a ring R such that

(a) $\operatorname{Spec}(R)$ is not irreducible, but connected.

Proof. $R = k[x, y]/(xy)$ as $\operatorname{Spec}(R)$ is the union of the two coordinate axes in $\mathbb{A}^2$. □

(b) $\operatorname{Spec}(R)$ is not connected. Can $\operatorname{Spec}(R)$ have infinitely many components?

Proof. $R = k[x] \oplus k[y]$ (p.72 Mumford). For an example with infinitely many connected components, consider $\operatorname{Spec}(R)$ for $R = \prod_{i=1}^{\infty} k$. □

(c) $\operatorname{Spec}(R)$ is not Noetherian. Can you choose R to be Noetherian?

Proof. $R := \bigoplus_{i=0}^{\infty} \mathbb{F}_2$ is not Noetherian but $\operatorname{Spec}(R)$ is Noetherian. One could also take the ring of power series $k[[x]]$. □

(d) **(For 5 bonus points.)**$\operatorname{Spec}(R)$ is Noetherian, but R is not Noetherian.

(e) $\operatorname{Spec}(R)$ has infinitely many points, such that for every pair of points one of them is contained in the closure of the other.

(4) (10 points.)

(a) Let $\phi : A \to B$ be a homomorphism of rings and let $f : Y \to X$ be the induced homomorphism of affine schemes, where $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$. Show that ϕ is injective if and only if the map of schemes $f^* : \mathcal{O}_X \to f_*\mathcal{O}_Y$ is injective. Furthermore, show that if ϕ is injective then f is dominant. i.e. $f(Y)$ is dense in X.

Proof. ($\Rightarrow$) If the map $\mathcal{O}_X \to f_*\mathcal{O}_Y$ is injective, then the map $A = \mathcal{O}_X(X) \to \mathcal{O}_Y(f^{-1}(X)) = \mathcal{O}_Y(Y) = B$ is injective.
($\Leftarrow$) If $\phi : A \to B$ is injective, then $A_x \to B_{\phi(x)}$ is inective for all $x \in A$. Therefore, as $f^{-1}(D(x)) = D(\phi(x))$, then the map $f^* : \mathcal{O}_X(D(x)) \to \mathcal{O}_Y(f^{-1}(D(x)))$ is injective for all $x \in A$. As the map is injective on a basis of open sets for X, it is injective on every stalk, and therefore an injective map of sheaves.
In order to prove that f is dominant, consider $x \in A$. If $D(x) \neq \emptyset$, then there exists $y \in \operatorname{Spec} B$ so that $x \notin \phi^{-1}y$ or, equivalently, that each non-empty $D(x) \subset \operatorname{Spec} A$ contains some $f(y)$. If $x \in \phi^{-1}q$ for all $y \in \operatorname{Spec} B$, then $\phi(x) \in y$ for all x, and is therefore nilpotent. Because ϕ is injective, therefore x is nilpotent and so, by the first part of the problem, $D(x) = \emptyset$. Therefore, $f(Y)$ is dense. □

(b) With the same notation as above, show that if ϕ is surjective, then f is a homeomorphism of Y onto a closed subset of X and $f^* : \mathcal{O}_X \to f_*\mathcal{O}_Y$ is surjective.

Proof. If ϕ is surjective, then $A/\operatorname{Ker}\phi \cong B$. Therefore, $\operatorname{Spec} A/\ker\phi = V(\operatorname{Ker}\phi) \subset \operatorname{Spec} A$ is closed. Suppose $s \in (f_*\mathcal{O}_Y)_p$ is represented by $t \in \mathcal{O}_Y(f^{-1}(U))$. Without loss of generality, assume that $U = D(a)$ and so $f^{-1}(U) = D(\phi(f))$. Therefore, $t \in B_{\phi(a)}$. As ϕ is surjective, then $A_x \to B_{\phi(x)}$ is surjective and so there exists $t \in \mathcal{O}_X(U)$ mapping to s. Therefore, as the induced maps on stalks are surjective, then f is a surjective map of sheaves. □

(c) Conversely, prove that if $f : Y \to X$ is a homeomorphism of Y onto a closed subset and $f^* : \mathcal{O}_X \to f_*\mathcal{O}_Y$ is surjective, then ϕ is surjective. (Hint: consider $X' = \operatorname{Spec}(A/\ker\phi)$ and use parts (a) and (b).)

Proof. If $X' = \operatorname{Spec}(A/\ker\phi)$ then there exists maps ψ, ρ such that $Y \xrightarrow{\psi} X' \xrightarrow{\rho} X$ where ρ is a homeomorphism onto a closed subset by part (b) and $\psi(Y)$ is dense in X' by part (a). As $\rho \circ \psi = f$ and $f(Y)$ is homeomorphic to a closed subset of X, then $\psi(Y) \subset X'$ is dense and closed. Therefore, $\psi(Y) = X'$. As both f and ρ are homeomorphisms, then ψ is a homeomorphism. We next show that ψ^* is an isomorphism. By part (b), ψ^* is injective. As f^* is surjective and the map $f^* : \mathcal{O}_X \to f_*\mathcal{O}_Y$ factors as $\mathcal{O}_X \xrightarrow{\rho^*} \rho\mathcal{O}_{X'} \xrightarrow{\rho_*(\psi^*)} \rho_*\psi^*\mathcal{O}_Y = f_*\mathcal{O}_Y$, then f^* is an isomorphism as it is both injective and surjective. Therefore, $\operatorname{Spec} B \cong \operatorname{Spec}(A/\ker\phi)$ and so $A/\ker\phi \to B$ is an isomorphism, and so ϕ is surjective. ☐

(5) (10 points.) Let X be a scheme.

(a) For $x \in X$, let $\mathcal{O}_x$ denote the local ring at x and let $\mathfrak{m}_x \subset \mathcal{O}_x$ be the maximal ideal. For an arbitrary field K, prove that defining a morphism of $\operatorname{Spec} K \to X$ is equivalent to assigning the inclusion map $k(x) \to K$ for a point $x \in X$.

Proof. If $(\phi, \phi^*) : \operatorname{Spec} K \to X$ is a morphism, then there exists a point $x = \phi((0)) \in X$ and a sheaf morphism $\phi^* : \mathcal{O}_X \to \phi_*\mathcal{O}_{\operatorname{Spec} K}$ which defines the map $\phi_x^* : \mathcal{O}_{X,x} \to \mathcal{O}_{\operatorname{Spec} K,(0)} = K$. As the maximal ideal $\mathfrak{m}_x \mapsto 0$, there is a morphism $k(x) \to K$ which is injective, as $k(x)$ is a field.
For a given point $x \in X$ and an inclusion $k(x) \hookrightarrow K$, define $\psi : \operatorname{Spec} K \to X$ by $\psi((0)) = x$ and $\phi_U^* : \mathcal{O}_X(U) \to \mathcal{O}_{\operatorname{Spec} K}(\psi^{-1}(U))$ for each open set $U \subset X$ by $\mathcal{O}_X(U) \to \mathcal{O}_{X,x} \to k(x) \hookrightarrow K$, where the first arrow is simply the zero map whenever $x \notin U$, and the last arrow is the inclusion into the direct limit when $x \in U$. ☐

(b) For a point $x \in X$, let the *Zariski tangent space* T_x to X and x be the dual of the $k(x)$-vector space $\mathfrak{m}_x/\mathfrak{m}_x^2$. If X is a scheme over k and $k[\epsilon]/\epsilon^2$ is the ring of *dual numbers* over k, prove that defining a k-morphism of $\operatorname{Spec} k[\epsilon]/\epsilon^2$ to X is equivalent to selecting an element $x \in X$ such that $k(x) = k$ and an element of T_x.

Proof. ($\Rightarrow$) Let $\phi : \operatorname{Spec} k[\epsilon]/\epsilon^2 = \{(\epsilon)\} \to X$ be a map of schemes over k. There is a point $\phi(\epsilon) = x \in X$ and a map of local rings

$$\begin{array}{ccc} \mathcal{O}_{X,x} & \longrightarrow & k[\epsilon]/\epsilon^2 \\ & \searrow \quad \nearrow & \\ & k & \end{array}$$

As $k(x) \to k[\epsilon]/\epsilon \cong k$ is an isomorphism, then $\phi_x(\mathfrak{m}_x) \subset (\epsilon)$. As $\epsilon^2 = 0$, define $\psi : \mathfrak{m}_x \to k$ by $\psi(z) = \phi_x(z)/\epsilon$. This map is well defined, k-linear, and annihilates $\mathfrak{m}_x^2$; it therefore defines an element of $\operatorname{Hom}(\mathfrak{m}_x/\mathfrak{m}_x^2, k)$.
($\Leftarrow$) Suppose there exists $x \in X$ with residue field k and a k-linear map $\psi : \mathfrak{m}_x \to k$ so that ψ annihilates $\mathfrak{m}_x^2$, ψ_x is a local homomorphism, and so that the above diagram commutes. Define a map $\phi : \operatorname{Spec} k[\epsilon]/\epsilon^2 \to X$ by $\phi((\epsilon)) = x$ and $\phi^*\mathcal{O}_X \to \mathcal{O}_{\operatorname{Spec} k[\epsilon]/\epsilon^2}$, where the first arrow is 0 if $x \notin U$ and is an inclusion in the direct limit otherwise. ☐