Algebraic Geometry I. Problem Sheet 5.

This problem sheet is to be submitted on Monday 26/11/2018 before the lecture.

Please email any comments or corrections to `mdawes@math.uni-bonn.de`.

It is possible to score a total of 50 points by answering the non-bonus problems. Additional points can be scored by answering the bonus problems. The total score (which may exceed 50) will count towards the final score for the semester and is given by the sum of the points scored for bonus and non-bonus problems.

Assume the field k is algebraically closed, and let R denote a commutative ring with 1.

(1) (10 points.) Prove the following.
  (a) A closed subset $V(\mathfrak{a}) \subset \mathrm{Spec}(A)$ is irreducible if and only if $\sqrt{\mathfrak{a}}$ is prime.
  (b) For every $\mathfrak{p} \in \operatorname{Spec} A$, the closure $\overline{\mathfrak{p}}$ is irreducible
  (c) We have $\mathfrak{q} \in \overline{\mathfrak{p}}$ if and only if $\mathfrak{p} \subset \mathfrak{q}$. In particular, $\overline{\mathfrak{p}} = V(\mathfrak{p})$.
  (d) Every closed irreducible subset $X \subset \operatorname{Spec} A$ has a unique generic point. Which point is it?

(2) (10 points.)
  (a) Let $f \in R$. Show that $D(f) = \varnothing$ if and only if f is nilpotent.
  (b) Show that the nilradical of A is equal to the Jacobson radical of R if and only if every non-empty open subset of $\operatorname{Spec} A$ contains a closed point of $\operatorname{Spec} A$.

(3) (10 points.) (Pathological examples.) Find a ring R such that
  (a) $\mathrm{Spec}(R)$ is not irreducible, but connected.
  (b) $\mathrm{Spec}(R)$ is not connected. Can $\mathrm{Spec}(R)$ have infinitely many components?
  (c) $\mathrm{Spec}(R)$ is not Noetherian. Can you choose R to be Noetherian?
  (d) **(For 5 bonus points.)**$\mathrm{Spec}(R)$ is Noetherian, but R is not Noetherian.
  (e) $\mathrm{Spec}(R)$ has infinitely many points, such that for every pair of points one of them is contained in the closure of the other.

(4) (10 points.)
  (a) Let $\phi : A \to B$ be a homomorphism of rings and let $f : Y \to X$ be the induced homomorphism of affine schemes, where $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$. Show that ϕ is injective if and only if the map of schemes $f^* : \mathcal{O}_X \to f_*\mathcal{O}_Y$ is injective. Furthermore, show that if ϕ is injective then f is dominant. i.e. $f(Y)$ is dense in X.
  (b) With the same notation as above, show that if ϕ is surjective, then f is a homeomorphism of Y onto a closed subset of X and $f^* : \mathcal{O}_X \to f_*\mathcal{O}_Y$ is surjective.
  (c) Conversely, prove that if $f : Y \to X$ is a homeomorphism of Y onto a closed subset and $f^* : \mathcal{O}_X \to f_*\mathcal{O}_Y$ is surjective, then ϕ is surjective. (Hint: consider $X' = \mathrm{Spec}(A/\ker\phi)$ and use parts (a) and (b).)

(5) (10 points.) Let X be a scheme.
  (a) For $x \in X$, let $\mathcal{O}_x$ denote the local ring at x and let $\mathfrak{m}_x \subset \mathcal{O}_x$ be the maximal ideal. For an arbitrary field K, prove that defining a morphism of $\operatorname{Spec} K \to X$ is equivalent to assigning the inclusion map $k(x) \to K$ for a point $x \in X$.
  (b) For a point $x \in X$, let the *Zariski tangent space* T_x to X and x be the dual of the $k(x)$-vector space $\mathfrak{m}_x/\mathfrak{m}_x^2$. If X is a scheme over k and $k[\epsilon]/\epsilon^2$ is the ring of *dual numbers* over k, prove that defining a k-morphism of $\operatorname{Spec} k[\epsilon]/\epsilon^2$ to X is equivalent to selecting an element $x \in X$ such that $k(x) = k$ and an element of T_x.