Algebraic Geometry I. Problem Sheet 4.

This problem sheet is to be submitted on Monday 19/11/2018 before the lecture.

Please email any comments or corrections to `mdawes@math.uni-bonn.de`.

It is possible to score a total of 50 points by answering the non-bonus problems. Additional points can be scored by answering the bonus problems. The total score (which may exceed 50) will count towards the final score for the semester and is given by the sum of the points scored for bonus and non-bonus problems.

Assume the field k is algebraically closed.

(1) (12 points.) Let X and Y be varieties. Prove the following are equivalent:
 (a) $k(X) \cong k(Y)$;
 (b) there exist non-empty open subvarieties $U \subset X$ and $V \subset Y$ which are isomorphic;
 (c) X and Y are birational.

In order to prove the result, we need the following theorem. (I expect a version was proved in lectures, but I'll include here in case it wasn't; if the theorem didn't occur in lectures, you shouldn't regard it as examinable.)

Theorem. *For varieties X and Y, there is a bijection between*
 (a) *the set of dominant rational maps from X to Y;*
 (b) *the set of k-algebra homomorphisms from $k(Y)$ to $k(X)$*
defined by the mapping $\alpha : \psi \mapsto (f \mapsto f \circ \psi)$ for $\psi : X \dashrightarrow Y$ dominant and $f \in k(Y)$.

Proof. We define an inverse to the mapping. Let $\rho : k(Y) \to k(X)$ be a homomorphism of k-algebras. The variety Y is covered by affine charts so, without loss of generality, assume that Y is affine. If $y_1, \ldots, y_n$ are generators for the affine coordinate ring $k[Y]$ as a k-algebra, then $\rho(y_1), \ldots, \rho(y_n) \in k(X)$. Therefore, there exists an open set $U \subset X$ on which $\rho(y_i)$ are regular on U. Therefore, ρ defines an injective homomorphism of k-algebras $k(Y) \to \mathcal{O}(U)$. For any variety X and any affine variety Y, there is a bijection of sets between $\mathrm{Hom}(X, Y)$ and $\mathrm{Hom}(k[Y], \mathcal{O}(X))$, and so there exists a morphism $\phi : U \to Y$ defining a dominant rational map from X to Y. One checks that ϕ defines a map from (b) to (a) which is inverse to α. □

We now prove the main result.

Proof. (c) $\implies$ (b): By definition, if X and Y are birational, then there are birational maps $\phi : X \dashrightarrow Y$ and $\psi : Y \dashrightarrow X$ so that $\psi \circ \phi = id_X$ and $\phi \circ \psi = id_Y$ (as rational maps). Suppose ϕ is represented by (ϕ, U) and ψ is represented by (ψ, V). Then $\psi \circ \phi$ is represented by $(\psi \circ \phi, \phi^{-1}(V))$ and as $\psi \circ \phi = id_X$ as a rational map, then $\psi \circ \phi$ is the identity on $\phi^{-1}(V)$. Similarly, $\phi \circ \phi$ is the identity on $\psi^{-1}(U)$. Therefore, by construction, the open sets $\phi^{-1}(\psi^{-1}(U)) \subset X$ and $\phi^{-1}(\phi^{-1}(V)) \subset Y$ are isomorphic under ϕ and ψ.
(b) $\implies$ (a): By definition of the function field.
(a) $\implies$ (c): Immediate from the previous theorem. □

(2) (14 points) Prove that the following varieties are birational but not isomorphic. Can you construct (or interpret) the birational maps between them geometrically?
 (a) $\mathbb{P}^2$;
 (b) $\mathbb{P}^1 \times \mathbb{P}^1$;
 (c) the surface in $\mathbb{P}^3$ defined by the cubic equation $x^2z + y^2w = xw^2 + yz^2$. (Showing the surface is not isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ is optional and can be omitted. Hint: Consider the lines $x = y = 0$ and $z = w = 0$.)

Proof. Because $\mathbb{P}^2 \supset \mathbb{A}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1 \supset \mathbb{A}^1 \times \mathbb{A}^1 \cong \mathbb{A}^2$, then $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ are birational by Problem 1. In order to prove $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ are non-isomorphic, let Y_1 and Y_2 be copies of $\mathbb{P}^1 \subset \mathbb{P}^2$ and let $Y_1^\bullet$, $Y_2^\bullet$ denote their affine cones in $\mathbb{A}^3$. As $\underline{0} \in Y_1^\bullet \cap Y_1^\bullet$ then $Y_1^\bullet \cap Y_2^\bullet \neq \emptyset$. Furthermore, $\dim Y_1^\bullet \cap Y_2^\bullet \geq 2+2-3 > 0$. Therefore, $Y_1 \cap Y_2 \neq \emptyset$. On the other hand, $\mathbb{P}^1 \times \mathbb{P}^1$ contains two non-intersecting copies of $\mathbb{P}^1$, and so the two varieties cannot be isomorphic.

Note that if $L_1, L_2 \subset \mathbb{P}^3$ are distinct lines and $a \in \mathbb{P}^3 \backslash L_1 \cup L_2$ then there exists a unique line $L(a)$ through L_1, L_2, a. In the case of the cubic $X := V(x^2z + y^2w = xw^2 + yz^2) \subset \mathbb{P}^3$, let L_1 be defined by $x = y = 0$ and L_2 by $z = w = 0$. One constructs a rational map $X \dashrightarrow L_1 \times L_2$ by $a \mapsto (L_1 \cap a, L_2 \cap a) \subset L_1 \times L_2)$, which is well defined away from $L_1 \cup L_2$. The inverse rational map $L_1 \times L_2 \dashrightarrow X$ is given by mapping $L_1 \times L_2 \ni (x, y) \mapsto L(a) \cap X$, which is well defined when L is not contained in X. □

(3) (12 points) A variety X is said to be *normal at a point* $P \in X$ if the ring $\mathcal{O}_p$ is integrally closed; X is said to be *normal* if it is normal at every point $P \in X$.

(a) Show that the variety $V(xy - z^2) \subset \mathbb{P}^2$ is normal.

Proof. The conic $V(xy-z^2)$ is isomorphic to $\mathbb{P}^1$, which is smooth and therefore normal. □

(b) Show that the cuspidal cubic $y^2 = x^3$ is not normal.

Proof. A 1-dimensional variety is normal if and only if it is smooth. The cuspidal cubic cannot be normal as it is singular at the origin. □

(c) Show that an affine variety Y is normal if and only if $k[Y]$ is integrally closed.

Proof. If $A \subset B$ are rings, C is the integral closure of A in B, and S is a multiplicatively closed subset of A, then a standard result in commutative algebra states that $S^{-1}C$ is the integral closure of $S^{-1}A$ in $S^{-1}B$. Therefore $\mathcal{O}(Y)$ is integrally closed if and only if $\mathcal{O}_p$ is integrally closed if and only if Y is normal. □

(d) If Y is an affine variety, show that there is a normal affine variety $\tilde{Y}$ (called the *normalisation* of Y) and a morphism $\pi : \tilde{Y} \to Y$ with the property that whenever Z is a normal variety and $\phi : Z \to Y$ is a dominant morphism, then there exists a unique morphism $\psi : Z \to \tilde{Y}$ so that $\phi = \pi \circ \psi$.

Proof. Let A be the integral closure of $k[Y]$ in $k(Y)$. By the finiteness of integral closure, A is a finitely generated reduced k-algebra and so corresponds to an affine variety $\tilde{Y}$. By the categorical correspondence, there is a morphism $\pi : \tilde{Y} \to Y$. The existance of the map ψ is immediate from the categorical correspondence as any other integrally closed $k[Z]$ satisfying the hypotheses must also contain $k[\tilde{Y}]$. □

You may use, without proof, the following theorem.

Theorem (Finiteness of Integral Closure) Let A be an integral domain and a finitely generated algebra over a field k. Let K be the quotient field of A, and let L be a finite algebraic extension of K. Then the integral closure A' of A in L is a finitely generated A-module, and also a finitely generated k-algebra.

(4) (12 points) A variety X is called *proper* (or *complete*) if for all varieties Y the projection

$$p_2 : X \times Y \to Y$$

is closed (i.e. p_2 maps closed subsets onto closed subsets). Properness for varieties is analogous to compactness for topological spaces. In fact, a Hausdorff topological space A is compact if and only if $A \times B \to B$ is closed for every topological space B. We will discuss properness in detail later in class. For now prove the following:

(a) The affine line $\mathbb{A}^1$ is not closed.

Proof. If $X = Y = \mathbb{A}^1$ then the image of the closed set $V(xy-1) \subset X \times Y \cong \mathbb{A}^2$ under the projection $X \times Y \to Y$ is $\mathbb{A}^1 \backslash \{(0)\}$, which is open. □

(b) Let X be a variety and let $U \subset X$ be a non-empty open subset not equal to X. Then U is not proper. (Your argument for (b) gives another proof for part (a). Why?)

Proof. We can actually prove a slightly stronger result. If $f : W \to Y$ is a morphism of varieties, then the graph $\Gamma_f = \{(w, f(w)) \in W \times Y \mid w \in W\}$ is closed in $W \times Y$. Therefore, if W is proper, then the image $f(W)$ of Γ_f under the projection $W \times Y \to Y$ is closed in Y. Now let $f : U \to X$ be the inclusion morphism. If U is proper, then the image $f(U) = U$ is closed in X. This is a contradiction as U is assumed to be open but neither $\emptyset$ nor X. □

(c) If X and Y are proper, then so is $X \times Y$.

Proof. Let Z be a variety and let $U \subset X \times Y \times Z$ be closed. By the universal property of products of varieties, we can factor the projection $p_2 : X \times Y \times Z \to Z$ as $p_2 = q_2' \circ q_2$ where q_2' and q_2 are the projections $q_2' : Y \times Z \to Z$ and $q_2 : X \times Y \times Z \to Y \times Z$. As X is proper, $q_2(U)$ is closed. As Y is proper, $q_2'(q_2(U))$ is closed. Therefore, $X \times Y$ is proper. □

(d) **(For 5 bonus points)** Let X be a proper variety. Then for every morphism $f : \mathbb{A}^1 \backslash \{0\} \to X$ there exists a morphism $f' : \mathbb{A}^1 \to X$ such that $f'|_{\mathbb{A}^1 \backslash \{0\}} = f$.