Algebraic Geometry I. Problem Sheet 4.

This problem sheet is to be submitted on Monday 19/11/2018 before the lecture.

Please email any comments or corrections to `mdawes@math.uni-bonn.de`.

It is possible to score a total of 50 points by answering the non-bonus problems. Additional points can be scored by answering the bonus problems. The total score (which may exceed 50) will count towards the final score for the semester and is given by the sum of the points scored for bonus and non-bonus problems.

Assume the field k is algebraically closed.

(1) (12 points.) Let X and Y be varieties. Prove the following are equivalent:
  (a) $k(X) \cong k(Y)$;
  (b) there exist non-empty open subvarieties $U \subset X$ and $V \subset Y$ which are isomorphic;
  (c) X and Y are birational.

(2) (14 points) Prove that the following varieties are birational but not isomorphic. Can you construct (or interpret) the birational maps between them geometrically?
  (a) $\mathbb{P}^2$;
  (b) $\mathbb{P}^1 \times \mathbb{P}^1$;
  (c) the surface in $\mathbb{P}^3$ defined by the cubic equation $x^2z + y^2w = xw^2 + yz^2$. (Showing the surface is not isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ is optional and can be omitted. Hint: Consider the lines $x = y = 0$ and $z = w = 0$.)

(3) (12 points) A variety X is said to be *normal at a point* $P \in X$ if the ring $\mathcal{O}_p$ is integrally closed; X is said to be *normal* if it is normal at every point $P \in X$.
  (a) Show that the variety $V(xy - z^2) \subset \mathbb{P}^2$ is normal.
  (b) Show that the cuspidal cubic $y^2 = x^3$ is not normal.
  (c) Show that an affine variety Y is normal if and only if $k[Y]$ is integrally closed.
  (d) If Y is an affine variety, show that there is a normal affine variety $\tilde{Y}$ (called the *normalisation* of Y) and a morphism $\pi : \tilde{Y} \to Y$ with the property that whenever Z is a normal variety and $\phi : Z \to Y$ is a dominant morphism, then there exists a unique morphism $\psi : Z \to \tilde{Y}$ so that $\phi = \pi \circ \psi$.

  You may use, without proof, the following theorem.

  Theorem (Finiteness of Integral Closure) Let A be an integral domain and a finitely generalted algebra over a field k. Let K be the quotient field of A, and let L be a finite algebraic extension of K. Then the integral closure A' of A in L is a finitely generated A-module, and also a finitely generated k-algebra.

(4) (12 points) A variety X is called *proper* (or *complete*) if for all varieties Y the projection

$$p_2 : X \times Y \to Y$$

  is closed (i.e. p_2 maps closed subsets onto closed subsets). Properness for varieties is analogous to compactness for topological spaces. In fact, a Hausdorff topological space A is compact if and only if $A \times B \to B$ is closed for every topological space B. We will discuss properness in detail later in class. For now prove the following:
  (a) The affine line $\mathbb{A}^1$ is not closed.
  (b) Let X be a variety and let $U \subset X$ be a non-empty open subset not equal to X. Then U is not proper. (Your argument for (b) gives another proof for part (a). Why?)

(c) If X and Y are proper, then so is $X \times Y$.

(d) **(For 5 bonus points)** Let X be a proper variety. Then for every morphism $f : \mathbb{A}^1 \backslash \{0\} \to X$ there exists a morphism $f' : \mathbb{A}^1 \to X$ such that $f'|_{\mathbb{A}^1 \backslash \{0\}} = f$.