Algebraic Geometry I. Problem Sheet 3.

This problem sheet is to be submitted no later than 5/11/18.

Please email any comments or corrections to `mdawes@math.uni-bonn.de`.

It is possible to score a total of 50 points by answering the non-bonus problems. Additional points can be scored by answering the bonus problems. The total score (which may exceed 50) will count towards the final score for the semester and is given by the sum of the points scored for bonus and non-bonus problems.

Assume the field k is algebraically closed.

(1) (10 points.) Let $C \subset \mathbb{P}^2$ be the projective variety defined by the cubic equation $zy^2 = x^2(x+z)$. Consider the point $P_0 := [0,0,1]$.

(a) Show that every line L in $\mathbb{P}^2$ passing through P_0 meets the cubic C in P_0 and precisely one more point Q. (The points Q and P_0 may coincide. Do they?).

Proof. For $\xi \in k$, let $\Pi_1 := \langle (1,\xi,0),(0,0,1)\rangle \subset k^3$ and $\Pi_2 := \langle (0,1,0),(0,0,1)\rangle \subset k^3$. Lines in $\mathbb{P}^2$ correspond to $\mathbb{P}(\Pi_1)$, $\mathbb{P}(\Pi_2)$. We calculate the respective intersections with C. Let $[\lambda : \lambda\xi : \mu] \in \mathbb{P}(\Pi_1)$ for $\lambda, \mu \in k$. Then $\mu\lambda^2\xi^2 = \lambda^2(\lambda+\mu)$ and so $\mu\lambda^2(\xi^2-1) = \lambda^2$. If $\lambda \neq 0$, there is a unique solution in μ, unless $\xi^2 = 1$, which has no solution (other than P_0). If $[0 : \lambda : \mu] \in \mathbb{P}(\Pi_2)$ then $\mu\lambda^2 = 0$. If $\lambda \neq 0$ then $\mu = 0$; if $\lambda = 0$, then solution is given by P_0. □

(2) (10 points.)

(a) Let X be a prevariety and let Y be an affine variety. Show that there is a bijection

$$\{f : X \to Y \mid f \text{ is a morphism of prevarieties}\} \cong \mathrm{Hom}_{k-alg}(\mathcal{O}_Y(Y), \mathcal{O}_X(X))$$

defined by sending f to the pullback map f^*.

Proof. By definition, if f is a morphism of pre-varieties, then if $g \in \mathcal{O}_Y(Y)$, then $f^*(g) := gf \in \mathcal{O}_X(X)$, which one checks belongs to $\mathrm{Hom}_{k-alg}(\mathcal{O}_Y(Y), \mathcal{O}_X(X))$, and so the map is well-defined. Conversely, suppose $h \in \mathrm{Hom}_{k-alg}(\mathcal{O}_Y(Y), \mathcal{O}_X(X))$ and take an affine open cover $U_i \subset k^n$ for Y. By the categorical correspondence, $f_i^* : \mathrm{res}_{Y,U_i} \mathcal{O}_Y(Y) \to \mathcal{O}_X(X)$ defines an affine morphism $g_i : X \to U_i \subset Y$, which agrees on $U_i \cap U_j$. Therefore, each of the maps g_i glue to define a morphism $g : Y \to X$. By applying the restriction maps to U_i, one checks that $g^*_{|U_i}$ agrees with f_i, therefore the map is surjective. Let $f_1 : X \to Y$ be a morphism of prevarieties, and define $(f_1)_i$ and $(g_1)_i$ as before. If $f^* = f_1^*$ then, using the categorical correspondence, each of the maps $(f_1)_i$ and f_i agree. Therefore, $f = f_1$, and the correspondence defines a bijection. □

(b) Let Y be an affine variety. Show that every map $\mathbb{P}^n \to Y$ is constant. (Hint: Problem Sheet 2.)

Proof. By Problem sheet 2, $\mathcal{O}_{\mathbb{P}^n}(\mathbb{P}^n) = k$. Therefore, $\mathrm{Hom}_{k-alg}(\mathcal{O}_Y(Y), \mathcal{O}_{\mathbb{P}^n}(\mathbb{P}^n)) = \mathrm{Hom}_{k-alg}(\mathcal{O}_Y(Y), k)$, which consists of precisely the constant maps to k. Therefore, by using the inverse to the pullback, constructed in the previous part of the problem, every pre-variety morphism $f : \mathbb{P}^n \to Y$ is constant. □

(c) Let X be a prevariety such that every map $X \to \mathbb{P}^1$ has closed image. Let Y be an affine variety. Show that every map $X \to Y$ is constant.

Proof. Suppose $Y \subset k^n$ and let f_i denote the restriction of f to the ith coordinate of k^n. Let $f_i' : X \to \mathbb{A}^1 \subset \mathbb{P}^1$ be defined by the composition of f and the natural inclusion of $\mathbb{A}^1$ in $\mathbb{P}^1$. By assumption, f_i' is closed. On the other hand, f_i' is continuous and X is connected, therefore $f_i'(X)$ is connected. As the only closed connected subsets of $\mathbb{P}^1$ are the singletons, then f is constant. □

(3) (10 points.) Show that the homogeneous polynomials $k[x_0, \dots, x_n]$ of degree d form a vector subspace of dimension $\binom{n+d}{d}$.

Proof. By definition, a homogeneous polynomial of degree d belongs to the k-vector space of homogeneous monomials spanned by the set

$$\left\{ x_0^{i_1} \dots x_n^{i_n} \mid i_n \in \mathbb{N}, \sum i_j = d \right\},$$

which is linearly independent over k. We can encode the indices $(i_0, \dots, i_n)$ by d * symbols and n + symbols. (For example, if $n = 2$ and $d = 6$ then ***+**+* would correspond to the partition $(3, 2, 1)$.) There are, therefore, $\binom{n+d}{d}$ unique partitions. □

(4) For $d, n \geq 1$, the d-th Veronese embedding of $\mathbb{P}^n$ is the map

$$\nu_d : \mathbb{P}^n \to \mathbb{P}^{N-1}$$

whose homogeneous coordinates are given by the $N = \binom{n+d}{d}$ monomials of degree d in the coordinates $x_0, \dots, x_n$ on $\mathbb{P}^n$.

(a) Prove that ν_d is a morphism.

Proof. This is immediate, because ν_d is defined by degree d homogeneous polynomials on $\mathbb{P}^n$ (and therefore on any open subset). □

(b) Show that the image of ν_d is defined by quadratic equations (find the equations).

Proof. We claim that $\nu_d(\mathbb{P}^n)$ is the projective variety

$$Z = V(\{z_I z_J - z_K z_L \mid I, J, K, L \in \mathbb{N}^{n+1}, I + J = K + L\}).$$

where if $I = (i_0, \dots, i^n)$ then $z_I := x_0^{i_0} \dots x_n^{i_n}$. If $I + J = K + L$, then $z_I z_J - z_k z_L$ vanishes on $\nu_d(\mathbb{P}^n)$ because

$$x^I x^J - x^K x^L = x^{I+J} - x^{K+L} = 0.$$

Therefore, $\nu_d(\mathbb{P}^n) \subset Z$. In order to show that $Z \subset \nu_d(\mathbb{P}^n)$, we construct an inverse morphism. If $z = [\dots : z_I : \dots] \in Z$, then one of the coordinates $z_{(d,0,\dots,0)}$, $z_{(0,d,\dots,0)}$, $\dots$, $z_{(0,\dots,0,d)}$ must be non-zero, otherwise all of the equations $z_I z_J = z_K z_L$ must be zero, which is a contradiction. Indeed, suppose $z_{(d,0,\dots,0)} = \dots = z_{(0,\dots,0,d)} = 0$ all vanish but some $z_{(i_0,\dots,i_n)} \neq 0$. Then (without loss of generality), we can assume $i_0 > 0$ is maximal (i.e. $z_{(j_0,\dots,j_n)} = 0$ for $j_0 > i_0$). As $i_0 < d$, there is an index i_1 so that $d > i_1 > 0$. Therefore, $z^2_{(i_0,\dots,i_n)} \neq 0$, and so $z_{(i_0+1,i_1-1,\dots,i_n)} z_{(i_0-1,i_1+1,\dots,i_n)} \neq 0$ and $z_{(i_0+1,i_1-1,\dots,i_n)} \neq 0$, which contradicts the maximality of i_0. Let the subset $U_i \subset Z$ be defined by the non-vanishing of a coordinate x_i^d. The subsets $U_0, \dots, U_n$ cover Z and we define a map

$$\psi : U_i \to \mathbb{P}^n$$

by

$$z \mapsto [z_{(1,0,\dots,d-1^i,\dots,0)} : z_{(0,1,0,\dots,d-1^i,0\dots,0)} : \dots : z_{(0,\dots,d-1^i,0,\dots,1)}]$$

for $z \in U_i$ (where the index i in the subscript denotes the index of the component, and not a power). The maps agree on $U_i \cap U_j$ and as

$$z_{(0,\dots,1^a,\dots,d-1^j,\dots,0)} z_{(0,\dots,d^i,\dots,0)} = z_{(0,\dots,1^a,\dots,d-1^i,\dots,0)} z_{(0,\dots,1^i,\dots,d-1^j,\dots,0)}$$

then

$$z_{(0,\dots,1^a,\dots,d-1^i,\dots,0)}$$

$$= \frac{z_{(0,\dots,d^i,\dots,0)}}{z_{(0,\dots,1^i,\dots,d-1^j,\dots,0)}} z_{(0,\dots,1^a,\dots,d-1^j,\dots,0)}$$

and so

$$[z_{(1,0,d-1^i,\dots,0)} : z_{(0,1,0,\dots,d-1^i,0,\dots,0)} : \dots : z_{(0,\dots,d-1^i,0,\dots,1)}] =$$
$$[z_{(1,0,\dots,d-1^j,\dots,0)} : z_{(0,1,0,\dots,d-1^j,0,\dots,0)} : \dots : z_{(0,\dots,d-1^j,0,\dots,1)}].$$

Therefore, we patch the ψ_i together to define a morphism $\psi : Z \to \mathbb{P}^n$. By a simple calculation, the morphism $\psi \circ \nu_d : \mathbb{P}^n \to \mathbb{P}^n$ is given by

$$[x_0 : \dots : x_n] \mapsto \nu_d(x) \mapsto [x_0 x_i^{d-1} : \dots : x_n x_i^{d-1}] = [x_0 : \dots : x_n],$$

which is the identity map. Similarly, one checks that $\nu_d \circ \psi$ is the identity map on Z. Therefore, $z = \nu_d(\psi(z))$ and so $\nu_d)$ is surjective and so $Z \subset \nu_d(\mathbb{P}^n)$ defines an isomorphism between $\mathbb{P}^n$ and Z. □

(c) Let $f \in k[x_0, \dots, x_n]$ be homogenous of degree d. Show that $\nu_d(V(f)) \subset \mathbb{P}^N$ is the intersection of $\nu_d(\mathbb{P}^N)$ with a linear subspace of $\mathbb{P}^N$.

Proof. Suppose

$$f = \sum a_{I_i} x^{I_i}$$

where x^{I_i} denotes a degree d monomial in $k[x_0, \dots, x_n]$ with index set $I_i \in \mathbb{N}^{n+1}$ and $a_I \in k$. Without loss of generality (reordering homogeneous coordinates, if neccesary) suppose $a_0 \neq 0$. Apply the to $\mathbb{P}^N$ the projective transformation defined by the matrix

$$\begin{pmatrix} a_{I_0} & a_{I_1} & \dots & a_{I_n} & \\ 0 & 1 & \dots & 0 & 0 \\ & & \dots & & \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}.$$

Then, in the new coordinates, $\nu_d(V(f))$ is precisely $\nu_d(\mathbb{P}^n) \cap V(z_{I_0})$. □

(d) Let $X \subset \mathbb{P}^n$ be a projective variety and let $f \in k[x_0, \dots, x_n]$ be homogenous of degree d. Conclude that $D(f) \cap X$ is affine.

Proof. By the last part of the problem, $\nu_d(D(f) \cap X)$ belongs to the affine chart U_0 and is Zariski closed, and therefore affine. □

(5) (10 points.) Let X be a variety and let U, V be two open subsets. Prove that $U \cap V$ is affine. Given an example to show this fails if X is only a pre-variety.

Proof. Consider the product $U \times V$ and let π_U and π_V denote the respective projections from $U \times V$ to U and V in X. By definition both $\pi_U(U \times V)$ and $\pi_V(U \times V)$ are closed. Furthermore, $\pi_U(U \times V) = \pi_V(U \times V) = U \cap V$, and so $U \times V$ is affine. In general, the statement fails if X is only a pre-variety. For example, if X is the plane with two origins then X contains two copies of $\mathbb{A}^2$, but their intersection is $\mathbb{A}^2 \backslash \{(0,0)\}$, which is not affine. □