Algebraic Geometry I. Problem Sheet 3.

This problem sheet is to be submitted on Monday 5/11/18 before the lecture.

Please email any comments or corrections to `mdawes@math.uni-bonn.de`.

It is possible to score a total of 50 points by answering the non-bonus problems. Additional points can be scored by answering the bonus problems. The total score (which may exceed 50) will count towards the final score for the semester and is given by the sum of the points scored for bonus and non-bonus problems.

Assume the field k is algebraically closed.

(1) (10 points.) Let $C \subset \mathbb{P}^2$ be the projective variety defined by the cubic equation $zy^2 = x^2(x+z)$. Consider the point $P_0 := [0, 0, 1]$.
   (a) Show that every line L in $\mathbb{P}^2$ passing through P_0 meets the cubic C in P_0 and precisely one more point Q. (The points Q and P_0 may coincide. Do they?)
   (b) Identify the set of lines in $\mathbb{P}^2$ through P_0 with the projective plane $\mathbb{P}^1$. Define a map of sets $\phi : \mathbb{P}^1 \to C$ by mapping a line L to the point Q as in (a). Show that the map ϕ is a morphism.
   (c) Show there exists a non-empty open set $U \subset \mathbb{P}^1$ such that $\phi : U \to C$ is an isomorphism onto its image.

(2) (10 points.)
   (a) Let X be a pre-variety and let Y be an affine variety. Show that there is a bijection
   $$\{f : X \to Y \mid f \text{ is a morphism of prevarieties}\} \cong \mathrm{Hom}_{k-alg}(\mathcal{O}_Y(Y), \mathcal{O}_X(X))$$
   defined by sending f to the pullback map f^*.
   (b) Let Y be an affine variety. Show that every map $\mathbb{P}^n \to Y$ is constant. (Hint: Problem Sheet 2.)
   (c) Let X be a pre-variety such that every map $X \to \mathbb{P}^1$ has closed image. Let Y be an affine variety. Show that every map $X \to Y$ is constant.

(3) (10 points.) Show that the homogeneous polynomials $k[x_0, \dots, x_n]$ of degree d form a vector subspace of dimension $\binom{n+d}{d}$.

(4) (10 points.) For all $d, n \geq 1$, the d-th Veronese embedding of $\mathbb{P}^n$ is the map
   $$\nu_d : \mathbb{P}^n \to \mathbb{P}^{N-1}$$
   whose homogeneous coordinates are given by the $N = \binom{n+d}{d}$ monomials of degree d in the coordinates $x_0, \dots, x_n$ on $\mathbb{P}^n$.
   (a) Prove that ν_d is a morphism.
   (b) Show that the image of ν_d is defined by quadratic equations (find the equations).
   (c) Let $f \in k[x_0, \dots, x_n]$ be homogeneous of degree d. Show that $\nu_d(V(f)) \subset \mathbb{P}^N$ is the intersection of $\nu_d(\mathbb{P}^N)$ with a linear subspace of $\mathbb{P}^N$.
   (d) Let $X \subset \mathbb{P}^n$ be a projective variety and let $f \in k[x_0, \dots, x_n]$ be homogeneous of degree d. Conclude that $D(f) \cap X$ is affine.

(5) (10 points.) Let X be a variety and let U and V be two open affine subvarieties. Prove that $U \cap V$ is affine. Given an example to show this fails if X is only a pre-variety.

(6) **(For 10 bonus points.)** Let X and Y be pre-varieties. We say that a morphism $f : X \to Y$ is *separated* if for all pre-varieties Z and all morphisms $h_1 : Z \to X$ and $h_2 : Z \to X$ such that $f \circ h_1 = f \circ h_2$ the set
   $$\{x \in X | h_1(x) = h_2(x)\}$$
   is closed.

In particular, X is separated if and only if the map to the point $X \to \{*\}$ is separated.

(a) Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of pre-varieties. Show that if $g \circ f$ is separated then f is separated.

(b) Show that if $f : X \to Y$ is separated then (the irreducible components of) every fiber of f is separated. Give an example that shows that the converse does not hold. (Hint: Line with two origins)