Algebraic Geometry I. Problem Sheet 2.

This problem sheet is to be submitted no later than 29/10/18.

Please email any comments or corrections to `mdawes@math.uni-bonn.de`.

It is possible to score a total of 50 points by answering the non-bonus problems. Additional points can be scored by answering the bonus problems. The total score (which may exceed 50) will count towards the final score for the semester and is given by the sum of the points scored for bonus and non-bonus problems.

Assume the field k is algebraically closed.

(1) (10 points.) Let $\phi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves where $\mathcal{F}$ and $\mathcal{G}$ are defined on the underlying topological space X.

(a) Show that $(\operatorname{Ker}\phi)_p = \operatorname{Ker}(\phi_p)$ and $(\operatorname{Im}\phi)_p = \operatorname{Im}(\phi_p)$ for all $p \in X$.

Proof. Note that

$$(\operatorname{Ker}\phi)_p = \lim_{\substack{\longrightarrow \\ U \in p}} (\operatorname{Ker}\phi)(U) = \lim{}_{U\in P} \operatorname{Ker}\psi_U,$$

which is a subgroup of $\mathcal{F}_p$, as is $\operatorname{Ker}\phi_p$. Therefore, we can prove equality inside $\mathcal{F}_p$. If $x \in (\ker)_P$ then pick (U, y) representing x with $y \in \operatorname{Ker}(\phi_U)$ The image of y in $\mathcal{F}_P$ is mapped to zero by ϕ_p. Conversely, if $x \in \operatorname{Ker}\phi_p$ then there exists (U, y) with $y \in \mathcal{F}(U)$ and $\phi_U(y) = 0$. Proceed similarly for $\operatorname{Im}(\phi)$. □

(b) Show that ϕ is injective (respectively surjective) if and only if the induced map on stalks is injective (respectively surjective) for all $p \in X$.

Proof. The morphism ϕ is injective (respectively surjective) if and only if $(\operatorname{Ker}\phi)_p = 0$ (respectively $(\operatorname{Im}\phi)_p = \mathcal{G}_p$) for all p. By the previous part of the question, this holds if and only if $\operatorname{Ker}\phi_p = 0$ (respectively $\operatorname{Im}\phi_p = \mathcal{G}_p$). i.e. if ϕ_p is injective (respectively surjective). □

(c) Show that the sequence of sheaves

$$\ldots \to \mathcal{F}^{i-1} \xrightarrow{\phi^{i-1}} \mathcal{F}^i \xrightarrow{\phi^i} \mathcal{F}^{i+1} \to \ldots$$

is exact if and only if the induced sequence on stalks is exact as a sequence of abelian groups for every point in X.

Proof. Note that

$$\operatorname{Im}\phi^{i-1} = \operatorname{Ker}\phi^i$$

if and only if

$$\operatorname{Im}\phi_p^{i-1} = (\operatorname{Im}\phi^{i-1})_p = (\ker\phi^i)_p = \ker\phi_p^i.$$

□

(d) Find an example where ϕ is surjective but, for some open set U, $\phi(U) : F(U) \to G(U)$ is not surjective. *(You could use the example given in class, but try to find your own.)*

Proof. Suppose $k = \mathbb{C}$ and let $\mathcal{F}_{0,1}$ the sheaf of regular functions on $\mathbb{P}^1$ vanishing at $[0 : 1]$, let $\mathcal{F}_{1,0}$ be the sheaf of regular functions on $\mathbb{P}^1$ vanishing at $[1 : 0]$. Let $\phi : \mathcal{F}_{0,1} \oplus \mathcal{F}_{1,0} \to \mathcal{O}_{\mathbb{P}^1}$ be defined on open sets U by $(f_1, f_2) \mapsto f_1 + f_2$. The map ϕ is surjective because it is surjective on every neighbourhood not containing both $[1 : 0]$, $[0 : 1]$. However, the map on global sections is not surjective because every regular function on $\mathbb{P}^1$ is constant.

Therefore, the global sections of $\mathcal{F}_{0,1}$ and $\mathcal{F}_{1,0}$ are $\{0\}$, but the global sections of $\mathcal{O}_{\mathbb{P}^1}$ is $\mathbb{C}$. □

(e) Show that ϕ is an isomorphism if and only if it is both injective and surjective.

Proof. The map ϕ is an isomorphism if and only if ϕ_p is an isomorphism for all p, which is true if and only if ϕ_p is injective and surjective for all p, which is true if and only if ϕ is injective and surjective. □

(2) (10 points.) Let $f : X \to Y$ be a continuous map of topological spaces. We saw in class that by starting with a sheaf $\mathcal{F}$ on X we can obtain a sheaf $f_*\mathcal{F}$ on Y known as the *pushforward sheaf.* There is also a construction in the opposite direction. If $\mathcal{G}$ is a sheaf on Y, then the *pullback sheaf* $f^{-1}\mathcal{G}$ is defined as the sheaf associated with the presheaf

$$U \mapsto \lim_{\substack{f(U)\subset V \\ V \text{ open}}} \mathcal{G}(V),$$

along with restriction morphisms from G.

Show that

(a) If $\mathcal{F}$ is a sheaf on X, show there is a natural map $f^{-1}f_*\mathcal{F} \to \mathcal{F}$.

Proof. Let $\phi_U : \lim_{f(U)\subset U} \mathcal{F}(f^{-1}(V)) \to \mathcal{F}(U)$ be defined by the collection of maps $\text{res}_{f^{-1}(V),U}$ for all V occurring in the direct limit. Because of the universal property of sheafification, there is a map $\tilde{\phi} : f^{-1}f_*\mathcal{F} \to \mathcal{F}$ that is functorial in $\mathcal{F}$. As f^{-1} defines a functor from sheaves on Y to sheaves on X, there is a map $\theta : \text{Hom}_Y(\mathcal{G}, f_*\mathcal{F}) \to \text{Hom}_X(f^{-1}\mathcal{G}, \mathcal{F})$ defined by $g \mapsto \tilde{\phi} \circ (f^{-1}g)$. □

(b) If $\mathcal{G}$ is a sheaf on Y, show there is a natural map $\mathcal{G} \to f_*f^{-1}\mathcal{G}$.

Proof. Define $\psi_U : \mathcal{G}(U) \to \lim_{f(f^{-1}(U))} \mathcal{G}(V)$ by the inclusion of $\mathcal{G}(U)$ in the direct limit. If we compose ϕ with the sheafication map, we obtain a map $\tilde{\phi} : \mathcal{G} \to f_*f^{-1}\mathcal{G}$ that is functorial in $\mathcal{G}$. We then obtain a map $\nu : \text{Hom}_X(f^{-1}\mathcal{G}, \mathcal{F}) \to \text{Hom}_Y(\mathcal{G}, f_*\mathcal{F})$ defined by $g \mapsto (f_*g) \circ \phi$. □

Use the maps in (a) and (b) to show there is a natural bijective correspondence

$$\text{Hom}_X(f^{-1}\mathcal{G}, \mathcal{F}) = \text{Hom}_Y(\mathcal{G}, f_*\mathcal{F}).$$

Proof. We check that $\nu \circ \theta = id$ and $\theta \circ \nu = id$ by a simple calculation on stalks. □

(3) (10 points.) Calculate the ring of algebraic functions on $\mathbb{A}^2\backslash\{(0,0)\}$. Conclude that $\mathbb{A}^2\backslash\{(0,0)\}$ is not an affine variety.

Proof. Let $D_x := \{(x,y) \in \mathbb{A}^2 \mid x \neq 0\}$ and $D_y := \{(x,y) \in \mathbb{A}^2 \mid y \neq 0\}$. Let f be a regular function on $\mathbb{A}^2\backslash\{(0,0)\}$. The restriction f_x of f to D_x is regular, therefore $f_x = g_x/x^n$ for some $g_x \in k[x,y]$ where g_x is not divisible by x when $n > 0$. Similarly, the restriction f_y of f to D_y is regular, therefore $f_y = g_y/y^m$ for some $g_y \in k[x,y]$ where g_y is not divisible by y when $m > 0$. Therefore, on $D_x \cap D_y$, $y^m g_x = x^n g_y$. The ring $k[x,y]$ is a unique factorisation domain, and so $m = n = 0$ and $f = g_x = g_y$. Therefore, f is defined at $(0,0)$, and so $f \in k[x,y]$ and $k[\mathbb{A}^2\backslash(0,0)] \subset k[x,y]$. Conversely, $k[x,y] \subset k[\mathbb{A}^2\backslash(0,0)]$. If $\mathbb{A}^2\backslash(0,0)$ is affine, then every proper ideal $\mathfrak{a} \subset k[\mathbb{A}^2\backslash(0,0)]$ defines a non-empty subset. However, if $\mathfrak{a} = (x,y)$ then $V(\mathfrak{a}) = (0,0)$ which is not contained in $\mathbb{A}^2\backslash(0,0)$. □

(4) (10 points.) Show that every algebraic function on $\mathbb{P}^n$ is constant. Hence, $\mathcal{O}_{\mathbb{P}^n}(\mathbb{P}^n) = k$.

Proof. We invoke a topological argument based on the compactness and irreducibility of $\mathbb{P}^n$. Let $f : \mathbb{P}^n \to \mathbb{A}^1$ be an algebraic function, and let $g : \mathbb{P}^n \to \mathbb{P}^1$ be the natural extension of f defined by the embedding $\mathbb{A}^1 \subset \mathbb{P}^1$. Consider the graph $\Gamma_g := \{(a, b) \mid g(a) = b\} \subset \mathbb{P}^n \times \mathbb{P}^1$. The image of the natural projection $\Gamma_f \to \mathbb{P}^1$ is closed, and so the image of g must also be closed. As the point at infinity is not contained in $g(\mathbb{P}^n)$, then $g(\mathbb{P}^n)$ is a closed set of $\mathbb{A}^1$. Therefore, $g(\mathbb{P}^n) = \{x_1, \ldots, x_m\}$ is a finite subset of $\mathbb{A}^1$. As $\mathbb{P}^n = \cup_{i=1}^m f^{-1}(x_i)$, then $\mathbb{P}^n = f^{-1}(x_0)$ for some distinguished $x_0 \in \{x_1, \ldots, x_m\}$; if not, we contradict the irreducibility of $\mathbb{P}^n$. Therefore, f must be constant. Therefore, $\mathcal{O}_{\mathbb{P}^n}(\mathbb{P}^n) = k$. □

(5) (10 points.)

(a) Show that any algebraic isomorphism $\mathbb{A}^1 \to \mathbb{A}^1$ is of the form

$$x \mapsto ax + b$$

for some $a \in k^*$ and some $b \in k$.

Proof. Apply the categorical correspondence. If $f \in \mathrm{Aut}(X)$ then, by the categorical correspondence, f^* is an automorphism the k-algebra $k[x]$. Therefore (for surjectivity reasons) $f^*(x)$ must be of degree 1 (i.e. $f^*(x) = ax+b$ for some $a \in k^*$, $b \in k$). Conversely, any map $x \mapsto ax + b$ of the above form is invertible as a k-algebra homomorphism. □

(b) Show that any algebraic isomorphism $\mathbb{P}^1 \to \mathbb{P}^1$ is of the form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : [x : y] \mapsto [ax + by : cx + dy]$$

for some

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2, k) := \{M \in \mathrm{Mat}_2(k) \mid \det M \in k^*\}.$$

Proof. A simple check shows that any $M \in \mathrm{GL}(2, k)$ acting as in the statement of the question is algebraic and invertible, and so the image of $\mathrm{GL}(2, k)$ belongs to $\mathrm{Aut}(\mathbb{P}^1)$. Conversely, suppose $f \in \mathrm{Aut}(\mathbb{P}^1)$. As any point $[a : b] \in \mathbb{P}^1$ can be transformed to $[0 : 1]$ by an element of $M \in \mathrm{GL}(2, k)$ then, without loss of generality, we can assume that $f \circ M$ fixes $[0 : 1]$. As $\mathbb{P}^1$ is the disjoint union $\mathbb{A}^1 \cup \{[1 : 0]\}$, then f defines an algebraic map $\mathbb{A}^1 \to \mathbb{A}^1$. By the previous part of the problem, $f \circ M$ acts by $[x : 1] \mapsto [a'x + b' : 1]$, and so f acts by $[x : y] \mapsto [ax + b : cx + d]$ where

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2, k).$$

□

(c) **(For 10 bonus points.)** Find all algebraic isomorphisms $\mathbb{P}^2 \to \mathbb{P}^2$.

(6) **(For 10 bonus points.)** The notion of a (pre)sheaf on a topological space can be formalised as follows.

A *Grothendieck topology* $(\mathcal{C}, \mathrm{Cov}_{\mathcal{C}})$ consists of a category $\mathcal{C}$ with a set $\mathrm{Cov}_{\mathcal{C}}$ consisting of collections $\{\pi_i : U_i \to U\}_i$ of morphisms in $\mathcal{C}$ (called *coverings of* U) satisfying the following conditions:

(a) Any isomorphism $\phi : U \xrightarrow{\sim} U$ in $\mathcal{C}$ defines a covering $\{U \to U\} \in \mathrm{Cov}_{\mathcal{C}}$.

(b) Suppose we are given a covering $\{\pi_i : U_i \to U\}_i \in \mathrm{Cov}_{\mathcal{C}}$ and for each i a covering $\{\pi_{ij} : U_{ij} \to U_i\}_j \in \mathrm{Cov}_{\mathcal{C}}$. Then, $\{\pi_i \circ \pi_{ij} : U_{ij} \to U\} \in \mathrm{Cov}_{\mathcal{C}}$ is a covering.

(c) If $\{\pi_i : U_i \to U\}_i$ is a covering and $V \to U$ is a morphism in $\mathcal{C}$, then $\{\tilde{\pi}_i : U_i \times_U V \to V\}_i$ is a covering.

(In particular, we assume the fibre products in (b) and (c) exist.)

Example. Take a finite group G and consider the category **G-Sets** consisting of sets S with a left G-action $G \times S \to S$. The morphisms in **G-Sets** are maps commuting with the G-action. The group G comes with a natural left G-action, and we denote the corresponding object in **G-Sets** by $\langle G \rangle$. One can show that the collections of $\{S_i \to S\}_i$ with $\bigcup S_i \to S$ surjective define a Grothendieck topology on **G-sets**.

Show that,

(a) For a topological space X, the category of open sets Ouv_X, whose objects are open subsets $U \subset X$ and morphisms are inclusions, comes with a natural Grothendieck topology given by the usual open coverings $U = \bigcup U_i$. Show that the notions of presheaf, sheaf, stalk, morphism of (pre)sheaves, etc., can be phrased entirely in terms of this Grothendieck topology.

(b) In the category **G-Sets**, show that the collections of $\{S_i \to S\}_i$ with $\bigcup S_i \to S$ define a Grothendieck topology.

(c) Show that any sheaf $\mathcal{F}$ on **G-Sets** defines a set $\mathcal{F}(\langle G \rangle)$ endowed with a natural left G-action.

(d) The final object in **G-Sets** consists of a set $\{*\}$ consisting of a single element. If $\mathcal{F}$ is a sheaf, show that the space of sections $\Gamma(\mathcal{F}, \{*\})$ is the fixed point set $\mathcal{F}(\langle G \rangle)^G$.