Algebraic Geometry I. Problem Sheet 2.

This problem sheet is to be submitted no later than 29/10/18.

Please email any comments or corrections to `mdawes@math.uni-bonn.de`.

It is possible to score a total of 50 points by answering the non-bonus problems. Additional points can be scored by answering the bonus problems. The total score (which may exceed 50) will count towards the final score for the semester and is given by the sum of the points scored for bonus and non-bonus problems.

Assume the field k is algebraically closed.

(1) (10 points.) Let $\phi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves where $\mathcal{F}$ and $\mathcal{G}$ are defined on the underlying topological space X.

(a) Show that $(\operatorname{Ker}\phi)_p = \operatorname{Ker}(\phi_p)$ and $(\operatorname{Im}\phi)_p = \operatorname{Im}(\phi_p)$ for all $p \in X$.

(b) Show that ϕ is injective (respectively surjective) if and only if the induced map on stalks is injective (respectively surjective) for all $p \in X$.

(c) Show that the sequence of sheaves

$$\ldots \to \mathcal{F}^{i-1} \xrightarrow{\phi^{i-1}} \mathcal{F}^{i} \xrightarrow{\phi^{i}} \mathcal{F}^{i+1} \to \ldots$$

is exact if and only if the induced sequence on stalks is exact as a sequence of abelian groups for every point in X.

(d) Find an example where ϕ is surjective but, for some open set U, $\phi(U) : F(U) \to G(U)$ is not surjective. *(You could use the example given in class, but try to find your own.)*

(e) Show that ϕ is an isomorphism if and only if it is both injective and surjective.

(2) (10 points.) Let $f : X \to Y$ be a continuous map of topological spaces. We saw in class that by starting with a sheaf $\mathcal{F}$ on X we can obtain a sheaf $f_*\mathcal{F}$ on Y known as the *pushforward sheaf*. There is also a construction in the opposite direction. If $\mathcal{G}$ is a sheaf on Y, then the *pullback sheaf* $f^{-1}\mathcal{G}$ is defined as the sheaf associated with the presheaf

$$U \mapsto \lim_{\substack{f(U)\subset V \\ V \text{ open}}} \mathcal{G}(V),$$

along with restriction morphisms from G.

Show that

(a) If $\mathcal{F}$ is a sheaf on X, show there is a natural map $f^{-1}f_*\mathcal{F} \to \mathcal{F}$.

(b) If $\mathcal{G}$ is a sheaf on Y, show there is a natural map $\mathcal{G} \to f_*f^{-1}\mathcal{G}$.

Use the maps in (a) and (b) to show there is a natural bijective correspondence

$$\operatorname{Hom}_X(f^{-1}\mathcal{G}, \mathcal{F}) = \operatorname{Hom}_Y(\mathcal{G}, f_*\mathcal{F}).$$

(3) (10 points.) Calculate the ring of algebraic functions on $\mathbb{A}^2\backslash\{(0,0)\}$. Conclude that $\mathbb{A}^2\backslash\{(0,0)\}$ is not an affine variety.

(4) (10 points.) Show that every algebraic function on $\mathbb{P}^n$ is constant. Hence, $\mathcal{O}_{\mathbb{P}^n}(\mathbb{P}^n) = k$.

(5) (10 points.)

(a) Show that any algebraic isomorphism $\mathbb{A}^1 \to \mathbb{A}^1$ is of the form

$$x \mapsto ax + b$$

for some $a \in k^*$ and some $b \in k$.

(b) Show that any algebraic isomorphism $\mathbb{P}^1 \to \mathbb{P}^1$ is of the form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : [x : y] \mapsto [ax + by : cx + dy]$$

for some

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2, k) := \{M \in \mathrm{Mat}_2(k) \mid \det M \in k^*\}.$$

(c) **(For 10 bonus points.)** Find all algebraic isomorphisms $\mathbb{P}^2 \to \mathbb{P}^2$.

(6) **(For 10 bonus points.)** The notion of a (pre)sheaf on a topological space can be formalised as follows.

A *Grothendieck topology* $(\mathcal{C}, \mathrm{Cov}_{\mathcal{C}})$ consists of a category $\mathcal{C}$ with a set $\mathrm{Cov}_{\mathcal{C}}$ consisting of collections $\{\pi_i : U_i \to U\}_i$ of morphisms in $\mathcal{C}$ (called *coverings of* U) satisfying the following conditions:

(a) Any isomorphism $\phi : U \xrightarrow{\sim} U$ in $\mathcal{C}$ defines a covering $\{U \to U\} \in \mathrm{Cov}_{\mathcal{C}}$.

(b) Suppose we are given a covering $\{\pi_i : U_i \to U\}_i \in \mathrm{Cov}_{\mathcal{C}}$ and for each i a covering $\{\pi_{ij} : U_{ij} \to U_i\}_j \in \mathrm{Cov}_{\mathcal{C}}$. Then, $\{\pi_i \circ \pi_{ij} : U_{ij} \to U\} \in \mathrm{Cov}_{\mathcal{C}}$ is a covering.

(c) If $\{\pi_i : U_i \to U\}_i$ is a covering and $V \to U$ is a morphism in $\mathcal{C}$, then $\{\tilde{\pi}_i : U_i \times_U V \to V\}_i$ is a covering.

(In particular, we assume the fibre products in (b) and (c) exist.)

Example. Take a finite group G and consider the category **G-Sets** consisting of sets S with a left G-action $G \times S \to S$. The morphisms in **G-Sets** are maps commuting with the G-action. The group G comes with a natural left G-action, and we denote the corresponding object in **G-Sets** by $\langle G \rangle$. One can show that the collections of $\{S_i \to S\}_i$ with $\bigcup S_i \to S$ surjective define a Grothendieck topology on **G-sets**.

Show that,

(a) For a topological space X, the category of open sets Ouv_X, whose objects are open subsets $U \subset X$ and morphisms are inclusions, comes with a natural Grothendieck topology given by the usual open coverings $U = \bigcup U_i$. Show that the notions of presheaf, sheaf, stalk, morphism of (pre)sheaves, etc., can be phrased entirely in terms of this Grothendieck topology.

(b) In the category **G-Sets**, show that the collections of $\{S_i \to S\}_i$ with $\bigcup S_i \to S$ define a Grothendieck topology.

(c) Show that any sheaf $\mathcal{F}$ on **G-Sets** defines a set $\mathcal{F}(\langle G \rangle)$ endowed with a natural left G-action.

(d) The final object in **G-Sets** consists of a set $\{*\}$ consisting of a single element. If $\mathcal{F}$ is a sheaf, show that the space of sections $\Gamma(\mathcal{F}, \{*\})$ is the fixed point set $\mathcal{F}(\langle G \rangle)^G$.