Algebraic Geometry I. Solutions to Problem Sheet 1.

This problem sheet is to be submitted no later than 22/10/18.

Please email any comments or corrections to `mdawes@math.uni-bonn.de`.

It is possible to score a total of 50 points by answering the non-bonus problems. Additional points can be scored by answering the bonus problems. The total score (which may exceed 50) will count towards the final score for the semester and is given by the sum of the points scored for bonus and non-bonus problems.

Assume the field k is algebraically closed.

(1) (10 points.) Prove that each of the following sets is an affine algebraic set and determine the vanishing ideal $I(X)$.

(a) $X = \{p\} \subset \mathbb{C}^n$, for some $p \in \mathbb{C}^n$.

Proof. Let $p = (a_1, \ldots, a_n) \in \mathbb{A}^n$. We will show that $I(X) = I$ where $I := (x_1 - a_1, \ldots, x_n - a_n)$. Clearly, $I \subset I(X)$ and $V(I) = X$. Suppose $f \in \mathbb{C}[x_1, \ldots, x_n] \cap I(X)$. As $x_i \equiv a_i \bmod I$, then $f(x_1, \ldots, x_n) \equiv f(a_1, \ldots, a_n) = 0 \bmod I$ and so $I(X) = I$. Moreover, as $f \equiv f(p) \bmod I$, then $\mathbb{C}[x_1, \ldots, x_n]/I(X) = \mathbb{C}$. □

(b) $X = \{(t, t^2, t^3) \mid t \in \mathbb{C}\}$.

Proof. Let $I = (x_1^2 - x_2, x_1^3 - x_2)$. Clearly $I \subset I(X)$ and $V(I) = X$. On the other hand, $\mathbb{C}[x_1, x_2, x_3](x_1^2 - x_2, x_1^3 - x_3)$ is isomorphic to $\mathbb{C}[x_1]$ under the map $f(x_1, x_2, x_3) \mapsto f(x_1, x_1^2, x_1^3)$, as in the case $X = \{p\}$. Suppose $f \in \mathbb{C}[x_1, x_2, x_3] \cap I(X)$. Then $f(x_1, x_2, x_3) \mapsto f(x_1, x_1^2, x_1^3)$ under the above isomorphism. Therefore, $f \equiv 0 \bmod I$ and so $f \in I$. Therefore, $I(X) = (x_1^2 - x_2, x_1^3 - x_3)$. □

(c) $X = \{(t^2, t^3) \mid t \in \mathbb{C}\}$.

Proof. Let $I = (x_2^2 - x_1^3)$. Clearly, $I \subset I(X)$. I claim that $V(I) = X$. Suppose $x_1, x_2 \in \mathbb{A}^2$ satisfy $x_2^2 = x_1^3$ and select $t \in \mathbb{C}$ so that $t^2 = x$. As $x_2^2 = (t^3)^2$, then $x_2 = t^3$ or $x_2 = -t^3 = (-t)^3$. Therefore, $(x_1, x_2) = (t^2, t^3)$ or $(x_1, x_2) = (t^2, -t^3)$. As a vector space, $\mathbb{C}[x_1, x_2]/I \cong \mathbb{C}[x_1] \oplus \mathbb{C}[x_1].x_2$. Under this isomorphism, $f \in \mathbb{C}[x_1, x_2] \cap I(X)$ is given by $f(x_1, x_2) = g_1(x_1) + g_2(x_1)x_2$ and satisfies $g_1(t^2) + g_2(t^2)t^3 = 0$. However, if we expand the polynomials $g_1(t^2)$ and $g_2(t^2)t^3$, we see that $g_1(t^2)$ contains only even powers of t, and $g_2(t^2)t^3$ contains only odd powers. Therefore, both g_1 and g_2 are zero and so $I(X) = I$. □

Conclude that no two are isomorphic.

Proof. To show the rings are non-isomorphic as $\mathbb{C}$-algebras, note that $\mathbb{C}$ is 1-dimensional, while the other two are infinite dimensional. The ring $\mathbb{C}[x_1]$ is Euclidean, and therefore a unique factorisation domain, however in $\mathbb{C}[x_1, x_2]/(x_2^2 - x_1^3)$, $y^2 = x^3$. However, y is neither a scalar multiple of x nor x^2 (use a degree argument), and one concludes $\mathbb{C}[x_1, x_2]/(x_2^2 - x_1^3)$ is not a UFD. □

(d) **(For 4 bonus points)** $X = \{(t^3, t^4, t^5) \mid t \in \mathbb{C}\}$.

(2) (5 points.) Consider the ring $A = k[x, y]/(y^2 - x^3 + x)$. Determine elements $z_1, \ldots, z_m$ of A so that A is a finite $k[z_1, \ldots, z_m]$-algebra.

Proof. If $B \subset A$ are algebras, then A is a finite B-algebra if there exists a finite set $\{a_1, \ldots, a_n\}$ so that

$$A = B.a_1 + \ldots + B.a_n.$$

Take a monomial $x^i y^j \in A$. As $y^2 = x^3 - x$ in A then

$$x^i y^j = \begin{cases} x^i(x^3 - x)^{j/2} & \text{if } j \text{ is even} \\ x^i(x^3 - x)^{(j-1)/2} y & \text{if } j \text{ is odd} \end{cases}$$

Therefore A is a finite $k[x]$-algebra. □

(3) (10 points.) Let $F : X \to Y$ be a morphism of affine varieties.

(a) Show that the image of F is dense in Y if and only if F^* is injective.

Proof. Suppose F is dominant. If $g \in \operatorname{Ker} F^*$ then $g(F(x)) = 0$ for all $x \in X$. Therefore, $F(X) \subset V(g) \cap Y$. As $\overline{F(X)} = Y$, then $V(g) \cap Y = Y$ and so $g = 0$.
If F is not dominant, then $\overline{F(X)} \subset Y$ is a proper subset. Therefore, there exists $0 \neq g \in k[Y]$ so that $\overline{F(X)} \subset V(g)$. However $F^*(g) = 0$. □

(b) Show that F is an isomorphism onto a Zariski closed set if and only if F^* is surjective.

Proof. The inclusion map

$$i : F(X) \hookrightarrow Y$$

(which is, of course, a polynomial map) corresponds to the restriction map

$$i^* : k[Y] \to k[F(X)]$$

on coordinate rings, and is surjective. The map i^* induces the isomorphism of k-algebras

$$k[Y]/I_Y(F(X)) \cong k[F(Y)].$$

where $I_Y(F(X)) \subset k[Y]$ is the vanishing ideal of $F(X)$ in $k(Y)$ Conversely, $\operatorname{Ker} F^* = I_X(Y) \subset k(Y)$ and so the map F^* factors as

$$k[Y] \xrightarrow{i^*} k[F(X)] \xrightarrow{\cong} k[Y]).$$

□

(4) (10 points.) Let $X \subset \mathbb{A}^n$ be an affine algebraic set.

(a) Let $f \in k[X] = k[x_1, \ldots, x_n]/I(X)$. Show that the set

$$D(f) := \{x \in X \mid f(x) \neq 0\} \subset X$$

is open with respect to the Zariski topology on X. (Recall that the Zariski topology on X is the topology induced by the Zariski topology on $\mathbb{A}^n$.)

Proof. Let $g \in k[x_1, \ldots, x_n]$ be a polynomial representing f and let $J := (I(X), x_{n+1}g - 1]$ be an ideal in $k[x_1, \ldots, x_n, x_{n+1}]$. One then checks that $U := X \backslash D(f)$ is isomorphic to the affine variety $W := V(J) \subset \mathbb{A}^{n+1}$ under the morphisms $p : W \to U$ defined by $(x_1, \ldots, x_n, t) \mapsto (x_1, \ldots, x_n)$ and $q : U \to W$ defined by $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, 1/g(x_1, \ldots, x_n))$, which are obviously inverse. □

(b) Show that the sets $D(f)$ for $f \in k[X]$ form a basis of the Zariski topology of X.

Proof. Apply the Hilbert basis theorem and the previous result to $k[X]$. □

(c) Given $f \in k[X]$, show that the map $f : X \to k$ obtained by evaluating f is continuous with respect to the Zariski topology on X and k.

Proof. As $k[\mathbb{A}^1]$ is a principal ideal domain, if $Y \subset k$ is closed, then $I(Y) = (p)$. Therefore, $f^{-1}(Y) = V(p(f))$, which is closed. □

(5) (5 points.) Prove that the radical of a homogeneous ideal in $k[x_0, \ldots, x_n]$ is homogeneous.

Proof. Let $\mathfrak{a} \subset k[x_0, \dots, x_n]$ be a homogeneous ideal and let $\sqrt{\mathfrak{a}}$ be the radical. For $f \in \sqrt{\mathfrak{a}}$, let $f(d)$ denote the degree d component of f. Suppose $\mathfrak{a}$ is generated by homogeneous polynomials h_{d_i} of degree d_i. Let

$$h = \sum_i h_{d_i} f_i.$$

The homogeneous components $h(d)$ of h are given by

$$h(d) = \sum h_{d_i} f_i(d - d_i)$$

where the sum is over i such that $d_i \le d$. Therefore, $h(d) \in \mathfrak{a}$.

If $h \in \sqrt{\mathbb{A}}$ then $h^n \in \mathfrak{a}$ for some n. Therefore, $h^n = h(d)^n +$ lower degree terms $\in \mathfrak{a}$. As $h(d)^n \in \mathfrak{a}$, then $h(d) \in \sqrt{\mathfrak{a}}$ and so

$$h - h(d) = h(d-1) + \dots \in \sqrt{\mathfrak{a}}.$$

Therefore, by induction, each $h(d) \in \sqrt{\mathfrak{a}}$. Therefore, $\sqrt{\mathfrak{a}}$ is a homogeneous ideal. □

(6) (10 points.) Identify $\mathbb{A}^n$ with the open set $U_0 := \{x_0 \neq 0\} \subset \mathbb{P}^n$ via the map

$$\phi : (x_1, \dots, x_n) \mapsto [1 : x_1 : \dots : x_n].$$

(a) Show that ϕ is a homeomorphism with respect to the Zariski topology on $\mathbb{A}^n$ and $\mathbb{P}^n$.

Proof. Obvious. □

(b) For a polynomial $p(x_1, \dots, x_n)$ of degree d, define $\theta(p) = x_0^d p(x_1/x_0, \dots, x_n/x_0)$. Let $Y \subset \mathbb{A}^n$ be an affine algebraic set and let $\overline{Y}$ be its closure in $\mathbb{P}^n$. Show that $I(\overline{Y})$ is the ideal generated by $\theta(I(Y))$.

Proof. If $g \in k[x_1, \dots, x_n]$ is of degree d, then $(\theta g)(x_0, \dots, x_n) = x_0^d g(x_1/x_0, \dots, x_n/x_0)$. If $g \in I(Y)$, then $\theta g \in I(\overline{Y})$ and so $I(\overline{Y}) \supset \theta I(Y)$. If $h \in I(Y)$ then we can assume h is homogenous and so if $g(x_1, \dots, x_n) = h(1, x_1, \dots, x_n)$ then $h = \theta g$ and so $I(\overline{Y})$ is generated by $\theta(I(Y))$. □

(c) Let $Y \subset \mathbb{A}^3$ be the affine algebraic set of Problem (1)(b). Calculate generators for $I(\overline{Y})$ where $\overline{Y}$ is the projective closure of Y in $\mathbb{P}^3$. Show that if $g_1, \dots, g_n$ are generators for $I(Y)$, then $\theta(g_1), \dots, \theta(g_n)$ do not necessarily generate $I(\overline{Y})$.

Proof. As calculated earlier $I(Y) = (x_2 - x_1^2, x_3 - x_1^3)$. Therefore, $\theta(x_2 - x_1^2) = x_0 x_2 - x_1^2$ and $\theta(x_3 - x_1^3) = x_0^2 x_3 - x_1^3$. However, $x_1 x_3 - x_2^2 \in I(\overline{Y})$ but $x_1 x_3 - x_2^2 \notin (\theta(x_2 - x_1^2), \theta(x_3 - x_1^3))$. □

(7) **(For 5 bonus points.)** Let $X = \{(t, e^t) \mid t \in \mathbb{C}\}$ be the graph of the exponential function. Is X an algebraic set? (Hint: Calculate $I(X)$.)