Problem Set 10, due Jan 28, 50 points

Algebraic Geometry I, Winter 18/19

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Vector bundles

Let X be a scheme and let $\mathcal{E}$ be a locally free $\mathcal{O}_X$-module of rank r. Recall that we defined the vector bundle associated to $\mathcal{E}$ to be

$$\mathbb{V}(\mathcal{E}) = \text{Spec Sym}^\bullet(\mathcal{E}^\vee),$$

and that $\mathbb{V}$ gave rise to a covariant equivalence from the category of locally free sheaves of rank r to the category of vector bundles of rank r.

Problem 0.(Warm-up) Show that $\mathbb{V}(\mathcal{O}_X^{\oplus r}) = X \times \mathbb{A}^r$.

Problem 1. Let $X = \mathbb{P}^1_{\mathbb{C}} = \text{Proj } A$ where $A = \mathbb{C}[x, y]$, and let

$$\varphi : \mathcal{O}_X(-1) \to \mathcal{O}_X$$

be the morphism of $\mathcal{O}_X$-modules induced by the morphism of A-modules $A(-1) \to A$ that sends $1 \mapsto x$.

(a) Compute the kernel and cokernel of φ. Conclude (once more) that the category of locally free sheaves is not abelian.

(b) Describe $\mathbb{V}(\varphi)$. Is it injective, surjective, a closed immersion?

(c) Consider the dual sequence

$$0 \to K \to \mathcal{O}_X \xrightarrow{\varphi^\vee} \mathcal{O}_X(-1)^\vee \to Q \to 0.$$

Apply Spec $\circ\,\text{Sym}^\bullet$ to this sequence. Describe the corresponding sequence of maps of schemes.

Problem 2. (Automorphisms of $\mathbb{P}^n$) Let $A = k[x_0, \ldots, x_n]$ and let $X = \text{Proj } A = \mathbb{P}^n_k$ be n-dimensional projective space over the field k. In this problem we show that the automorphisms of $\mathbb{P}^n_k$ are linear, that is, that they come from k-linear automorphisms of the k-vector space A_1. We assume (and we will see later in class) that $\text{Pic}\, \mathbb{P}^n_k \cong \mathbb{Z}$ generated by $\mathcal{O}_X(1)$. Let $\sigma : X \to X$ be an automorphism.

a) Show that we have an isomorphism $\lambda : \sigma^*\mathcal{O}_X(1) \to \mathcal{O}_X(1)$.

b) Consider the composition

$$\tau : \Gamma(X, \mathcal{O}_X(1)) \xrightarrow{\sigma^*} \Gamma(X, \sigma^*\mathcal{O}_X(1)) \xrightarrow{\Gamma(X,\lambda)} \Gamma(X, \mathcal{O}_X(1))$$

where σ^* is the pullback of sections along σ. Show that τ is an k-linear isomorphism of the vector space $\Gamma(X, \mathcal{O}_X(1)) = A_1$.

c) Let $\tilde{\tau} = \text{Sym}(\tau)$ be the induced graded automorphism of A. Show that σ is the automorphism of X induced by $\tilde{\tau}$.

d) Conclude that $\text{Aut}_k(\mathbb{P}^n_k) = \text{PGL}_k(n + 1)$.

Proof. See Hartshorne p. 151. □

Regularity

Problem 3. Let k be algebraically closed. Find all the singular closed points of the following plane curves:

(a) the node $y^2 = x^2 + x^3$
(b) the cusp $y^2 = x^3$
(c) the tacnode $y^2 = x^4$.

(See also Hartshorne Problem I.5.1 for more examples and pictures)

Proof. We shall just do (a) and (b), as the approach is similar for (c).

For (a), let $f = y^2 - x^2 - x^3$. The Jacobian is given by

$$J(f) = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix} = \begin{pmatrix} -x(2+3x^2) \\ 2y \end{pmatrix}.$$

Since k is algebraically closed, then every closed point is of the form (x, y) and (x, y) is a singular point if and only if the Jacobian vanishes, which occurs if and only if $y = 0$ and $x = 0$ or $x = \pm\sqrt{-2}$. However, only $(x, y) = (0, 0)$ belongs to $V(f)$. (Be careful, though: here the result is true in any characteristic, but in general might not be. Think about the possible complications if $\operatorname{char} k = 2$ or 3.)

For (b), let $f = y^2 - x^3$. The Jacobian is given by

$$J(f) = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix} = \begin{pmatrix} -3x^2 \\ 2y \end{pmatrix}.$$

Which is zero only at the point $(x, y) = (0, 0) \in V(f)$. □

Problem 4. (Jacobian criterion for projective hypersurfaces) Assume k is algebraically closed. Show that the hypersurface $V(f) \subset \mathbb{P}^n_k$ is regular if and only if the system of equations

$$f = \frac{\partial f}{\partial x_0} = \ldots = \frac{\partial f}{\partial x_n} = 0$$

has no solutions. If the degree of f is not divisible by the characteristic of k show that the equation $f = 0$ is redundant here (Hint: Use Euler's equation $\deg(f) f = \sum_i x_i \partial f / \partial x_i$.)

Proof. We can actually prove a stronger result. Suppose $Y \subset \mathbb{P}^n$ is an projective variety of dimension r whose ideal is generated by$f_1, \ldots f_t \in k[x_0, \ldots, x_n]$. Let $p \in Y$ be a point given by homogeneous coordinates $[a_0 : a_1 : \ldots : a_n]$. I claim that P is non-singular at P if and only if the Jacobian

$$J(f_1, \ldots, f_t) = \left(\left(\frac{\partial f_i}{\partial x_j} \right) (a_0, \ldots, a_n) \right)_{i,j=0,\ldots,n}$$

is of rank $n - r$. We prove the result for each standard affine chart U_i of $\mathbb{P}^n_k$, starting with U_0.

As each f_i is homogeneous, then rank J is invariant under rescaling $(a_0, \dots, a_n)$. Therefore assume $a_0 = 1$ (corresponding to U_0) and define $g_i(x_1, \dots, x_n) := f_i(1, x_1, \dots, x_n)$. Then the Jacobian $(\partial x_i f_j)(p)_{i,j=0,\dots,n}$ is simply the Jacobian $(\partial x_i f_j)(p)_{\substack{i=1,\dots,n \\ j=0,\dots,n}}$ with $(\partial x_0 f_i)(p)_{i=0,\dots,n}$ added to the top row. Therefore, by using Euler's identity, the matrices have the same rank. Therefore, by using the Jacobian criterion for non-singularity in the affine case, Y is non-singular on the chart U_0 if and only if $(\partial f_i/\partial x_i)(p)$ has rank $n-r$. One then applies the same argument to the other charts U_i.

The result in the question then follows as a special case. $\square$

Problem 5.

a) Show that Spec $k[x, y, z]/(x^2 - yz)$ is normal but singular.

Solution.
Let $A = k[x, y, z]/(x^2 - yz)$ and $V = \text{Spec } A$. By the Jacobian criterion the variety V is singular at 0, hence singular. We need to show normality. We give two different proofs in the case char $k \neq 2$. We omit the case char $k = 2$ which can be proven by using Serre's criterion for normality.

For the first proof we follow Hartshorne Exercise II.6.4 and prove the following stronger statement:

Lemma 1. *Let k be of charactieristic $\neq 2$. Let $f \in k[x_1, \dots, x_n]$ be a square-free non-constant polynomial, i.e. in the unique factorization of f into irreducible polynomials, there are no repeated factors. Then the ring $A = k[x_1, \dots, x_n]/(z^2 - f)$ is integrally closed.*

Since A is integral and finite over $k[x_1, \dots, x_n]$ its fraction field is

$$\begin{aligned} K &:= \text{Frac} A \\ &= A \otimes_{k[x_1,\dots,x_n]} k(x_1, \dots, x_n) \\ &= k(x_1, \dots, x_n)[z]/(z^2 - f). \end{aligned}$$

Since $z^2 - f$ is irreducible in $k(x_1, \dots, x_n)[z]$ and k is not of characteristic 2, the extension $K/k(x_1, \dots, x_n)$ is Galois with Galois action $z \mapsto -z$. For every $\alpha = g + hz \in K$ the minimal polynomial is

$$\begin{aligned} m_\alpha &= (X - (g + hz))(X - (g - hz)) \\ &= X^2 - 2gX + (g^2 - h^2 f). \end{aligned}$$

If $\alpha \in A$ then $g, h \in k[x_1, \dots, x_n]$ so m_α is defined over $k[x_1, \dots, x_n]$. Conversely, if m_α is defined over $k[x_1, \dots, x_n]$ then $2g, g^2 - h^2 f \in k[x_1, \dots, x_n]$ so by the assumption on the characteristic $g, h^2 f \in k[x_1, \dots, x_n]$, since f is square-free hence $h \in k[x_1, \dots, x_n]$ (For the last step, factor h, f into irreducible polynomials; since f is square-free it can not cancel all denominators of h^2). By the Lemma below, we have that $\alpha \in K$ is integral over $k[x_1, \dots, x_n]$ if and only if its minimal polynomial is defined over $k[x_1, \dots, x_n]$, hence by the argument above if and only if $\alpha \in A$. We conclude that A is the integral closure of $k[x_1, \dots, x_n]$ in K and hence integrally closed.

Lemma 2. *In the situation above, if $\alpha \in K$ is integrally dependend on $k[x_1, \dots, x_n]$ then its minimal polynomial is defined over $k[x_1, \dots, x_n]$.*

Proof of Lemma. Let

$$p(X) = X^n + a_{n-1}X^{n-1} + \dots + a_1 X + a_0$$

be the minimal polynomial of α over $L = k(x_1, \dots, x_n)$. In particular, the coefficients a_i of $p(X)$ lie in L. We show $a_i \in k[x_1, \dots, x_n]$. Let $\alpha_0 = \alpha, \alpha_1, \dots, \alpha_\ell$ be the roots of $p(X)$ in some splitting field. Let $\tilde{P}(X)$ be the monic polynomial with coefficients in $k[x_1, \dots, x_n]$ satisfied by α. Then P devides $\tilde{P}$, so $\tilde{P}(\alpha_i) = 0$ for all i. Hence the α_i are integrally dependend over $k[x_1, \dots, x_n]$. The coefficients a_i are symmetric polynomials in the α_i and therefore integrally dependend on $k[x_1, \dots, x_n]$. But $a_i \in L$ and $k[x_1, \dots, x_n]$ is integrally closed in L so $a_i \in k[x_1, \dots x_n]$. (The key here is that the base $k[x_1, \dots, x_n]$ is integrally closed. The Lemma holds more generally.) □

We give a second proof of the normality of $A = k[x, y, z]/(xy - z^2)$. We have

$$A = k[s^2, st, t^2]$$

via the identification $x = s^2, y = st, z = t^2$. The ring $k[s^2, st, t^2]$ is the ring of invariants of the $\mathbb{Z}_2$-action

$$s \mapsto -s, \; t \mapsto -t.$$

Since $k[s, t]$ is integrally closed, the ring of invariants $k[s, t]^G$ under any action of a finite group G is integrally closed (if $\alpha \in \operatorname{Frac} k[s, t]^G$ is integrally depended on $k[s, t]^G$ then it lies in $k[s, t]$ by normality of $k[s, t]$ but since it is also invariant under G it lies in $k[s, t]^G$). □

b) Assume k is algebraically closed and let $n \geq 1$. Show that the curve Proj $k[x, y, z]/(x^n + y^n + z^n)$ is regular if and only if n is prime to the characteristic of k.

Proof. Let $f := x^n + y^n + z^n$ and let $V = \text{Proj } k[x, y, z]/(x^n + y^n + z^n)$. By question 4, V is regular if and only if

(1)
$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = \frac{\partial f}{\partial z} = f = 0$$

has no solution. If $(\operatorname{char} k, n) = 1$ then, as

(2)
$$\frac{\partial f}{\partial x} = nx^{n-1}, \qquad \frac{\partial}{\partial y} = ny^{n-1}, \qquad \frac{\partial f}{\partial z} = nz^{n-1},$$

the only solution in k to (1) is $x = y = z = 0$. However, as $[0 : 0 : 0] \notin \mathbb{P}^3_k$, then V is regular.

If V is not regular, then there exists $[a : b : c] \in V$ satisfying (1). As it cannot be the case that $a = b = c = 0$ then, by (2), $n = 0 \in k$. Therefore $(\operatorname{char} k, n) \neq 1$. □

c) Assume k is algebraically closed. Show that if two varieties X, Y over k are regular, then their product $X \times_k Y$ is regular. Show that this is false in general if k is not algebraically closed (Hint: Use the Jacobi criterion. For the last part consider $k = \mathbb{F}_p(u)$, $X = Y = \operatorname{Spec} k[x]/(x^p - u)$.)

Proof. As regularity is a local property, we can assume X and Y are affine. Then $X \times_k Y = \operatorname{Spec} k[X] \otimes_k \operatorname{Spec} k[Y]$. To be explicit, suppose $k[X] = k[x_1, \ldots, x_n]/(f_1, \ldots, f_t)$ and $k[Y] = k[y_1, \ldots, y_m]/(g_1, \ldots, g_s)$. Then $k[X] \otimes_k k[Y] \cong k[x_1, \ldots, x_n, y_1, \ldots, y_m]/(f_1, \ldots, f_t, g_1, \ldots, g_s)$. Therefore, the Jacobian J of $X \times_k Y$ at some point $p \in X \times_k Y$ is of the form

$$J = \begin{pmatrix} J(f_1, \ldots, f_t) & 0 \\ 0 & J(g_1, \ldots, g_s). \end{pmatrix}$$

By assumption, $\operatorname{rank} J(f_1, \ldots, f_t) = n - t$ and $\operatorname{rank} J(g_1, \ldots, g_s) = m - s$. Therefore, $\operatorname{rank} J = (n + m) - (t + s)$. Therefore, $X \times_k Y$ is non-singular. □