Problem Set 10, due Jan 28, 50 points

Algebraic Geometry I, Winter 18/19

Vector bundles

Let X be a scheme and let $\mathcal{E}$ be a locally free $\mathcal{O}_X$-module of rank r. Recall that we defined the vector bundle associated to $\mathcal{E}$ to be

$$\mathbb{V}(\mathcal{E}) = \text{Spec Sym}^\bullet(\mathcal{E}^\vee),$$

and that $\mathbb{V}$ gave rise to a covariant equivalence from the category of locally free sheaves of rank r to the category of vector bundles of rank r.

Problem 0.(Warm-up) Show that $\mathbb{V}(\mathcal{O}_X^{\oplus r}) = X \times \mathbb{A}^r$.

Problem 1. Let $X = \mathbb{P}^1_{\mathbb{C}} = \text{Proj } A$ where $A = \mathbb{C}[x, y]$, and let

$$\varphi : \mathcal{O}_X(-1) \to \mathcal{O}_X$$

be the morphism of $\mathcal{O}_X$-modules induced by the morphism of A-modules $A(-1) \to A$ that sends $1 \mapsto x$.

(a) Compute the kernel and cokernel of φ. Conclude (once more) that the category of locally free sheaves is not abelian.

(b) Describe $\mathbb{V}(\varphi)$. Is it injective, surjective, a closed immersion?

(c) Consider the dual sequence

$$0 \to K \to \mathcal{O}_X \xrightarrow{\varphi^\vee} \mathcal{O}_X(-1)^\vee \to Q \to 0.$$

Apply Spec $\circ$ Sym$^\bullet$ to this sequence. Describe the corresponding sequence of maps of schemes.

Problem 2. (Automorphisms of $\mathbb{P}^n$) Let $A = k[x_0, \dots, x_n]$ and let $X = \text{Proj } A = \mathbb{P}^n_k$ be n-dimensional projective space over the field k. In this problem we show that the automorphisms of $\mathbb{P}^n_k$ are linear, that is, that they come from k-linear automorphisms of the k-vector space A_1. We assume (and we will see later in class) that $\text{Pic}\, \mathbb{P}^n_k \cong \mathbb{Z}$ generated by $\mathcal{O}_X(1)$. Let $\sigma : X \to X$ be an automorphism.

a) Show that we have an isomorphism $\lambda : \sigma^*\mathcal{O}_X(1) \to \mathcal{O}_X(1)$.

b) Consider the composition

$$\tau : \Gamma(X, \mathcal{O}_X(1)) \xrightarrow{\sigma^*} \Gamma(X, \sigma^*\mathcal{O}_X(1)) \xrightarrow{\Gamma(X,\lambda)} \Gamma(X, \mathcal{O}_X(1))$$

where σ^* is the pullback of sections along σ. Show that τ is an k-linear isomorphism of the vector space $\Gamma(X, \mathcal{O}_X(1)) = A_1$.

c) Let $\tilde{\tau} = \text{Sym}(\tau)$ be the induced graded automorphism of A. Show that σ is the automorphism of X induced by $\tilde{\tau}$.

d) Conclude that $\text{Aut}_k(\mathbb{P}^n_k) = \text{PGL}_k(n + 1)$.

Regularity

Problem 3. Let k be algebraically closed. Find all the singular closed points of the following plane curves:

(a) the node $y^2 = x^2 + x^3$
(b) the cusp $y^2 = x^3$
(c) the tacnode $y^2 = x^4$.

(See also Hartshorne Problem I.5.1 for more examples and pictures)

Problem 4. (Jacobian criterion for projective hypersurfaces) Assume k is algebraically closed. Show that the hypersurface $V(f) \subset \mathbb{P}^n_k$ is regular if and only if the system of equations

$$f = \frac{\partial f}{\partial x_0} = \ldots = \frac{\partial f}{\partial x_n} = 0$$

has no solutions. If the degree of f is not divisible by the characteristic of k show that the equation $f = 0$ is redundant here (Hint: Use Euler's equation $\deg(f)f = \sum_i x_i \partial f/\partial x_i$.)

Problem 5.

a) Show that Spec $k[x, y, z]/(x^2 - yz)$ is normal but singular.
b) Assume k is algebraically closed and let $n \geq 1$. Show that the curve Proj $k[x, y, z]/(x^n + y^n + z^n)$ is regular if and only if n is prime to the characteristic of k.
c) Assume k is algebraically closed. Show that if two varieties X, Y over k are regular, then their product $X \times_k Y$ is regular. Show that this is false in general if k is not algebraically closed (Hint: Use the Jacobi criterion. For the last part consider $k = \mathbb{F}_p(u)$, $X = Y = \text{Spec } k[x]/(x^p - u)$.)