Algebraic Geometry I. Problem Sheet 1

This problem sheet is to be submitted no later than 22/10/18.

Please email any comments or corrections to `mdawes@math.uni-bonn.de`.

It is possible to score a total of 50 points by answering the non-bonus problems. Additional points can be scored by answering the bonus problems. The total score (which may exceed 50) will count towards the final score for the semester and is given by the sum of the points scored for bonus and non-bonus problems.

Assume the field k is algebraically closed.

(1) (10 points.) Prove that each of the following sets is an affine algebraic set and determine the vanishing ideal $I(X)$. Conclude that no two are isomorphic. Do your answers change over a non-algebraically closed field?
  (a) $X = \{p\} \subset \mathbb{C}^n$, for some $p \in \mathbb{C}^n$.
  (b) $X = \{(t, t^2, t^3) \mid t \in \mathbb{C}\}$.
  (c) $X = \{(t^2, t^3) \mid t \in \mathbb{C}\}$.
  (d) **(For 4 bonus points)** $X = \{(t^3, t^4, t^5) \mid t \in \mathbb{C}\}$.

(2) (5 points.) Consider the ring $A = k[x, y]/(y^2 - x^3 + x)$. Determine elements $z_1, \ldots, z_m$ of A so that A is a finite $k[z_1, \ldots, z_m]$-algebra.

(3) (10 points.) Let $F : X \to Y$ be a morphism of affine varieties.
  (a) Show that the image of F is dense in Y if and only if F^* is injective.
  (b) Show that F is an isomorphism onto a Zariski closed set if and only if F^* is surjective.

(4) (10 points.) Let $X \subset \mathbb{A}^n$ be an affine algebraic set.
  (a) Let $f \in k[X] = k[x_1, \ldots, x_n]/I(X)$. Show that the set

$$D(f) := \{x \in X \mid f(x) \neq 0\} \subset X$$

  is open with respect to the Zariski topology on X. (Recall that the Zariski topology on X is the topology induced by the Zariski topology on $\mathbb{A}^n$.)
  (b) Show that the sets $D(f)$ for $f \in k[X]$ form a basis of the Zariski topology of X.
  (c) Given $f \in k[X]$, show that the map $f : X \to k$ obtained by evaluating f is continuous with respect to the Zariski topology on X and k.

(5) (5 points.) Prove that the radical of a homogeneous ideal in $k[x_0, \ldots, x_n]$ is homogeneous.

(6) (10 points.) Identify $\mathbb{A}^n$ with the open set $U_0 := \{x_0 \neq 0\} \subset \mathbb{P}^n$ via the map

$$\phi : (x_1, \ldots, x_n) \mapsto [1 : x_1 : \ldots : x_n].$$

  (a) Show that ϕ is a homeomorphism with respect to the Zariski topology on $\mathbb{A}^n$ and $\mathbb{P}^n$.
  (b) For a polynomial $p(x_1, \ldots, x_n)$ of degree d, define $\theta(p) = x_0^d p(x_1/x_0, \ldots, x_n/x_0)$. Let $Y \subset \mathbb{A}^n$ be an affine algebraic set and let $\overline{Y}$ be its closure in $\mathbb{P}^n$. Show that $I(\overline{Y})$ is the ideal generated by $\theta(I(Y))$.
  (c) Let $Y \subset \mathbb{A}^3$ be the affine algebraic set of Problem (1)(b). Calculate generators for $I(\overline{Y})$ where $\overline{Y}$ is the projective closure of Y in $\mathbb{P}^3$. Show that if $g_1, \ldots, g_n$ are generators for $I(Y)$, then $\theta(g_1)$, …, $\theta(g_n)$ do not necessarily generate $I(\overline{Y})$.

(7) **(For 5 bonus points.)** Let $X = \{(t, e^t) \mid t \in \mathbb{C}\}$ be the graph of the exponential function. Is X an algebraic set? (Hint: Calculate $I(X)$.)