Practice problems for final exam
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Problem 1. For which of the following constructions of sheaves does the natural pre-sheaf needs to be sheafified? If yes, give an example.
Kernel, pushforward, pullback, cokernel, image, tensor product, direct sum, direct product, Hom sheaf, dual (of $\mathcal{O}_X$-module).

Problem 2. Hartshorne I.5.1 and I.5.2. For the given examples calculate also the sheaf of relative differentials with respect to the projection to one of the coordinate axis (for 5.1) or coordinate planes (5.2).

For the following problem you might have to review a bit about *Gaussian integers*. For example see the Wikipedia page. In particular, let $p \in \mathbb{Z}$ be a prime number. Then in $\mathbb{Z}[i]$,

- (p) is a prime ideal if $p \equiv 3 (\text{mod } 4)$
- $(p) = (\pi_1)(\pi_2)$ for distinct prime ideals $(\pi_1), (\pi_2)$ if $p \equiv 1 (\text{mod } 4)$.
- $(2) = (1 + i)^2$.

Problem 3. Let $X = \text{Spec } \mathbb{Z}[i] = \text{Spec } \mathbb{Z}[x]/(x^2 + 1)$.

(1) Describe all the points of X and the Zariski topology on it.
(2) Consider the morphim $\pi : X \to \text{Spec } \mathbb{Z}$. Calculate the fibers of π.
(3) Calculate the value of $f = 15$ at the point $(1 + i)$.
(4) Calculate the dimension of X. Is X reduced, irreducible, integral, normal, regular?
(5) Find the Zariski tangent space at the point $(1 + i)$. What is its dimension?
(6) Determine $\pi_* \mathcal{O}_X$. What is the dimension of the fiber of $\pi_* \mathcal{O}_X$ at the closed points? Is $\pi_* \mathcal{O}_X$ locally free?
(7) Calculate $\Omega_{X/\text{Spec } \mathbb{Z}}$. Find the support of $\Omega_{X/\text{Spec } \mathbb{Z}}$.
(8) Calculate the base change $X' = X \times_{\text{Spec } \mathbb{Z}} \text{Spec } \mathbb{C}$. What does your map π become after base change?
(9) Combine all the information above and draw a picture of X and the map π. Find Mumford's drawing of X in the red book (its hiding in plain sight!) and check if your intuition matches his.

Problem 4. Let $X = \text{Spec } A$. Find the support of the quasi-coherent sheaf $\tilde{M}$ on $\text{Spec } A$ for the following A-modules M.

(a) $M = A$
(b) $M = A/I$ for an ideal $I \subset A$
(c) $M = \mathbb{C}$ if $A = \mathbb{C}[x, y]$ and x, y act on M by $a, b \in \mathbb{C}$ respectively.
(d) $M = M_1 \oplus M_2$
(e) $M = A_f$ for some $f \in A$.

Problem 5. Decide using the valuative criterion whether the following morphisms are separated or proper.

a) $\mathbb{A}^1_k \to \operatorname{Spec} k$
b) $\mathbb{P}^1_k \to \operatorname{Spec} k$
c) $\operatorname{Spec} k[t] \to \operatorname{Spec} k[x,y]/(x^2-y^3)$ induced by $x \mapsto t^3, y \mapsto t^2$.
d) $\operatorname{Spec} k[x,y]/(x^2-y^3) \to \operatorname{Spec} k[x]$ induced by $x \mapsto x$.

Can you do the same directly, i.e. without the valuative criterion?

Problem 6. Hartshorne II.5.2. You can find a discussion of this example also in Mumford's Red Book, page 142.

Problem 7. Let X be a scheme and let $x \in X$. Show that the image of the natural morphism $\operatorname{Spec} \mathcal{O}_{X,x} \to X$ is the set of points of X that specialize to x.

Problem 8. Let $\mathcal{F}$ be a cohernet sheaf on Noetherian scheme. Using Nakayama's Lemma show that $x \mapsto \varphi(x) = \dim_{k(x} \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} k(x)$ is semi-upper continuous

Problem 9. Describe $\operatorname{Proj} k[x,y]/(x^2)$.

Problem 10. Consider $X = \operatorname{Proj} k[x,y,z,w]/(wz-xy, wy-x^2, xz-y^2)$. Is X smooth over k? Is it irreducible? Which curve is X? What is $\mathcal{O}_X(1)$ on this curve?