Practice problems (for Midterm I)
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Let k be an algebraically closed field.

(1) True or false?

(a) If a topological space is irreducible then it is connected.

Proof. True. If X is irreducible, then $X \neq X_1 \cup X_2$ for any proper closed subsets $X_1, X_2 \subset X$. On the other hand, if X is connected then $X \neq X_1 \cup X_2$ for any proper closed subsets $X_1, X_2 \subset X$ such that $X_1 \cap X_2 = \emptyset$. □

(b) The intersection of any two affine varieties in k^n is an affine variety.

Proof. False. While the intersection of any two affine varieties is an affine *set*, it need not be irreducible, and hence fail to be a variety. For example, the intersection of irreducible varieties $V(y - x^3) \cap V(y-1) = \{(1,1), (-1,1)\}$, which is not irreducible. □

(c) The intersection of any two projective varieties in $\mathbb{P}^n$ is a projective variety.

Proof. False (becase irreducibility can fail). The intersection of the irreducible varieties $V(z^2y - x^3), V(y - z) \subset \mathbb{P}^2$ is given by $\{[1 : 1 : 1], [-1 : 1 : 1], [0 : 1 : 0], [1 : 0 : 0]\}$, which is not irreducible. □

(d) If $f : X \to k$ is regular, then it is continuous (with respect to the Zariski topology).

Proof. Take a finite affine open cover $\{U_i\}$ of X. Then $f^{-1}_{|U_i}(a) = U_i - V(g - ah)$, which is closed. Therefore, $f^{-1}(a)$ is given by the finite union of closed sets, and is therefore closed. □

(e) X is irreducible if and only if every two non-empty open subsets of X intersect.

Proof. True. If $U \subset X$ is open, then U is dense. Therefore, $U \cap V$ is non-empty for any other non-trivial open subset $V \subset X$ □

(f) Let $f : \mathbb{A}^n \to k$ be regular and non-constant. Then every irreducible component of $V(f)$ has dimension $n - 1$.

Proof. True. Take an irreducible component Z of $V(f)$ and let $U \subset \mathbb{A}^n$ be open so that $\emptyset \neq U \cap Z$. Let g be the restriction of f in $\Gamma(U, \mathcal{O}_{\mathbb{A}^n}(U))$. On the one hand, $Z \cap U$ corresponds to a prime ideal $P \subset \Gamma(U, \mathcal{O}_{\mathbb{A}^n}(U))$. On the other hand, Z is a maximal irreducible subset of $V(f)$, and so $Z \cap U$ is a maximal irreducible subset of $V(g)$. Therefore, the ideal P is maximal and contains f, and so $V(f)$ has dimension $n - 1$. □

(g) A non-empty open subset of a variety is a variety.

Proof. True. Let $\emptyset \neq U \subset X$ be an open subset in a variety X. As varieties are irreducible, then U is connected. Moreover, U has an open cover of affine varieties; as X is noetherian, then we can assume this cover is finite and so U is a prevariety. A prevariety is a variety if and only if the image $\Delta(X)$ of the diagonal morphism

$$\Delta : X \to X \times X$$

is closed. As $\Delta(X)$ is closed, then $\Delta(U)$ is closed and so U is a variety. □

(h) Trick question: The empty set (with the canonical sheaf) is a variety.

(2) A matrix $A \in M_2(k)$ is *nilpotent* if $A^2 = 0$ or, equivalently, if both the trace and determinant are zero. Therefore, if
$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$
and
$$\mathfrak{a} := (a^2 + bc, d^2 + bc, (a+d)b, (a+d)c)$$
$$\mathfrak{b} := (ad - bc, a + d),$$
then
$$V(\mathfrak{a}) = V(\mathfrak{b}) = \{A \in M_2(k) \mid A \text{ nilpotent}\}.$$
Show that
(a) $\operatorname{rad} \mathfrak{a} = \mathfrak{b}$;
(b) $V(\mathfrak{b})$ is irreducible in $M_2(k)$ (identified with $\mathbb{A}^4(k)$).

Proof. It suffices to show that $\mathfrak{b}$ is prime. If so, then $V(\mathfrak{b})$ is irreducible. Furthermore, $\mathfrak{b} = \operatorname{rad} \mathfrak{b}$ and so, by the Nullstellensatz, $\operatorname{rad} \mathfrak{a} = \operatorname{rad} \mathfrak{b} = \mathfrak{b}$.
$$V(\mathfrak{b}) = V(\mathfrak{a}) = V(ad - bc) \cap V(a + d)$$
$$\cong V(a^2 + bc) \subset \mathbb{A}^3$$
under the isomorphism $(w, x, y, z) \mapsto (w, x, y, -w)$. The ideal $(a^2 + bc)$ is prime if and only if $k[a, b, c]/(a^2 + bc)$ is an integral domain. As
$$k[a, b, c]/(a^2 + bc) \cong k[t^2, st, s^2] \subset k[s, t]$$
and $k[s, t]$ is an integral domain, then $(a^2 + bc)$ is prime and the result follows. □

(3) (Sheaves)
(a) Let $\mathcal{F}$ be a sheaf on a topological space X and let $s, t \in \mathcal{F}(U)$ be two sections over an open set $U \subset X$. Show that the set of points $x \in U$ such that $s_x = t_x \in \mathcal{F}_x$ is an open subset of U.

Proof. By definition of the direct limit, for all $x \in X$ such that $s_x = t_x \in \mathcal{F}_x$, there exists $x \in U_x \subset U$ where U_x open such that $s_{|U_x} = t_{|U_x}$ implies $s_y = t_y$ for all $y \in U_x \subset U$ and so $\{x \in U \mid s_x = tx \in \mathcal{F}_x\} = \bigcup_{s_x = t_x \in \mathcal{F}_x} U_x$ is open. □

(a) If $\mathcal{F}$ is a sheaf of abelian groups, one defines the support $\operatorname{Supp}(s)$ of a section $s \in \mathcal{F}(U)$ as the set of points $x \in U$ such that $0 \neq s_x \in \mathcal{F}_x$. Show that $\operatorname{Supp}(s)$ is a closed subset of U.

Proof. Let $0 \in \Gamma(X, \mathcal{F})$ be the zero section. Then $\{x \in \mathcal{F} \mid s_x = 0_x\}$ is open, and so $\{x \in \mathcal{F} \mid s_x \neq 0_x\}$ is closed. □

(b) We define the support of $\mathcal{F}$, $\operatorname{Supp}(\mathcal{F})$ to be $\{x \in X | \mathcal{F}_x \neq 0\}$. Give an example that shows that $\operatorname{Supp}(\mathcal{F})$ does not have to be a closed subset.

Proof. Let $X = \mathbb{A}^1$ and, for $x \in X$, let $\mathcal{F}_x$ be the sheaf whose sections on open $U \subset X$ are given by
$$\mathcal{F}_x(U) = \begin{cases} \mathcal{O}_x(U) & \text{if } x \in U \\ 0 & \text{otherwise.} \end{cases}$$
Now let $K \subset X$ be open and define the sheaf $\mathcal{F}$ by
$$\mathcal{F} := \prod_{x \in K} \mathcal{F}_x.$$
Then $\operatorname{Supp}(\mathcal{F}) = K$, which is closed. □

(4) Give an example that shows that a bijective morphism $f : X \to Y$ does not have to be an isomorphism.

Proof. Let $X = \mathbb{A}^1$, $Y = \mathbb{A}^2$ and let $f : \mathbb{A}^1 \to \mathbb{A}^2$ by $(t) \mapsto (t^2, t^3)$. Then the image of f is the curve $V(y^3 - x^2)$ (which is irreducible). The morphism is injective onto its image and corresponds to the inclusion of rings

$$f^* : k[T^2, T^3] \to k[T].$$

However, there can be no inverse because $k[T]$ is not isomorphic to $k[T^2, T^3]$. □

(5) Describe geometrically the morphisms of affine varieties which correspond to the ring homomorphisms

(i) $\varphi : k[x, y] \to k[t], x \mapsto t, y \mapsto t$

Proof. corresponds to the map $\mathbb{A}^1 \to \mathbb{A}^2$ defined by $(t) \mapsto (t, t)$ (which one can verify by calculating the pullback). □

(ii) $\psi : k[t] \to k[x, y], t \mapsto x + y$.

Proof. Corresponds to the map $\mathbb{A}^2 \to \mathbb{A}^1$ defined by $(t, s) \mapsto (t + s)$. □

6. (Bonus) In Problem set 3 we have seen that an open subset of the projective variety $C \subset \mathbb{P}^2$ defined by the cubic equation $zy^2 = x^2(x + z)$ is isomorphic to an open subset of $\mathbb{P}^1$.

If $n \geq 4$, does there exist a projective variety $C \subset \mathbb{P}^2$ defined by a degree n equation and an open subset $U \subset C$ such that U is isomorphic to an open subset of $\mathbb{P}^1$? (In this case we say C is rational)