Practice problems (for Midterm I)

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Let k be an algebraically closed field.

1. True or false?
   (a) If a topological space is irreducible then it is connected.
   (b) The intersection of any two affine varieties in k^n is an affine variety.
   (c) The intersection of any two projective varieties in $\mathbb{P}^n$ is a projective variety.
   (d) If $f : X \to k$ is regular, then it is continuous (with respect to the Zariski topology).
   (e) X is irreducible if and only if every two non-empty open subsets of X intersect.
   (f) Let $f : \mathbb{A}^n \to k$ be regular and non-constant. Then every irreducible component of $V(f)$ has dimension $n-1$.
   (g) A non-empty open subset of a variety is a variety.
   (h) Trick question: The empty set (with the canonical sheaf) is a variety.
2. Görtz, Wedhorn, Exercise 1.11.
3. (Sheaves)
   (a) Let $\mathcal{F}$ be a sheaf on a topological space X and let $s, t \in \mathcal{F}(U)$ be two sections over an open set $U \subset X$. Show that the set of points $x \in U$ such that $s_x = t_x \in \mathcal{F}_x$ is an open subset of U.

   If $\mathcal{F}$ is a sheaf of abelian groups, one defines the support $\mathrm{Supp}(s)$ of a section $s \in \mathcal{F}(U)$ as the set of points $x \in U$ such that $0 \neq s_x \in \mathcal{F}_x$. Show that $\mathrm{Supp}(s)$ is a closed subset of U.

   (b) We define the support of $\mathcal{F}$, $\mathrm{Supp}(\mathcal{F})$ to be $\{x \in X | \mathcal{F}_x \neq 0\}$. Give an example that shows that $\mathrm{Supp}(\mathcal{F})$ does not have to be a closed subset.

4. Give an example that shows that a bijective morphism $f : X \to Y$ does not have to be an isomorphism.

5. Describe geometrically the morphisms of affine varieties which correspond to the ring homomorphisms

   (i) $\varphi : k[x, y] \to k[t], x \mapsto t, y \mapsto t$
   (ii) $\psi : k[t] \to k[x, y], t \mapsto x + y$.

6. (Bonus) In Problem set 3 we have seen that an open subset of the projective variety $C \subset \mathbb{P}^2$ defined by the cubic equation $zy^2 = x^2(x + z)$ is isomorphic to an open subset of $\mathbb{P}^1$.

If $n \geq 4$, does there exist a projective variety $C \subset \mathbb{P}^2$ defined by a degree n equation and an open subset $U \subset C$ such that U is isomorphic to an open subset of $\mathbb{P}^1$? (In this case we say C is rational)