Algebraic Geometry Fall 2018, Christmas Pset

Lecturer: Georg Oberdieck

Points: All problems are bonus problems. 20 points in total, 10 each.

1. Let C be a normal proper curve over a field k. Prove that there is a closed immersion $i : C \to \mathbb{P}^N_k$ for some $N \geq 1$.

Hint: Let $x \in K$ be a non-zero element in the function field of C, and let R be the integral closure of $k[x]$ in K. Then R is finitely generated over k, hence of the form

$$R = k[x_1, \ldots, x_m]/I$$

for some $m \geq 1$ and ideal I. Show that composition

$$U_x = \operatorname{Spec} R \hookrightarrow \mathbb{A}^m_k \hookrightarrow \mathbb{P}^m_k$$

from the open set $U_x \subset C$ extends to a morphism $f_x : U_x \to \mathbb{P}^m_k$ (use properness of $\mathbb{P}^n$). Now prove that

$$C \xrightarrow{f_x \times f_{1/x}} \mathbb{P}^m \times \mathbb{P}^n \xrightarrow{Segre} \mathbb{P}^N$$

is a closed immersion.

Remark. This exercise shows that C is isomorphic to a closed subscheme of projective space. As we will see later this implies that C is the vanishing locus in $\mathbb{P}^N$ of a set of homogeneous equations. The same does not hold for all proper varieties.

2. One of the harder problems in algebraic geometry is to show that two varieties are not isomorphic, or not birational. For example, as we have seen in Problem set 4 that it is not completely elementary to show that a cubic surface is not isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. Another instance is the rationality problem, i.e. to decide whether a given variety is rational. For example, it is a famous open problem to decide whether the (very general) cubic hypersurface in $\mathbb{P}^5$ is rational or not. In the following exercise you will show that the cubic curve $y^2 = x^3 - x$ is not rational. This is the first curve that we encounter in this class which we can prove to be non-rational.

Solve problems 6.1 and 6.2 in Hartshorne Chapter I. (Remark: A curve over a field is non-singular if and only if it is normal, and for the problem this equivalence is all that you will need. You may also assume that $k[x, y]/(y^2 - x^3 + x)$ is integrally closed in its fraction field.)