The Eliashberg-Gromov Theorem

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This set of notes is based on [1] §12.2 and §2.4.

1 The Eliashberg-Gromov Theorem

Our goal is to prove the following theorem, assuming the existence of a symplectic capacity.

Theorem 1 (Eliashberg-Gromov Theorem)**.** For every symplectic manifold (M, ω) the group $\mathrm{Symp}(M, \omega)$ is C^0-closed in $\mathrm{Diff}(M)$. [1]

In other words, if a sequence $\{\varphi_j\}$ of symplectomorphisms converges to φ uniformly **and** φ is a diffeomorphism, then φ is a symplectomorphism.

The proof of the theorem is based on the following facts, which will be proved in later sections.

Definition 1 (Ellipsoids)**.** An ellipsoid is of the form

$$E(r) = \{z \in \mathbb{C}^n \mid \sum_{i=1}^{n} \left|\frac{z_i}{r_i}\right|^2 \leq 1\}$$

for some $0 \leq r_1 \leq \cdots \leq r_n$.

Facts. (1) Existence of a symplectic capacity c.

(2) A homeo of $\mathbb{R}^{2n}$ which is a limit of ellipsoid-capacity-preserving (ecp) continuous self-maps, under the uniform convergence on compact sets, on $\mathbb{R}^{2n}$ is also ecp. (Lemma 12.2.3) See Section 2

[1] Let M, N be topological spaces. Let $C^0(M, N)$ be the set of continuous maps $M \to N$. Let

$$\mathcal{B} = \{V(K, U) \mid K \subset M \text{ compact}, U \subset N \text{ open}\},$$

where

$$V(K, U) = \{f \in C^0(M, N) \mid f(K) \subset U\}.$$

The C^0-topology is the topology with $\mathcal{B}$ as a subbase.

(3) An ecp diffeomorphism of $\mathbb{R}^{2n}$ is either a symplectomorphism or an anti-symplectomorphism. (Proposition 12.2.2) See Section 3.

This follows from its affine version: Endomorphisms of $\mathbb{R}^{2n}$ that preserve linear symplectic width of ellipsoids are (anti-)symplectic. (Theorem 2.4.2, Theorem 2.4.4)

For Fact (1), an example is given by the following. The **Gromov width**

$$c(M,\omega) := \sup\{\pi r^2 \mid B^{2n}(r) \text{ embeds symplectically in } M\}$$

is a symplectic capacity, in the sense that it assigns to each symplectic manifold (M,ω) a nonnegative number or ∞ and satisfies the following properties.

(monotonicity) If there is a symplectic embedding $(M_1,\omega_1) \hookrightarrow (M_2,\omega_2)$ and $\dim M_1 = \dim M_2$, then $c(M_1,\omega_1) \le c(M_2,\omega_2)$.

(conformality) $c(M,\lambda\omega) = \lambda c(M,\omega)$.

(normalized) $c(B^{2n}(1),\omega_0) = c(Z^{2n}(1),\omega_0) = \pi$.

We can extend it to define a capacity which assigns an arbitrary subset $A \subset (\mathbb{R}^{2n},\omega_0)$ a nonnegative number or ∞ by

$$c(A) := \inf\{c(U,\omega_0|_U) \mid A \subset U, \text{ and } U \text{ open in } \mathbb{R}^{2n}\}.$$

Then c satisfies the extended monotonicity property

$$A \subset B \quad\Longrightarrow\quad c(A) \le c(B).$$

Theorem (Eliashberg-Gromov Theorem, [1] Theorem 12.2.1)**.** For every symplectic manifold (M,ω) the group $\mathrm{Symp}(M,\omega)$ is C^0-closed in $\mathrm{Diff}(M)$.

Proof. It suffices to prove it for $(\mathbb{R}^{2n},\omega_0)$. Suppose a sequence of symplectomorphisms $\varphi_j : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ converges to $\varphi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ uniformly and φ is a diffeomorphism.

Consider a symplectic capacity. By monotonicity, the maps φ_j preserve the capacity of ellipsoids. Thus, by (2), φ preserves the Gromov width of ellipsoids.

By (3), φ is either a symplectomorphism or an anti-symplectomorphism. Suppose, to the contrary, that φ is an anti-symplectomorphism. Then $\psi_j := \varphi_j \times \mathrm{Id} \in \mathrm{Symp}(\mathbb{R}^{2n}\times\mathbb{R}^{2n}, \omega := \omega_0 \times \omega_0)$ converges uniformly to $\psi = \varphi \times \mathrm{Id} \in \mathrm{Diff}(\mathbb{R}^{2n}\times\mathbb{R}^{2n})$. By the same argument, $\psi^*\omega \in \{\pm\omega\}$. However, the antisymplecity of φ implies

$$(\varphi \times \mathrm{Id})^*(\omega_0 \times \omega_0) = (-\omega_0) \times \omega_0,$$

which is neither $\pm(\omega_0 \times \omega_0)$. $\square$

2 Proof of Fact (2)

It remains to prove (2) and (3).

Lemma 1 (Fact (2), [1] Lemma 12.2.3)**.** Let c be a nomalized symplectic capacity on $\mathbb{R}^{2n}$. Let $\psi_j : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be a sequence of continuous maps converging to a homeomorphism $\psi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$, uniformly on compact sets. Assume that ψ_j preserves the capacity of ellipsoids for every j. Then ψ preserves the capacity of ellipsoids.

Proof. WLOG, consider ellipsoids centered at zero. We claim that, for every ellipsoid E and $\forall \lambda < 1$, there exists a $j_0 > 0$ such that

$$\psi_j(\lambda E) \subset \psi(E) \subset \psi_j(\lambda^{-1} E) \qquad \forall j \geq j_0. \tag{1}$$

Denote $f_j := \psi^{-1} \circ \psi_j$. Then $f_j \to \mathrm{Id}$ uniformly on compact sets. Thus, for j large, we have $f_j(\lambda E) \subset E$, implying the first inclusion

$$\psi_j(\lambda E) \subset \psi(E).$$

Suppose the second inclusion fails. Then there exists $y_0 \in E$ and $\psi(y_0) \in \psi(E) \setminus \psi_j(\lambda^{-1}E)$ for all large j; i.e. $y_0 \in E$ and

$$f_j(x) \neq y_0 \quad \forall x \in \lambda^{-1} E.$$

Then the map $F_j : \lambda^{-1}\partial E \to S^{2n-1}$ defined by

$$F_j(x) := \frac{f_j(x) - y_0}{|f_j(x) - y_0|}$$

can be extended to all of $\lambda^{-1}E$, thus having degree 0.

On the other hand, since $\lambda < 1$ and $f_j \xrightarrow{C^0_{loc}} \mathrm{Id}$, there exists j_0 such that $\forall j > j_0$,

$$x \in \lambda^{-1}\partial E \quad \Rightarrow \quad f_j(x) \notin E.$$

Then F_j is homotopic to

$$G_j : \lambda^{-1}\partial E \to S^{2n-1}, \quad G_j(x) = \frac{f_j(x)}{|f_j(x)|}.$$

But G_j has degree 1. Contradiction. Therefore, the claim holds.

Since ψ_j preserves the capacity of ellipsoids, and by conformality of a capacity c, we have $c(\psi_j(\lambda E)) = \lambda^2 c(E)$, and $c(\psi_j(\lambda^{-1}E)) = \lambda^{-2} c(E)$. By monotonicity, we have

$$\lambda^2 c(E) \leq c(\psi(E)) \leq \lambda^{-2} c(E).$$

Since this holds for all $\lambda < 1$, we get $c(\psi(E)) = c(E)$. □

3 Proof of Fact (3)

Definition 2 (linear symplectic width)**.** The *linear symplectic width* of an arbitrary subset $A \subset \mathbb{R}^{2n}$ is defined by

$$w_L(A) = \sup\{\pi r^2 \mid (\Psi + z_0)(B^{2n}r) \subset A \quad \forall \Psi \in \mathrm{Sp}(2n), \quad \forall z_0 \in \mathbb{R}^{2n}\}.$$

Definition 3 (Linear nonsqueezing property)**.** A matrix $\Psi \in GL_{2n\times 2n}(\mathbb{R})$ is said to have the *linear nonsqueezing property* if, for every linear symplectic ball B of radius r and every linear symplectic cylinder Z of radius R, we have $\Psi B \subset Z \Rightarrow r \leq R$.

Lemma 2 (Affine rigidity, [1] Theorem)**.** Let $\Psi \in GL_{2n\times 2n}(\mathbb{R})$ be a nonsingular matrix such that Ψ and Ψ^{-1} have the linear nonsqueezing property. Then Ψ is either symplectic or anti-symplectic.

Sketch of proof. Assume that Ψ is neither symplectic nor anti-symplectic. Then we may find some $u_1, v_1 \in \mathbb{R}^{2n}$ such that

$$0 < \lambda^2 = |\omega_0(\Psi^T u_1, \Psi^T v_1)| < \omega_0(u_1, v_1) = 1$$

for some $0 < \lambda < 1$. Let

$$u_1' = \lambda^{-1}\Psi^T u_1, \qquad v_1' = \lambda^{-1}\Psi^T v_1.$$

Complete u_1, v_1 and u_1', v_1' to symplectic bases

$$B = \{u_1, v_1, \dots, u_n, v_n\}, \qquad B' = \{u_1', v_1', \dots, u_n', v_n'\}$$

of $\mathbb{R}^{2n}$, respectively.

Let $\Phi, \Phi' \in \mathrm{Sp}(2n)$ be the matrices mapping the standard basis $\{e_1, f_1, \dots, e_n, f_n\}$ to B and B', respectively. Then $A = (\Phi')^{-1}\Psi^T\Phi$ satisfies

$$Ae_1 = \lambda e_1, \qquad Af_1 = \lambda f_1.$$

This implies that A^T maps $B^{2n}(1)$ to the cylinder $Z^{2n}(\lambda)$ with $\lambda < 1$. This contradicts that Ψ has the linear nonsqueezing property. □

Theorem 2 (Affine version of Fact (3), Theorem 2.4.4)**.** Let $\psi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be a linear map. Then the following are equivalent.

(i) Ψ preserves the linear symplectic width of ellipsoids centered at zero.

(ii) The matrix is either symplectic or anti-symplectic; i.e. $\Psi^*\omega_0 = \pm\omega_0$.

Sketch of proof. ((ii) $\Rightarrow$ (i)) Follows from affine non-squeezing ([1] Theorem 2.4.1).

((i) $\Rightarrow$ (ii)) Assume (i) . We want to show that both Ψ and Ψ^{-1} have the linear nonsqueezing property.

Let B be a linear symplectic ball of radius r and Z a linear symplectic cylinder of radius R such that $\Psi(B) \subset Z$. Then $w_L(B) = w_L(\psi B) \leq w_L(Z) = \pi R^2$, implying $r \leq R$.

Note that Ψ is nonsingular because otherwise $w_L(\Psi B) = 0$, which contradicts that Ψ preserves the linear symplectic width.

Moreover, Ψ^{-1} also preserves w_L because $w_L(E) = w_L(\Psi\psi^{-1}E) = w_L(\Psi^{-1}(E))$. So Ψ^{-1} also satisfies the linear nonsqueezing property.

By Lemma 2, this implies (ii). □

Corollary 1 (Fact (3), [1] Proposition 12.2.2)**.** Let $\psi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be a diffeomorphism and let c be a symplectic capacity on $\mathbb{R}^{2n}$. Then the following are equivalent.

(i) ψ preserves the capacity of ellipsoids; i.e. it satisfies $c(\psi(E)) = c(E)$ for every ellipsoid $E \subset \mathbb{R}^{2n}$.

(ii) ψ is either a symplectomorphism or an anti-symplectomorphism; i.e. it satisfies $\psi^*\omega_0 = \pm\omega_0$.

Sketch of proof. We will only show ((i) $\Rightarrow$(ii)). Let c be a symplectic capacity on $\mathbb{R}^{2n}$. Let $\psi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be a diffeomorphism that preserves the capacity of ellipsoids. We may assume $\psi(0) = 0$. Then there exists a sequence of ecp diffeomorphisms ψ_t of $\mathbb{R}^{2n}$ that converges uniformly on compact sets to $\Psi = D_0\psi$. Here we may take $\psi_t(z) = \frac{1}{t}\psi(tz)$. By Fact (2), Ψ preserves the capacity of ellipsoids. By Theorem 2, $\Psi^*\omega_0 = \pm\omega_0$. Similarly, $(D_z\psi)^*\omega_0 = \pm\omega_0$ for any $z \in \mathbb{R}^{2n}$. By continuity, the sign is independent of z. □

References

[1] Dusa McDuff and Dietmar Salamon. *Introduction to symplectic topology.* Third. Oxford Graduate Texts in Mathematics. Oxford University Press, Oxford, 2017, pp. xi+623.