Homework 3 (optional)

(1) (Exercise 3.1.19, McDuff–Salamon 3rd edition; we used this in our proof of Weinstein's Lagrangian neighborhod theorem) For $L \subset M$ a Lagrangian, show that there is an isomorphism

$$T_{(q,0)}T^*L \simeq T_qL \oplus T_q^*L$$

such that for $(u, \eta), (v, \sigma) \in T_qL \oplus T_q^*L$, we have

$$-d\lambda_{\mathrm{can},(q,0)}((u, \eta), (v, \sigma)) = \sigma(u) - \eta(v).$$

To produce such a decomposition, consider the "horizontal vectors", which are tangent to the zero section, and "vertical vectors", which project to zero.

(2) (Exercise 1.1.20, McDuff–Salamon 3rd edition) On $\mathbb{R}^{2n}$, prove that the Poisson bracket satisfies

$$\{F, G\} = \omega_0(X_F, X_G).$$

(3) (Exercise 1.1.22, McDuff–Salamon 3rd edition) We call a Hamiltonian system on $\Omega \subset \mathbb{R}^{2n}$ *integrable* if there exist n smooth functions $F_1, \ldots, F_n \colon \Omega \to \mathbb{R}$ satisfying the following conditions:

- $\{F_i, H\} = 0$ for all i. That is, each F_i is an integral of motion.
- $\{F_i, F_j\} = 0$ for all i, j. (We say that the F_i's *Poisson commute.*)
- The vectors $\nabla F_1(z), \ldots, \nabla F_n(z)$ are linearly independent for every $z \in \Omega$. (We say that the F_i's are *independent.*)

(The notion of an integrable system continues to make sense for a general symplectic manifold.)

Consider the *harmonic oscillator*, the Hamiltonian system on $\mathbb{R}^{2n}$ with Hamiltonian

$$H \coloneqq \sum_{i=1}^{n} a_i \big(x_i^2 + y_i^2\big),$$

for $a_1, \ldots, a_n > 0$.

(a) Find the solutions of this system.
(b) Prove that this system is integrable away from the origin.
(c) What are all the periodic solutions on a given energy surface $H = c$, $c > 0$?

(4) (Exercise 3.1.9, McDuff–Salamon 3rd edition) Suppose that (M, ω) is $2n$-dimensional. Prove that the Poisson bracket of two functions $F, G \colon M \to \mathbb{R}$ satisfies

$$\{F, G\}\frac{\omega^{\wedge n}}{n!} = dF \wedge dG \wedge \frac{\omega^{\wedge(n-1)}}{(n-1)!}.$$

From this deduce that if F has compact support, then their Poisson bracket has mean value zero, i.e. $\int_M \{F, G\}\omega^{\wedge n} = 0$.