# Homework 1 (optional)

(1) (Exercise 2.11, McDuff–Salamon) Let $A \in GL(2n;\mathbb{R})$ be a nondegenerate skew-symmetric matrix (i.e. $\det A \neq 0$ and $A = -A^T$) and set $\omega(u, v) := u^T Av$. Prove that a symplectic basis for $(\mathbb{R}^{2n}, \omega)$ can be constructed in terms of the real and imaginary parts of the eigenvectors of $A$. (Recall that self-adjoint matrices can be diagonalized.)

(2) (Exercises 2.13 and 2.15, McDuff–Salamon)

- Show that if $\beta$ is a skew-symmetric bilinear form on a real vector space $W$, there exists a basis $u_1, \dots, u_n, v_1, \dots, v_n, w_1, \dots, w_p$ of $W$ such that $\beta(u_i, v_j) = \delta_{ij}$ and all other pairings vanish. The integer $2n$ is called the *rank* of $\beta$.
- Show that any hyperplane $W$ in a symplectic vector space $(V, \omega)$ is coisotropic. (Use the previous part to find a nonzero $w \in W \cap W^\omega$. What can you say about the span of $w$?)

(3) Recall that we gave two definitions of the canonical 1-form $\lambda_{can}$ on a cotangent bundle $T^*L$:

- The first definition was in local coordinates $x_1, \dots, x_n$ on $L$. We defined $\lambda_{can} := \sum_{j=1}^n y_j dx_j$, where $y_j$ is the fiber coordinate corresponding to $x_j$.
- The second definition was coordinate-independent. We denoted by $\pi$ the projection $T^*L \to L$, which induces a map

$$\pi^* \colon T^*(T^*L) \to T^*L, \qquad \pi^*_{(p,\xi)} \colon T^*_{(p,\xi)}(T^*L) \to T^*_p L.$$

  We defined $\lambda_{can}$ by

$$\lambda_{can,(p,\xi)} := \pi^*_{(p,\xi)}\xi.$$

  That is, for $V \in T_{(p,\xi)}T^*L$,

$$\lambda_{can,(p,\xi)}(V) := \xi_p(\pi_{*,(p,\xi)}(V)).$$

Prove that the first definition of $\lambda_{can}$ is well-defined. You can do this in one of two ways (or both ways, if you have energy to spare):

- Consider how the first definition of $\lambda_{can}$ behaves under coordinate transformations.
- Show that the two definitions agree locally.