SEMINAR PLAN: MORSE THEORY WITH A VIEW TOWARDS FLOER THEORY (S2D1/S4D1)

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1. Summary

A smooth real-valued function on a smooth manifold is said to be Morse if all of its critical points are non-degenerate. In other words, the "Hessian matrix" of second derivatives must be non-singular at all critical points. It is an important fact that Morse functions exist on any smooth manifold.

Morse functions have the virtue that one can precisely describe how the topology of the sub-level sets changes when one passes a critical point. This leads to a rich theory relating the topology of smooth manifolds with the study of Morse functions on them. This theory was pioneered by M. Morse in the mid twentieth century and was arguably the key tool in the spectacular development of differential topology in the 1950s and 60s due to Bott, Smale, Milnor and many others.

There is another approach to Morse theory which is *analytical* rather than *topological*. Its roots are murky to the instructor, but it was at least partly popularized by Witten in the early 1980s, for reasons having to do with high energy physics.

This analytical approach to Morse theory does not yield any new applications in topology. However, in the mid 1980s, Andreas Floer initiated an infinite-dimensional generalization which we now refer to as Floer theory (or Morse-Floer theory). Floer theory produces a package of invariants which are of central interest in many parts of geometry and topology, particularly in differential topology and symplectic geometry. The development of Floer theory has changed the face of many areas of mathematics, and continues to be a central topic today.

While Floer theory is analytically much deeper than Morse theory, both theories share the same formal properties. A solid grasp of the analytical approach to Morse theory is a huge help, and arguably a pre-requisite, for studying Floer theory.

The main goal of the seminar is to understand the analytical approach to Morse theory. Towards the end the semester, we will move on to discussing some basic symplectic geometry. The last talks will be about Floer theory and (some of) its applications to symplectic geometry. We will not discuss the analytical foundations of Floer theory, and focus only on the formal structure and analogies with Morse theory.

2. Instructions and advice for speakers

Please be advised that the following instructions apply only to this particular seminar! Other instructors may have different expectations regarding both the preparation and content of seminar talks.

Preparatory meetings. If you are speaking in week n, then I would like to meet briefly on the Friday of week $n-2$ and $n-1$. *Here n runs over the weeks during which the seminar meets.* The meetings will take place in my office right after the seminar (so around 13:50) and will end no later than 13:10, so as not to conflict with classes starting at 14:15. On the Friday of week $n-2$, please bring a talk outline. On the Friday of week $n-1$, please your talk notes.

If you are speaking in the first two weeks, please contact me to fix a meeting time.

Timing. All talks should be "board talks" (i.e. please no slides). Talks should last 90 minutes, including a five minute break. Since there will almost surely be questions during the talk, I would suggest to plan to speak for roughly 80 minutes, while keeping some material "in reserve", in case there are fewer questions than usual. *It is extremely important to end on time.* If you go over time, I will ask you to stop after a few minutes, and this will likely affect your grade.

How to interpret the talk plan. Whenever I refer to a particular statement number in the talk plan (e.g. Thm 1.2.5, Cor 2.1.9, etc), then this means that I expect you to include this statement in your talk. At the minimum, this means that you should write this statement on the board and "unpack" what the statement is saying. Depending on the context, you can also include an example and/or picture and/or proof. In contrast, if I just ask you to cover a particular section (e.g."cover §2.1(d), (e) and (f)"), then it is your task to identify the most important points (and omit the parts which you consider less important, unless I explicitly requested you to discuss them).

General advice. Your goal should be to convey to your audience the best possible understanding of the assigned material in the allotted time. My advice is to focus especially on explaining *definitions* and *statements*. I encourage you to draw pictures and to give copious examples and non-examples. It is not the case that you need to supply a proof for every theorem which you discuss: depending on the context, it may be appropriate to give a complete proof, or just a sketch, or a proof by picture, or to omit the proof entirely. It is your task to strike a good balance!

3. Plan of talks

Unless otherwise indicated, all references below refer to the book of Audin–Damian [1].

Part 1: Morse theory.

Morse functions (11.04.2025 double header). Define degenerate/non-degenerate critical points and state the definition of a Morse function; see §1.1(a). Illustrate the aforementioned notions with copious pictures and examples (you should include the examples of §1.4 (b) and §1.4(c) (see also the introduction to Part 1 and §1.1(b) in [1], and [2, pp. 1-5]). State and explain the statement of Theorem 1.2.5. Give an idea of the proof, following §1.2(a), 1.2(b).

The Morse lemma (11.04.2025 double header). Prove the Morse lemma; see §1.3(a) or [2, pp. 6-7]. Define Morse charts (p. 14). Deduce that non-degenerate critical points are isolated Cor. 1.3.2. Define gradients and pseudo-gradients §2.1. State that pseudo-gradients exist for all Morse functions on all manifolds; time allowing, discuss the proof; §2.1(c).

Public holiday, no seminar (18.04.2025).

Morse functions and handle attachment (25.04.2025). The goal of this talk is to explain the "classical" perspective on Morse theory (in particular: at a critical point of index k, the sublevel sets change by attaching a k-cell). This talk should cover §2.1(d–f). In particular: Prop. 2.1.5, Thm. 2.1.7 and Cor. 2.1.9, and Thm. 2.1.11. (ideally prove the first two...). Please make sure to define stable/unstable manifolds. You may also wish to consult [2]. This material is not really relevant for Floer theory, but is very important in many parts of geometry and topology.

The Smale condition (02.05.2025). The goal of this talk is to introduce the *Smale condition*: this is a transversality condition, which implies that moduli spaces of Morse trajectories are smooth manifolds. Cover §2.2 and define $\mathcal{L}(a, b)$ (the moduli space of trajectories); explain why it is a smooth manifold when the Smale condition holds. Give examples and non-examples of data satisfying the Smale condition (see §. 2.1 and ideally §2.2(d)). State Thm. 2.2.5 (Kupka–Smale theorem) and discuss the proof.

The Morse complex (09.05.2025). Define the Morse complex (over $\mathbb{Z}/2$) and do the first and third examples of §3.1(c). Explain why d is well-defined and why $d^2 = 0$. The main ingredients are *compactness* (Thm. 3.2.2) and *transversality* (Smale condition) and *gluing* (Thm. 3.2.7). Along the way, you should introduce the natural compactification $\mathcal{L}(a, b)$ of the space of gradient trajectories by *broken* trajectories.

Invariance (16.05.2025). The main goal of this talk is to explain the proof of Thm. 3.4.2, which establishes the well-definedness of Morse homology on compact manifolds. You should also state, as a corollary of this isomorphism, that the minimal number of critical points of a Morse function is bounded from below by the sum of the (mod 2) Betti numbers. I suggest to begin the talk with a few computations of the Morse complex (e.g. from §3.1(c), in coordination with the previous speaker) as a warmup and review for the audience.

Comparison to cellular homology (23.05.2025). Review the definition of cellular homology. The main task in this talk is to cover Thm. 4.9.3, which states that the Morse homology of a compact manifold is isomorphic to its cellular homology. Note that the proof is too long to be explained completely, so the talk should focus on explaining the strategy and key steps.

Part 2: Towards Floer theory.

Basic symplectic geometry (30.05.2025). Cover §5.1–5.4. Introduce symplectic vector spaces. Define symplectic manifolds. Examples. State the Darboux theorem. Define Hamiltonian vector fields, Hamiltonian systems. Make sure to cover in particular Proposition 5.4.2, Definition 5.4.4 (non-degenerate orbits) and Proposition 5.4.5. Note that §5.4 is very important, so I would suggest that you devote at least half of the talk to it.

Instructor away, no seminar (06.06.2025).

Semester break, no seminar (13.06.2025).

The index of a non-degenerate orbit (20.05.2025). The goal of this talk is to explain how to associate an *index* to a non-degenerate orbit of a Hamiltonian. First cover §5.5: i.e. define almost-complex structures and the notion of *compatibility*. Make sure to state Cor. 5.5.6. Then skip to §7, which is the main focus of this talk. You won't be able to cover everything, but try to explain the main steps in the construction of the index (see pp. 189-190) and to give some examples and/or computations. Make sure to state Cor. 7.2.2.

Instructor away, no seminar (27.06.2025).

The action functional and Floer's equation (04.07.2025). This talk should cover §6.3, §6.4 and the first part of §6.5(a). The first third of the talk should introduce the symplectic action functional. Under the assumptions of p. 156, define the symplectic action functional on the (contractible) loop space (i.e. cover §6.3. Make sure to cover Prop. 6.3.3).

In the second third of the talk: explain how to define a metric on the loop space. Introduce Floer's equation. Explain how to interpret solutions to Floer's equation as the negative gradient trajectories of the symplectic action functional. State remark 6.4.1. See §6.4.

In the last third of the talk: define the notion of *energy* and prove that it decreases along trajectories. §6.5.a. Cover Remark 6.5.2.

Arnold's conjecture and Floer homology (11.07.2025). The goal of this talk is introduce the Arnold Conjecture, and to summarize the Floer homology proof under the assumptions of page 156 (i.e. the version stated as Theorem 6.1.2). You should start by introducing the Arnold conjecture, making sure to cover Prop. 6.1.5, 6.1.6.

The rest of the talk should be an introduction to Floer homology. For each step in the construction of Morse homology, state the analog in Floer homology (in particular, you should introduce the Floer chain complex). Along the way, you should introduce the moduli space of solutions to Floer equation $\mathcal{M}$ on p. 166. Make sure to discuss the role of compactness, transversality and gluing (although you obviously are not expected to give proofs).

This talk should preferably be given by an experienced (and/or highly motivated) student. A good reference is §6.2 of [1]. Another good reference is [3].

Compactness (18.07.2025). The goal of this talk is to give an idea of the proof of Corollary 9.1.9. The main ingredient is Theorem 9.1.7(c), which itself relies on Theorem 6.5.4 (Gromov compactness). It is unrealistic to explain the proofs of these statements in detail, so the goal of the talk should be to convey the structure and key ideas of the proof (in particular, make sure to explain the importance of Assumption 6.2.1 in the proof of Theorem 6.5.4.). I also suggest to begin the talk by briefly reminding the audience of the relevance of Corollary 9.1.9 to the definition of Floer homology (which will already have been explained in the previous talk).

References

[1] Michele Audin and Mihai Damian, *Morse theory and Floer homology*, Springer, 2014.
[2] John Willard Milnor, *Morse theory*, Princeton university press, 1963.
[3] Dietmar Salamon, *Lectures on Floer homology*, https://people.math.ethz.ch/~salamon/PREPRINTS/floer.pdf.