Characterizations of Pretameness

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Content

Set forcing

Let M be a countable transitive model of ZFC. In set forcing, we use a partial order $\mathbb{P} \in M$ to construct a new model $M[G]$, where G is $\mathbb{P}$-generic over M.

Theorem

If M is a model of ZFC and G is $\mathbb{P}$-generic over M then $M[G] \models \mathrm{ZFC}$.

Set forcing

The proof uses

Theorem (Forcing theorem)

1. *The forcing relation* $p \Vdash^M_{\mathbb{P}} \varphi(\sigma_0, \ldots, \sigma_{n-1})$ *is definable over* M ***(Definability lemma)****.*
2. *If* $M[G] \models \varphi(\sigma_0^G, \ldots, \sigma_{n-1}^G)$ *then there is* $p \in G$ *such that* $p \Vdash^M_{\mathbb{P}} \varphi(\sigma_0, \ldots, \sigma_{n-1})$ ***(Truth lemma)****.*

Class forcing

One can generalize forcing and consider (definable) **proper classes** $\mathbb{P} \subseteq M$.

Observation

Let $\mathbb{P} = \mathrm{Col}(\omega, \mathrm{Ord}^M)$ denote the forcing notion whose conditions are finite functions $p : \mathrm{dom}(p) \to \mathrm{Ord}^M$, $\mathrm{dom}(p) \subseteq \omega$ finite, ordered by reverse inclusion. Then $\mathbb{P}$ adds a cofinal function $\omega \to \mathrm{Ord}^M$. In particular, Replacement fails.

Class forcing

... but it can get even worse:

Theorem (Holy, K., Lücke, Njegomir, Schlicht 2015)

Let M be a countable transitive model of ZF^-. There is a partial order $\mathbb{P} \subseteq M$ which is definable over M such that $\mathbb{P}$ does not satisfy the forcing theorem for atomic formulae over M.

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Motivation

Question

1. *Under what conditions does a class forcing satisfy the forcing theorem?*
2. *How can we characterize the preservation of the axioms of* ZFC *(resp.* $\mathrm{ZF}(C)^-$*?)*

A general setting for class forcing

We study class forcing in a **second-order** context.

Definition

We denote by GB^- the theory in the two-sorted language with variables for sets and classes, with

- set axioms given by ZF^- with class parameters allowed in the schemata of Separation and Collection
- class axioms of extensionality, foundation and first-order class comprehension (i.e. involving only set quantifiers).

Somtimes we additionally assume that C contains a **good well-order** $\prec$ of M, i.e. $\prec$ is a global well-order such that $\{y \mid y \prec x\} \in M$ for each $x \in M$.

Examples are $\langle M, \mathrm{Def}(M)\rangle$, where M is a countable transitive model of ZF^-, and models of Kelley-Morse class theory KM.

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Class forcing extensions

From now on, let $\mathbb{M} = \langle M, \mathcal{C} \rangle$ be a countable transitive model of GB^-. A **class forcing** $\mathbb{P} = \langle P, \leq_{\mathbb{P}}, \mathbb{1}_{\mathbb{P}} \rangle$ is a preorder such that $\leq_{\mathbb{P}}, P \in \mathcal{C}$. $\mathbb{P}$-**names** are defined in the usual way by recursion.

- $M^{\mathbb{P}}$ denotes the set of $\mathbb{P}$-names which are in M (**set names**).
- $\mathcal{C}^{\mathbb{P}}$ denotes the set of $\mathbb{P}$-names which are in $\mathcal{C}$ (**class names**).

A filter G is $\mathbb{P}$-**generic** over $\mathbb{M}$ if it meets all dense subsets of M which are in $\mathcal{C}$. Evaluations of names are defined as usual. We set $\mathbb{M}[G] = \langle M[G], \mathcal{C}[G] \rangle$, where

- $M[G] = \{\sigma^G \mid \sigma \in M^{\mathbb{P}}\}$
- $\mathcal{C}[G] = \{\Gamma^G \mid \Gamma \in \mathcal{C}^{\mathbb{P}}\}$.

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The forcing theorem

Let $\mathbb{P}$ be a class forcing. We write $p \Vdash_{\mathbb{P}} \varphi(\sigma, \Gamma)$ if for every $\mathbb{P}$-generic filter G, $\mathbb{M}[G] \models \varphi(\sigma^G, \Gamma^G)$.

Definition

We say that $\mathbb{P}$ satisfies the **forcing theorem** over $\mathbb{M}$, if for every $\mathcal{L}_\in$-formula $\varphi(x, C)$ allowing class parameters and for every $\Gamma \in \mathcal{C}^{\mathbb{P}}$,

1. $\{\langle p, \sigma \rangle \in P \times M^{\mathbb{P}} \mid p \Vdash_{\mathbb{P}} \varphi(\sigma, \Gamma)\} \in \mathcal{C}$ (**definability lemma**)
2. whenever G is $\mathbb{P}$-generic over $\mathbb{M}$ and $\sigma \in M^{\mathbb{P}}$ and $\Gamma \in \mathcal{C}^{\mathbb{P}}$ such that $\mathbb{M}[G] \models \varphi(\sigma^G, \Gamma^G)$ then there is $p \in G$ with $p \Vdash_{\mathbb{P}} \varphi(\sigma, \Gamma)$ (**truth lemma**).

Pretameness

The following notion was introduced by Sy Friedman.

Definition

We say that class forcing $\mathbb{P}$ for $\mathbb{M}$ is **pretame** for $\mathbb{M}$ if for every $p \in \mathbb{P}$ and for every sequence of dense classes $\langle D_i \mid i \in I \rangle$ such that $I \in M$ and $\{\langle p, i \rangle \mid i \in I \wedge p \in D_i\} \in \mathcal{C}$, there is $q \leq_{\mathbb{P}} p$ and $\langle d_i \mid i \in I \rangle \in M$ such that for every $i \in I$, $d_i \subseteq D_i$ and d_i is predense below q.

Pretameness

Theorem (S. Friedman)

Let $\mathbb{M}$ be a model of GB^- such that either $M \models$ Power set, or $\mathcal{C}$ contains a good well-order. Then the following statements hold for every notion of class forcing $\mathbb{P}$:

1. *If $\mathbb{P}$ is pretame then $\mathbb{P}$ satisfies the forcing theorem.*
2. *If $\mathbb{P}$ is pretame and G is $\mathbb{P}$-generic over $\mathbb{M}$ then $\mathbb{M}[G]$ satisfies GB^-.*
3. *Suppose that for every $p \in \mathbb{P}$ there is a $\mathbb{P}$-generic filter G such that $p \in G$ and $\mathbb{M}[G] \models \mathrm{GB}^-$, then $\mathbb{P}$ is pretame.*

Pretameness

2. If $\mathbb{P}$ is pretame and G is $\mathbb{P}$-generic over $\mathbb{M}$ then $\mathbb{M}[G]$ satisfies GB^-.

Sketch of the proof.

Suppose that $\mathbb{M}[G] \models \forall x \in \sigma^G \exists y \varphi(x, y, \Gamma^G)$. Take $p \in G$ such that $p \Vdash_{\mathbb{P}} \forall x \in \sigma \exists y \varphi(x, y, \Gamma)$. For $\langle \tau, q \rangle \in \sigma$ let

$$D_{\tau,q} = \{r \leq_{\mathbb{P}} p \mid \exists \pi \in M^{\mathbb{P}}(r \Vdash_{\mathbb{P}} \varphi(\tau, \pi, \Gamma)) \vee r \perp_{\mathbb{P}} q\}.$$

Then each $D_{\tau,q}$ is dense below p. Take $r \in G$ and $\langle d_{\tau,q} \mid \langle \tau, q \rangle \in \sigma \rangle \in M$ such that each $d_{\tau,q} \subseteq D_{\tau,q}$ is predense below r. Let $\alpha \in \mathrm{Ord}^M$ minimal such that for each $\langle \tau, q \rangle \in \sigma$ and each $s \in d_{\tau,q}$ with $s \leq_{\mathbb{P}} q$ there is $\pi \in V_\alpha^M$ with $s \Vdash_{\mathbb{P}} \varphi(\tau, \pi, \Gamma)$.Put

$$\mu = \{\langle \pi, s \rangle \mid \pi \in V_\alpha^M \wedge \exists \langle \tau, q \rangle \in \sigma(s \in d_{\tau,q} \wedge s \leq_{\mathbb{P}} q \wedge s \Vdash_{\mathbb{P}} \varphi(\tau, \pi, \Gamma))\}.$$

Then $\mathbb{M}[G] \models \forall x \in \sigma^G \exists y \in \mu^G \varphi(x, y, \Gamma^G)$. $\dashv$

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Failures of Separation

Recall that in every $\mathrm{Col}(\omega, \mathrm{Ord})$-generic extension $\mathbb{M}[G]$ there is a cofinal function $F : \omega \to \mathrm{Ord}^M$. Actually, even Separation fails: Let G be $\mathbb{P}$-generic over M for $\mathbb{P} = \mathrm{Col}(\omega, \mathrm{Ord})$. Consider

$$X = \{n \in \omega \mid F(n) \text{ even}\}.$$

Let $\dot{F} \in C^{\mathbb{P}}$ be a class name for F, $\sigma \in M^{\mathbb{P}}$ and $p \in G$ with $p \Vdash_{\mathbb{P}} \sigma = \{n \in \check{\omega} \mid \dot{F}(\check{n}) \text{ even}\}$. Let $\alpha = \mathrm{rank}(\sigma)$ and $q \leq_{\mathbb{P}} p$ in G such that $q(n) = \alpha$ for some $n \in \omega$. Let $\pi : \mathbb{P} \to \mathbb{P}$ swap α and $\alpha + 1$. Then $\pi^*(\sigma) = \sigma$ where π^* is the map $M^{\mathbb{P}} \to M^{\mathbb{P}}$ derived from π. Note that $G' = \pi'' G$ is $\mathbb{P}$-generic with $\pi(q) \in G'$ and $\sigma^G = \sigma^{G'}$. But

$$n \in \sigma^G \iff \alpha \text{ even} \iff \alpha + 1 \text{ odd} \iff n \notin \sigma^{G'}.$$

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In $\mathrm{Col}(\omega, \mathrm{Ord})$-generic extensions Separation fails.

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Question

Does the preservation of Separation in a class-generic extension already imply the preservation of Replacement?

Separation implies Replacement

Theorem

Let $\mathbb{M} = \langle M, \mathcal{C} \rangle$ be a countable transitive model of GB^- such that $\mathcal{C}$ contains a good well-order $\prec$. Let $\mathbb{P} \in \mathcal{C}$ be a class forcing which satisfies the forcing theorem and let G be $\mathbb{P}$-generic over $\mathbb{M}$. If $\mathbb{M}[G]$ satisifies Separation, then $\mathbb{M}[G]$ also satisfies Replacement.

To prove this, we first need

Lemma

Suppose that M satisfies Power Set, or $\mathcal{C}$ contains a good well-order. Let $\mathbb{P}$ be a class forcing and G be $\mathbb{P}$-generic over $\mathbb{M}$. Then Replacement fails in $\mathbb{M}[G]$ if and only if there is $\kappa \in \mathrm{Ord}^M$ such that $\mathcal{C}[G]$ contains a cofinal function $\kappa \to \mathrm{Ord}^M$.

Sketch of the proof.

Suppose that $\mathbb{M}[G] \models \forall x \in \sigma^G \exists y\, \varphi(x, y, \Gamma^G)$ and consider

$$F(x) = \min\{\alpha \in \mathrm{Ord}^M \mid \exists \mu \in (V_\alpha)^M \cap M^{\mathbb{P}}\, \varphi(x, \mu^G, \Gamma^G)\}$$

for $x \in \sigma^G$. If F is not cofinal in Ord^M then $\mathrm{ran}(F) \subseteq \alpha$ for some $\alpha \in \mathrm{Ord}^M$. But then $\mathbb{M}[G] \models \forall x \in \sigma^G \exists y \in \tau^G\, \varphi(x, y, \Gamma^G)$, where $\tau = \{\langle \mu, \mathbb{1}_{\mathbb{P}} \rangle \mid \mu \in (V_\alpha)^M \cap M^{\mathbb{P}}\}$. $\dashv$

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Separation implies Replacement

Theorem

Let $\mathbb{M} = \langle M, \mathcal{C} \rangle$ be a countable transitive model of GB^- such that $\mathcal{C}$ contains a good well-order $\prec$. Let $\mathbb{P} \in \mathcal{C}$ be a class forcing which satisfies the forcing theorem and let G be $\mathbb{P}$-generic over $\mathbb{M}$. If $\mathbb{M}[G]$ satisifies Separation, then $\mathbb{M}[G]$ also satisfies Replacement.

To prove this, we first need

Lemma

Suppose that M satisfies Power Set, or $\mathcal{C}$ contains a good well-order. Let $\mathbb{P}$ be a class forcing and G be $\mathbb{P}$-generic over $\mathbb{M}$. Then Replacement fails in $\mathbb{M}[G]$ if and only if there is $\kappa \in \mathrm{Ord}^M$ such that $\mathcal{C}[G]$ contains a cofinal function $\kappa \to \mathrm{Ord}^M$.

Sketch of the proof.

Suppose that $\mathbb{M}[G] \models \forall x \in \sigma^G \exists y\, \varphi(x, y, \Gamma^G)$ and consider

$$F(x) = \min\{\alpha \in \mathrm{Ord}^M \mid \exists \mu \in (\mathrm{V}_\alpha)^M \cap M^{\mathbb{P}}\, \varphi(x, \mu^G, \Gamma^G)\}$$

for $x \in \sigma^G$. If F is not cofinal in Ord^M then $\mathrm{ran}(F) \subseteq \alpha$ for some $\alpha \in \mathrm{Ord}^M$. But then $\mathbb{M}[G] \models \forall x \in \sigma^G \exists y \in \tau^G\, \varphi(x, y, \Gamma^G)$, where $\tau = \{\langle \mu, \mathbb{1}_{\mathbb{P}} \rangle \mid \mu \in (\mathrm{V}_\alpha)^M \cap M^{\mathbb{P}}\}$. $\dashv$

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WLOG suppose that $\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}} \dot{F} : \check{\kappa} \to \mathrm{Ord}^M$ cofinal. Let $\langle C_\gamma \mid \gamma \in \mathrm{Ord}^M \rangle \in \mathcal{C}$ be a sequence of classes of ordinals such that each C_γ has one of the forms

$$A_{p,\alpha} = \{\beta \in \mathrm{Ord}^M \mid \exists q \leq_{\mathbb{P}} p(q \Vdash_{\mathbb{P}} \dot{F}(\check{\alpha}) = \check{\beta})\} \in \mathcal{C}$$

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for $p \in \mathbb{P}, \alpha < \kappa$ and $\tau \in M^{\mathbb{P}}$ such that C_γ is a proper class, and each such $A_{p,\alpha}, B_{p,\alpha,\tau}$ appears unboundedly often in the enumeration. There is $D \in \mathcal{C}$ such that $C_\gamma \cap D$ and $C_\gamma \setminus D$ are proper classes for each $\gamma \in \mathrm{Ord}^M$. If Separation holds in $\mathbb{M}[G]$ then there is $\tau \in M^{\mathbb{P}}$ and $p \in G$ such that $p \Vdash_{\mathbb{P}} \tau = \{\alpha < \check{\kappa} \mid \dot{F}(\alpha) \in \check{D}\}$. Observe that there is $\alpha < \kappa$ such that $A_{p,\alpha}$ is a proper class. But then $A_{p,\alpha} \cap D = B_{p,\alpha,\tau}$ is a proper class. But then $B_{p,\alpha,\tau} \setminus D = \emptyset$, a contradiction. ⊣

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Motivation

In set forcing, if there is a dense embedding $\mathbb{P} \to \mathbb{Q}$ then $\mathbb{P}$ and $\mathbb{Q}$ have the same generic extensions.

Observation

Let $\mathrm{Col}_*(\omega, \mathrm{Ord})$ denote the suborder of $\mathrm{Col}(\omega, \mathrm{Ord})$ of conditions p with $\mathrm{dom}(p) \in \omega$. Clearly, $\mathrm{Col}_*(\omega, \mathrm{Ord})$ is dense in $\mathrm{Col}(\omega, \mathrm{Ord})$. However, $\mathrm{Col}(\omega, \mathrm{Ord})$ collapses all M-cardinals but $\mathrm{Col}_*(\omega, \mathrm{Ord})$ does not add any new sets.

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How can we characterize class forcings $\mathbb{P}$ which have the same generic extensions as all other class forcings into which $\mathbb{P}$ embeds densely?

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The extension maximality principle

Definition

A notion of class forcing $\mathbb{P}$ satisfies the **extension maximality principle (EMP)** over $\mathbb{M} \models \mathrm{GB}^-$ if for every notion of class forcing $\mathbb{Q}$ such that $\mathbb{P}$ is dense in $\mathbb{Q}$ and for every $\mathbb{Q}$-generic filter G over $\mathbb{M}$, $M[G] = M[G \cap \mathbb{P}]$.

Theorem

Suppose that $\mathbb{P} \in \mathcal{C}$ is a notion of class forcing which satisfies the forcing theorem and that $\mathcal{C}$ contains a good well-order. Then $\mathbb{P}$ is pretame if and only if $\mathbb{P}$ satisfies the EMP.

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Sketch of the proof.

Suppose first that $\mathbb{P}$ is pretame. Let $\mathbb{P}$ be dense in $\mathbb{Q}$, G $\mathbb{Q}$-generic and $\sigma \in M^{\mathbb{Q}}$. For each $q \in \mathrm{tc}(\sigma) \cap \mathbb{Q}$ consider the dense set

$$D_q = \{p \in \mathbb{P} \mid p \leq_{\mathbb{Q}} q \vee p \perp_{\mathbb{Q}} q\}.$$

Take $p \in G \cap \mathbb{P}$ and $d_q \subseteq D_q$ in M predense below p. For each $\tau \in \mathrm{tc}(\{\sigma\}) \cap M^{\mathbb{Q}}$ let

$$\bar{\tau} = \{\langle \bar{\mu}, r \rangle \mid \exists s (\langle \mu, s \rangle \in \tau \wedge r \in d_s \wedge r \leq_{\mathbb{Q}} s)\}.$$

Then $\bar{\sigma} \in M^{\mathbb{P}}$ and $\sigma^G = \bar{\sigma}^{G \cap \mathbb{P}} \in M[G \cap \mathbb{P}]$. $\dashv$

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Sketch of the proof, continued.

Suppose $\mathbb{P}$ is non-pretame but satisfies the EMP. Let G be $\mathbb{P}$-generic such that Replacement fails in $\mathbb{M}[G]$. Then so does Separation. Take $\Gamma \in \mathcal{C}^{\mathbb{P}}, \sigma \in M^{\mathbb{P}}$ and $p \in G$ with $p \Vdash_{\mathbb{P}} \Gamma \subseteq \sigma$ such that there is no $q \in G$ and $\tau \in M^{\mathbb{P}}$ with $q \Vdash_{\mathbb{P}} \Gamma = \tau$. Let

$$\mathbb{Q} = \mathbb{P} \sqcup \{\sup A_\mu \mid \mu \in \mathrm{dom}(\sigma), A_\mu \neq \emptyset\},$$

where $A_\mu = \{q \leq_{\mathbb{P}} p \mid q \Vdash_{\mathbb{P}} \mu \in \Gamma\}$. Now let

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Then $\tau^H = \Gamma^H = \Gamma^G$, where H is the upwards closure of G in $\mathbb{Q}$. Contradiction. ⊣

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Motivation

As in set forcing, we are interested in preserving properties of forcing notions under dense embeddings.

Notation

Let Ψ be some property of notions of class forcing. We will say that a notion of class forcing $\mathbb{P}$ satisfies Ψ **densely**, if every notion of class forcing $\mathbb{Q}$ such that there is a dense embedding from $\mathbb{P}$ into $\mathbb{Q}$ satisfies the property Ψ.

We have seen that the forcing theorem may fail for class forcings. On the other hand, there are non-pretame forcings such as $\text{Col}(\omega, \mathrm{Ord})$ which do satisfy the forcing theorem.

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Which class forcings densely satisfy the forcing theorem?

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Pretameness and the forcing theorem

Theorem

Suppose that $\mathbb{M} \models \mathsf{GB}^-$ and $\mathcal{C}$ contains a good well-order but no first-order truth predicate. Then a class forcing $\mathbb{P}$ for $\mathbb{M}$ is pretame if and only if it densely satisfies the forcing theorem.

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Suppose that M satisfies Power Set, or $\mathcal{C}$ contains a good well-order. Let $\mathbb{P}$ be a class forcing and G be $\mathbb{P}$-generic over $\mathbb{M}$. Then Replacement fails in $\mathbb{M}[G]$ if and only if there is $\kappa \in \mathrm{Ord}^M$ such that $\mathcal{C}[G]$ contains a cofinal function $\kappa \to \mathrm{Ord}^M$.

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Pretameness and the forcing theorem

Theorem

Suppose that $\mathbb{M} \models \mathrm{GB}^-$ and $\mathcal{C}$ contains a good well-order but no first-order truth predicate. Then a class forcing $\mathbb{P}$ for $\mathbb{M}$ is pretame if and only if it densely satisfies the forcing theorem.

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Pretameness and the forcing theorem

Sketch of the proof, continued.

Now consider the $\mathbb{Q}$-name

$$\dot{E} = \{\langle \mathrm{op}(\check{\alpha}, \check{\beta}), p_{\alpha,\beta}\rangle \mid \alpha, \beta < \kappa\} \in M^{\mathbb{Q}}.$$

If G is $\mathbb{Q}$-generic over $\mathbb{M}$ then in $M[G]$, $\langle M, \in\rangle$ is isomorphic to $\langle \kappa, \dot{E}^G\rangle$, witnessed by $\dot{F}^G$. We translate $\mathcal{L}_\in$-formulae in the forcing language of $\mathbb{Q}$ to infinitary formulae by defining

$$(v_i = v_j)^*_{\vec{\alpha}} = (\check{\alpha}_i = \check{\alpha}_j)$$
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Sketch of the proof, continued.

Recall that we have an assignment $\varphi(v), \alpha \mapsto \varphi^*_\alpha$. Then we have

$$M \models \varphi(x) \Longleftrightarrow \forall\alpha < \kappa \forall p \in \mathbb{P}[p \Vdash_{\mathbb{P}} \dot{F}(\check{\alpha}) = \check{x} \to p \Vdash_{\mathbb{Q}} \varphi^*_\alpha].$$

We use

Lemma (Holy, K., Luecke, Njegomir, Schlicht)

If $\mathbb{Q}$ satisfies the forcing theorem for atomic formulae, then it also satisfies the forcing theorem for infinitary quantifier-free formulae.

Hence if $\mathbb{Q}$ satisfies the forcing theorem then C contains a first-order truth predicate for M. Contradiction. $\dashv$

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Motivation

Let $\mathbb{P}$ be a forcing notion. A **nice name** for a set of ordinals is a $\mathbb{P}$-name of the form $\bigcup_{\alpha<\gamma}\{\check{\alpha}\} \times A_\alpha$, where $A_\alpha \subseteq \mathbb{P}$ is a set-sized antichain and $\gamma \in \mathrm{Ord}^M$.

In set forcing, in $\mathbb{P}$-generic extensions every set of ordinals has a nice name. This motivates the following

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Consider the forcing notion $\mathbb{P} = \mathrm{Col}(\omega, \mathrm{Ord})$ and $\sigma = \{\langle \check{n}, \{\langle n, 0\rangle\}\rangle \mid n \in \omega\}$. There is a name for the complement of σ^G: Let

$$\tau_n = \check{n} \cup \{\langle \check{m}, \{\langle i, 0\rangle \mid n \leq i < m\}\rangle \mid m > n\}.$$

Then τ_n is a name for the least $m \geq n$ with $m \notin \sigma^G$. Hence $\tau = \{\langle \tau_n, \mathbb{1}_{\mathbb{P}}\rangle \mid n \in \omega\}$ is a name for $\omega \setminus \sigma^G$.

Suppose that $\mu = \bigcup_{n\in\omega}\{\check{n}\} \times A_n$ and $p \Vdash_{\mathbb{P}} \mu = \check{\omega} \setminus \sigma$. Take $n \notin \mathrm{dom}(p)$ and $\alpha > \mathrm{rank}(A_n)$ and put $q = p \cup \{\langle n, \alpha\rangle\}$. Then $q \Vdash_{\mathbb{P}} \check{n} \in \mu$ so there must be $r \in A_n$ which is compatible with q. But then $n \in \mathrm{dom}(r)$ and so $r(n) = \alpha$, a contradiction.

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$\omega \setminus \sigma^G$ has a $\mathbb{P}$-name but no nice $\mathbb{P}$-name.

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Definition

A forcing notion $\mathbb{P}$ is said to be **nice**, if for every $\gamma \in \mathrm{Ord}^M, \sigma \in M^{\mathbb{P}}$ and for every $\mathbb{P}$-generic filter G such that $\sigma^G \subseteq \gamma$ there is a nice name $\tau \in M^{\mathbb{P}}$ such that $\sigma^G = \tau^G$.

Let's consider some easy examples:

- $\mathrm{Col}(\omega, \mathrm{Ord})$ is not nice.
- Every M-complete Boolean algebra is nice: Given σ, γ as above put $\tau = \{\langle \check{\alpha}, [\![\check{\alpha} \in \sigma]\!]\rangle \mid \alpha < \gamma\}$.
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We have shown that pretame forcings are nice. However, there are also non-pretame forcings that are nice:

Since $\mathrm{Col}(\omega, \mathrm{Ord})$ satisfies the forcing theorem, it has a Boolean completion $\mathbb{B}$. Then $\mathbb{B}$ is nice but it is non-pretame, since it still adds a cofinal function $\omega \to \mathrm{Ord}^M$. Moreover, we have

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Let $\mathbb{M}$ be a model of KM. Then a class forcing $\mathbb{P}$ for $\mathbb{M}$ is pretame if and only if it is densely nice.

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We have seen that pretameness of a class forcing $\mathbb{P}$ is - under sufficient conditions on the ground model $\mathbb{M}$ - equivalent to each of the following properties:

- $\mathbb{P}$ preserves Replacement.
- $\mathbb{P}$ preserves Separation.
- $\mathbb{P}$ does not add a cofinal function from some ordinal κ into Ord^M.
- $\mathbb{P}$ satisfies the EMP.
- $\mathbb{P}$ densely satisfies the forcing theorem.
- $\mathbb{P}$ is densely nice.
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All properties above always hold for set forcings; this suggests that pretame forcings are the "right" class of class forcings to consider.

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Further results

A class forcing $\mathbb{P}$ is said to satisfy the Ord-**cc**, if all its antichain are set-sized, i.e. elements of M.

We can strenghten many previously considered properties and obtain characterize class forcings $\mathbb{P}$ with the Ord-cc by

- $\mathbb{P}$ satisfies the strong extension maximality principle.
- $\mathbb{P}$ satisfies the maximality principle.
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- $\mathbb{P}$ has a unique Boolean completion.
- $\mathbb{P}$ has a Boolean completion $\mathbb{B}(\mathbb{P})$ such that every subclass of $\mathbb{B}(\mathbb{P})$ which is in $\mathcal{C}$ has a supremum in $\mathbb{B}(\mathbb{P})$.

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Open questions

There remain many open questions related to (pretame) class forcing:

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1. Is ZF^- enough to prove that pretame forcings satisfy the forcing theorem?
2. Is ZF^- enough to characterize pretameness via the preservation of Replacement?

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Does every class forcing which preserves Separation satisfy the forcing theorem?

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Thank you for your attention!

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