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Graduate Seminar on Algebraic Geometry (Modul S4A1)
Zeta functions in algebraic geometry

Thursday 2:15-3:45 pm, SR 0.008, **Winter term 2018/19**

This is the detailed plan for the seminar. Roughly, we follow the notes of Mustata, but some talks on background material are added. The program is subject to change. Depending on the background of the participants we may add (or drop certain) topics.
Prerequisites Algebraic geometry as in Hartshorne's book, Chapters II-III and some part of IV. However, the participants are expected to learn quickly some more material not covered by Hartshorne's book or by the classes on algebraic geometry during the last academic year.

Should you be interested in the seminar, please send an email to Daniel Huybrechts or to René Mboro before September 24th explaining your background and naming two of the talks you would be interested to give. The talks will be distributed by the end of September.
Every speaker is to contact René Mboro at least two weeks before the talk with a detailed plan what to cover and how. He will also be the person to contact should you encounter any problems in yours or anyone else's talk or for references. Each talk is 90 min and they should be prepared accordingly. Questions during and after the talks are welcome.

1. Frobenius: Absolute, arithmetic, geometric.
Date: 25/10, **Speaker:** René Mboro
This talk collects all standard facts about the Frobenius morphism that come up throughout the seminar. Start with [10, Sec. 2.1], see also [6, Ch. IV.2]. This should be done with great care and presented in the clearest form possible, as it is crucial for large parts of the seminar. Explain the Artin–Schreier and the Kummer sequence [10, Examples 4.21 & 4.22]. Recall that the Frobenius morphism is not smooth [6, Example II. 10.5.1]. Show that the pull-back of a line bundle $\mathcal{L}$ under the Frobenius satisfies $F^*\mathcal{L} \cong \mathcal{L}^p$. Apply this to Frobenius split varieties and show that $H^i(X, \mathcal{L}) = 0$ for all $i > 0$ and any ample line bundle $\mathcal{L}$, see [9, p. 1]. Maybe something can be explained or recalled about the étale topology/cohomology which will be used later. If time is too short for this, we will have to accept this as black box.

2. Weil conjectures: Statements and easy examples.

Date: 08/11, **Speaker:** Yijie Diao

Define the Zeta function $Z(X,t)$ for a variety X defined over $\mathbb{F}_q$ and state the Weil conjectures, see [6, App. C.1] for a quick introduction. Section 3 in [10] contains more details on the formal aspects. (Don't waste time on the formal Proposition 2.6.) Compare the Zeta function with the Riemann Zeta function, see [6, Exercise App. C.5.4]. The better analogy will come later for arithmetic varieties. Explain the two examples in [10, Sect. 2.6]. In Remark 2.9 discuss the equality $Z(X,p^{-s}) = \sum a_n/p^{sn}$ with a_n the number of effective cycles of degree n. This comes up again in Section 6.5.

In the definition of varieties with polynomial count it is enough to assume that $P \in \mathbb{C}[y]$ (and not $P \in \mathbb{Z}[y]$), see [7]. In there, one also finds the notion of zeta-equivalent varieties. (For those who know about Hodge numbers: It is surprising that zeta-equivalent varieties not only have the same Betti numbers but also the same Hodge numbers.)

3. Weil conjectures for curves: The first non-trivial examples.

Date: 15/11, **Speaker:** Parthiv Basu

Follow [10, Ch. 3]. Lemma 3.8 corresponds to [6, Exercise V.1.10] and Theorem 3.6 to [6, Exercise App. C.5.7]. State the Hodge index theorem (needed in Proposition 3.9) and the adjunction formula for general surfaces, see [6, Ch. V.1].

Elliptic curves: Add Hasse's proof of the Riemann hypothesis for elliptic curves, see [6, Exercise IV.4.16] or [13, Ch. V].

Show that the Weil conjectures for curves imply the Weil conjectures for all rational surfaces.

4. Weil cohomology theories: The formal set-up.

Date: 22/11, **Speaker:** Solomiya Mizyuk

This should cover the material of [10, Ch 4] which can be find in many other sources. The first main result is Theorem 4.7 which deduces from the formal definition of a Weil cohomology theory the Lefschetz trace formula. Then prove rationality and the functional equation assuming the existence of a Weil cohomology theory for varieties over $\mathbb{F}_q$ (Theorem 4.11 and Theorem 4.14, cf. [6, App. C]). Étale cohomology yields a Weil cohomology theory for varieties over $\mathbb{F}_q$. This should just be remarked as a black box. Hint: Focus on the proofs of these three results. Writing out the formal definition of a Weil cohomolog theory can take long, but it is a little dry. The problem in this talk will be to find the right balance between the two parts.

5. L-functions: Outline of the main steps.

Date: 06/12, **Speaker:** Thorsten Beckmann

The Hasse–Weil Zeta function $Z(X,t)$ was defined for X defined over a finite field. The global version

$$L_X(s) = \prod_p Z(X_p, p^{-s})$$

is for arithmetic varieties X, i.e. schemes defined over $\mathrm{Spec}(\mathbb{Z})$ with X_p the fibre over $p \in \mathrm{Spec}(\mathbb{Z})$. Not much is known in general. The aim here is to show that $L_X(s)$ defines is a holomorphic function for $\mathrm{Re}(s) > \dim(X)$, cf. [10, Cor. 6.28]. There are two main ingredients: (i) General theory of Dirichlet series, see [10, Sec. 6.4] or the classic [12, Ch. VI.2] and (ii) Lang–Weil estimates in [10, Sec. 6.2].
When X_p is a smooth projective variety, the Weil conjectures give us information about the factors $Z(X_p, p^{-s})$. However, usually some of the fibres are not smooth in which case one needs to bound the number rational points otherwise. This makes (ii) technically more involved. In this talk one should try to give an outline of the results in Section 6.5. Results from 6.2-6.3 on Lang–Weil estimates and from 6.4 on Dirichlet series should be stated clearly but not proved.

6 & 7. Fulton's trace formula and applications: This needs a team of two people, In particular, the second speaker should take over some portions of the first part.
Date: 13/12 & 20.12. **Speakers:** Jiadong Han & Willem de Muinck Keizer
Formulation, proof, and analogies.
The formula computes the number of $\mathbb{F}_q$-rational points modulo p:

$$|X(\mathbb{F}_q)| \equiv \sum (-1)^i \mathrm{tr}(F|H^i(X, \mathcal{O}_X)) \mod p$$

It is a special case of a formula for coherent F-modules (Theorem 5.4) and eventually reduces to the localization theorem (Theorem 5.8) expressing the inverse of $K_\bullet^F(X(\mathbb{F}_q)) \to K_\bullet^F(X)$ as $\sum \mathrm{tr}(F(x))$.
The talk should cover Section 5.1 and 5.1 in [10], see also [1] for further details. Compare Theorem 5.8 to the holomorphic Lefschetz theorem [3, p. 421], which will later be generalized by the Woods Hole formula.
The proof can also be cast in the language of crystals. Introduce this notion as explained in [11, Sec. 1] and, in particular, the technique of taking quotients by Serre subcategories. (There F-modules are called τ-sheaves.) In [11, Sec. 3] the Grothendieck group of F-sheaves is viewed (conceptually more clearly) as the Grothendieck group of the abelian category $\mathrm{Coh}_{\mathrm{crys}}(X)$ of crystals. Fulton's trace formula (or rather Theorem 5.4) is generalized by the Woods Hole trace formula:

$$\sum_{x \in f^{-1}(y)(\mathbb{F}_q)} \mathrm{tr}(F|\mathcal{M}(x)) = \mathrm{tr}(F|Rf_*\mathcal{M}(y)),$$

where $f\colon X \to Y$ is a suitable morphism, $y \in Y(\mathbb{F}_q)$ and $\mathcal{M} \in \mathrm{Coh}_{\mathrm{crys}}(X)$. At least, state the formula, explain all terms and how to view Theorem 5.4 as the special case $Y = \mathrm{Spec}(\mathbb{F}_q)$. The story is part of Grothendieck's function-sheaf correspondence.

Supersingular varieties and Chevalley–Warning.
Fulton's trace formula can be used to characterize supersingular Calabi–Yau hypersurfaces (i.e. smooth hypersurfaces in $\mathbb{P}^n$ of degree $n+1$). Note that the equivalence of (i) and (ii) in [10, Prop. 5.15] works for general Calabi–Yau varieties, see [1] for the definition. The case of

hypersurfaces is generalized to complete intersections in [10, Exercise 5.19]. For the special case of supersingular elliptic curves see [13], see also [10, Exercise 5.18].
The theorem of Chevalley–Warning states that for hypersurfaces (or, more generally, complete intersections) of degree $< n+1$ the number of points is always $\equiv 1 \mod p$, see [1, Sec. 5.1] for a proof and references. An improvement, giving formulas $\mod q^n$, is due to Ax–Katz. State the result.

8. Grothendieck ring of varieties: Bittner's description and zero divisors.
Date: 17/01 **Speaker:** Louis Jaburi
Follow [10, Sec. 7.1] and introduce the Grothendieck ring of varieties $K_0(\mathrm{Var}(k))$. When discussing the multiplicativity for Zariski locally trivial fibration, give an example that shows that this not true for étale locally trivial fibrations. If time permits, it would be good to at least indicate the main ideas of the proof of Theorem 7.10. At least explain the assertion in Theorem 7.11 in detail. Example 7.13 needs mixed Hodge theory which is beyond the scope of the seminar, but the results should nevertheless be stated. In the discussion of Theorem 7.21 mention that there are in fact examples of non-isomorphic elliptic curves E_1, E_2 for which there exists another elliptic curve E with $E_1 \times E \cong E_2 \times E$. This yields easier examples of zero divisors. More recently other examples of zero divisors have been found, in particular $\mathbb{L}$ itself is a zero divisor on $K_0(\mathrm{Var}(k))$. Add a description of $[\mathrm{Gl}(n,k)] \in K_0(\mathrm{Var}(k))$ and at least mention the one for $[\mathrm{Gr}(n,m)]$, see [8].

9. Kapranov's Zeta function: Rationality of the Zeta function on a motivic level.
Date: 24/01 **Speaker:** Denis Nesterov
This talk should cover Sections 7.2-7.4 in [10]. Discuss the definition of

$$Z_{\mathrm{mot}}(X,t) \coloneqq \sum_{n\geq 0} [\mathrm{Sym}^n(X)] t^n$$

and explain, in particular, how it is related to the usual Zeta function for varieties over finite fields. We take the existence of the symmetric product $\mathrm{Sym}^n(X) = X^n/\mathfrak{S}_n$ of a quasi-projective variety X for granted. State and prove the rationality of the Zeta function for curves and at least state the fact that for surfaces the rationality holds exactly for surfaces of negative Kodaira dimension. (Skip the proof of the weaker version Proposition 7.38.) Note that in Section 7.2 quite some time is spent of the subte difference between varieties and quas-projective varieties (see [6, App. B] for an example), but this should not be the main concern of the talk. Better to spend more time on the proof of Theorem 7.33. Compare this part with the proof of the Weil conjectures for curves in Section 3.1.

11. General Woods Hole trace formula. Woods Hole trace formula.
Date: 31/01 **Speaker:** Lisa Li
This is a generalization of Fulton's trace formula which replaces the Frobenius by an endomorphism $f\colon X \to X$ with simple fixed points. For any coherent sheaf $\mathcal{F}$ and a homomorphism

$f^*\mathcal{F} \to \mathcal{F}$

$$\sum(-1)^i \mathrm{tr}(\varphi|H^i(X,\mathcal{F}) = \sum \frac{\mathrm{tr}(\varphi(x))}{\det(1-df|\Omega(x))}.$$

Follow [11, App. A]. This talk uses some more advanced machinery (Grothendieck–Verdier duality for morphisms) and should be best given by a participant with a background in derived categories.

References

[1] T. Bülles *Fulton's trace formula for coherent cohomology.* Bachelor thesis (2016). `http://www.math.uni-bonn.de/people/huybrech/Buelles.pdf`

[2] E. Gorsky *Adams operations and power structures.* Moscow Math. J. 9 (2009) 305–323.

[3] P. Griffiths, J. Harris *Principles of algebraic geometry.* Wiley & Sons, New York, (1978).

[4] S. Gusein-Zade, I. Luengo, A. Melle-Hernandez *A power structure over the Grothendieck ring of varieties.* MRL 11 (2004), 49–57.

[5] S. Gusein-Zade, I. Luengo, A. Melle-Hernandez *Power structure over the Grothendieck ring of varieties and generating series of Hilbert schemes of points.* Michigan Math. J. 54 (2006), 353–359.

[6] R. Hartshorne *Algebraic Geometry* GTM 52. Springer (1977).

[7] N. Katz *E-polynomials, zeta-equivalence, and polynomial-count varieties.* Appendix to Hausel, Rodriguez-Villegas: Mixed Hodge polynomials of character varieties. Invent. math. 174 (2008), 555–624.

[8] A. Kiefner *Zero divisors in the Grothendieck ring of varieties.* Bachelor thesis (2016). `http://www.math.uni-bonn.de/people/huybrech/Kiefner.pdf`

[9] V. Mehta, A. Ramanathan *Frobenius splitting and cohomology vanishing for Schubert varieties.* Ann. Math. 122 (1985), 27–40.

[10] M. Mustata *Zeta functions in algebraic geometry* `http://www-personal.umich.edu/~mmustata/zeta_book.pdf`

[11] L. Taelman *Sheaves and Functions Modulo p.* London Mathematical Society Lecture Note Series. Cambridge University Press, (2015). `https://staff.fnwi.uva.nl/l.d.j.taelman/beijing.pdf`

[12] J.-P. Serre *A course in arithmetics.* GTM 7. Springer (1973).

[13] J. Silverman *The arithmetic of elliptic curves.* GTM 106. Springer (1986).