Seminar on ‘Noether-Lefschetz and Gromov–Witten theory’.
Winter term 2009/10[1]

The aim of this seminar is to understand the relation between Noether–Lefschetz loci and Gromov–Witten invariants for Calabi–Yau threefolds given by Lefschetz pencils of K3 surfaces. The main emphasis should be put on the Noether–Lefschetz part and the hope would be to learn along the way some aspects of the theory of automorphic forms applied to K3 surfaces. The main reference is the paper of Maulik and Pandharipande [1], but we will have to learn the background from other sources. See also the survey [2] and the sequel [3].

The main topics that we should try to cover are:

I. Classical Noether-Lefschetz theory.

The classical Noether–Lefschetz locus for hypersurfaces in $\mathbb{P}^3$ is the locus where the Picard rank is bigger than one. One knows that the locus is proper for degree $d \geq 4$. There is a purely algebraic proof (due to Griffiths and Harris) and one that uses Hodge theory. For K3 surfaces the result is easier to prove using pure Hodge theory and standard facts about K3 surfaces. For a complete curve of quasi-polarized K3 surfaces the Noether–Lefschetz number counts the fibres with higher Picard rank, which can be understood as the intersection of this curve with the Noether–Lefschetz locus.

II. Noether-Lefschetz numbers and modular forms.

The main input for the paper [1] is a result of Borcherds that says that the generating series of all Heegner divisors has certain modular forms as coefficients (see the theorem on page 28 in [1]). So we will learn a few things about period domains, arithmetic quotients and Heegner divisors. The easiest Heegner divisor is related to automorphic forms. For K3 surfaces this has been used to show the isotriviality of complete families of K3 surfaces with constant Picard rank (Borchers et al).

We probably will not have time to go into the details of Borcherds’ papers, but we should at least dig out what is needed. There are other applications of automorphic forms to K3 surfaces. E.g. results of Gritsenko, Hulek and Sankaran on the Kodaira dimension of their moduli space. But probably, we won’t have time for this either.

III. Relation between NL numbers and Gromov–Witten invariants.

In [1] it is shown that GW invariants of families of K3 surfaces are related to NL numbers. At this occasion it would be interesting to review work of Beauville and Bryan, Leung on the Yau-Zaslov conjecture that counts rational curves on K3 surfaces. The final aim would be to understand Theorem 1 in [1] relating the GV invariants of a pencil of quartics with the NL numbers of this family.

IV. Application to K3 surfaces.

The paper [1] in particular computes the Noether–Lefschetz number explicitly for a pencil of quartics. This combines Borcherds’ work with its application to Noether–Lefschetz numbers and the relation between GV and NL.

Another application (less recent) of Borcherds’ techniques is the result that any complete family of K3 surfaces with constant Picard number is isotrivial [7]. See also [14, 19, 20, 21].

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Detailed plan of the meetings (subject to changes)

The following has to be read as a suggestion and, of course, the time given for each of the talks is approximative. We should feel free to change the order or to choose different topics. It seems that one more session would be good so that we could really cover [7]. Is December 22 is an option? Of course, we are free to schedule complementing talks to cover more material or discuss details. But the official meetings with Mainz should only take place every other week. The late start for the seminar is due to beginning of term in Mainz.

I. Meeting. 27 Oct 2009 (Bonn)

1. Introduction and survey (D. Huybrechts, 45min)

2. Noether–Lefschetz theorem. Hodge theoretic approach (H. Hartmann, 1h30)

The original Noether–Lefschetz theorem saying that the generic hypersurface in $\mathbb{P}^3$ of degree $d \geq 4$ contains only CI curves goes back to Noether who gave a very intuitive geometric argument (see the Introduction of [11]). Lefschetz later developed more Hodge theoretic arguments, in modern form this is in [10].

This talk should give an account of the proof following [23]. Define the Noether–Lefschetz locus for hypersurfaces and more generally for threefolds with a very ample system (Def. 15.31). State Theorem 15.33 and explain that the classical Noether–Lefschetz theorem is a consequence (Theorem 15.32). Prove Theorem 15.33 using the irreducibility of the monodromy representation (Theorem 15.27, Corollary 15.28). (How much do we want to recall about vanishing cycles?) The NL locus is often dense. Explain Proposition 17.20.

At the end, one might mention the recent result of Maulik, Poonen and Voisin [16] that for a family over $\bar{\mathbb{Q}}$ there also fibres defined over $\bar{\mathbb{Q}}$ with generic Picard number. (The proof in the complex case is easier than the p-adic situation.)

3. Noether–Lefschetz theorem. Algebraic approach (S. Rollenske, 45min)

The paper [11] contains a very concrete algebraic proof of the Noether–Lefschetz theorem for hypersurfaces in $\mathbb{P}^3$. It is based on a one-parameter degeneration of a hypersurface of degree d to the union of a hyperplane and a hypersurface of degree $d-1$ (or a certain blow-up of it, to ensure the smoothness of the whole family). A curve in the generic fibre is then followed to the degeneration the Picard group of which can be controlled.

Present the main construction of [11] explained in Section 2 a) and 2 b). Mention the remaining technical issue that necessitates further base change of the family, but leave out Section 2 c).

II. Meeting. 10 Nov 2009 (Bonn/Mainz)

1. Noether–Lefschetz locus for K3 surfaces (P. Sosna 1h)

Introduce the moduli space of marked K3 surfaces and the period map. State the main results of the theory (no proofs!): The Global Torelli theorem and the surjectivity of the period map. Explain the polarized version and the notion of (quasi-)polarized K3 surfaces. Discuss the Noether–Lefschetz locus for these moduli spaces via the period map. Emphasize that the NL locus is of codimension one. Sketch the standard results on the density of Kummer, elliptic and quartic surfaces. Use [4] or any of the other sources (eg. Section 4.8 in [12] for the density results).

2. Noether–Lefschetz numbers (A. Ferretti 45min)

The Noether–Lefschetz number for a complete family of quasi-projective K3 surfaces $X \longrightarrow C$ over a complete curve C roughly measures the number of points $\xi \in C$ for which $\mathrm{Pic}(X_\xi)$ goes up by a class of given square.

Define the divisors $P_{\Delta,\delta}$ and $D_{h,d}$ in [1]. Then discuss Sections 1.4 and 1.5 in [1]. In particular, one needs to prove the finiteness of the Noether–Lefschetz number (Proposition 1).

3. Discrete groups acting on period domains and Heegner divisors (S. Müller-Stach, 1h15)

The K3 lattice is of signature $(3, 19)$. Choosing a polarization this becomes $(2, 19)$. The two cases behave differently with respect to the action of the orthogonal group of the lattice on the period domain. In the first case, the action is not properly discontinuous. Also explain why in the second case we have to consider finite index subgroups.

Introduce the period domain $\mathbb{D} \subset \mathbb{P}(\Lambda_{\mathbb{C}})$ for a lattice Λ of signature $(2, n)$ and the action of the finite index subgroup $\Gamma \subset \mathrm{O}(\Lambda)$ acting trivially on the discriminant. Recall that $\mathbb{D}/\Gamma$ is quasi-projective (Baily–Borel, see eg. [18] for a discussion). For details in the case $n = 2$ see [9], which also contains a few general remarks on the Grassmann, projective and the tube domain model (see Section 2.4.)

See [9] Section 2.4.3 for the definition of Heegner divisors. Cover Section 4.3 in [1]

III. Meeting. 24 Nov 2009 (Mainz)

1. Borcherds' theorem on Heegner divisors (Xuanming Ye, 1h45)

The Heegner divisors of the last talk (recall the definition) can be put in a generating series. Viewed as an element of the Picard group of $\mathbb{D}/\Gamma$ with coefficients in power series of a formal parameter q, the coefficients turn out to be modular forms. This follows from work of Borcherds'.

Define the generating series $\Phi(q)$ ([1], p. 27). Discuss the notion of modular forms of half-integral weight ([1], Section 4.2) and state the main theorem ([1], p. 28). Explain why the Heegner divisors are Cartier (reference?). It would be good to get an impression of the proof of this result. I have not checked the literature carefully yet.

We have to compare the Heegner divisors with the divisors $D_{h,d}$ used to define the Noether–Lefschetz numbers. This is Lemma 4 in [1]. See also Section 8 in [15] for some geometric aspects related to K3 surfaces.

2. Application to NL numbers (P. Sosna 1h15)

The modularity of Borcherds' theorem allows one to explicitly compute NL numbers. There is a general result (see Section 4.4) and more precise results for families of quartics. The most explicit calculations for pencils of quartics are based on the relation between NL numbers and GW invariants, that will be explained later.

IV. Meeting. 8 Dec 2009 (Mainz)

1. Isotriviality of families of K3 surfaces (Kang Zuo 1h30)

A theorem of Borcherds, Katzarkov, Pantev and Shepherd-Barron says that any complete family of smooth projective K3 surfaces with constant Picard number must be isotrivial [7]. Roughly the idea is that automorphic forms are sections of an ample line bundle on the moduli space and that their zeros are related to jumps of the Picard rank.

There are other approaches (eg. Jorgenson, Todorov and maybe also Viehweg, Zuo?) and generalizations to higher dimensional HK manifolds due to Oguiso [19, 20].

2. Moduli spaces of Enriques surfaces (D. Huybrechts 1h30)

Borcherds' proved that the moduli space of Enriques surfaces is quasi-affine. His proof is based in his theory of infinite products etc. More recently, Pappas [22] gave a new proof which uses the Grothendieck–Riemann–Roch formula. We follow Pappas' automorphic free proof and explain how this implies the isotriviality of complete families of Enriques surfaces.

V. Meeting. 19 Jan 2010 (Bonn/Mainz)

1. GW and GV invariants (Ch. Lehn 1h)

We need a little recollection of GW invariants with a special emphasize on K3 fibred Calabi–Yau threefolds. Recall the concept of a perfect obstruction theory and the virtual fundamental class. (The reduced version for K3 surfaces is needed in the next talk.) Define the Gopakumar–Vafa invariants via the formula on page 15 [1]. Let us know what will be known then about the integrality of the GV invariants $n^X_{g,\gamma}$?

2. GW theory for K3 surfaces (H. Hartmann 1h)

The standard GW invariants for symplectic manifolds are trivial. In order to remedy this one has to work with a reduced obstruction theory. This needs a little deformation theory. The reduced virtual class allows one to define GW integrals $R_{g,\beta}$ (by integrating the top Chern class of the Hodge bundle). The BPS counts $r_{g,\alpha}$ are related to the $R_{g,\beta}$ in the same way as the GP invariants $n^X_{g,\gamma}$ are related to the GW invariants. Follow Section 2.2 and 2.3 of [1], but leave Conjecture 1 and 2 for the next talk.

3. Higher genus Yau–Zaslow conjecture (Ziyu Zhang 1h)

Discuss Conjecture 1 and 2 in [1]. Explain that Conjecture 2 specialize to the Yau–Zaslow formula for genus zero curves on K3 surfaces. Incorporate Section 3.4. How realistic is it to actually prove (one version of) the Yau–Zaslow conjecture (Beauville, Bryan/Leung,Lee/Leung)? See also [3] and a later talk.

VI. Meeting. 2 Feb 2010 (Bonn/Mainz)

1. NL, GV, BPS (NN 1h30)

Prove Theorem 1 in [1], which expresses the GV invariants $n^X_{g,\gamma}$ for the total space of a one-dimensional family of quasi-polarized family of K3 surfaces in terms of the NL numbers of this family. This takes up Section 3.1 in [1]. In [7] this is Theorem 2, generalized to families with more than one polarization.

2. NL numbers for pencils of quartics (NN)

Prove Theorem 2 in [1], which computes explicitly the NL number for pencil of quartics. This is contained in Section 0.7, 5.1 and 5.2.

3. Yau–Zaslov conjecture (NN)

We will not have time to prove Theorem 1 in [3], but it would be good to get an idea of the strategy of that paper. Maybe we can invite the local expert for a talk.

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