Algebraic Geometry I, Fall 2018

<p align="center">Lecturer: Georg Oberdieck</p>

General information

References:
D. Mumford: The Red book of varieties and schemes. Springer LN 1358.
U. Goertz, T. Wedhorn: Algebraic Geometry I. Vieweg.
R. Hartshorne: Algebraic Geometry GTM 52. Springer.
D. Eisenbud and J. Harris, The Geometry of Schemes, GTM 197, Springer.
Vakil: Foundations of algebraic geometry. Online lectures.
Shafarevich, Basic Algebraic Geometry I + II.

Schedule[1]

Oct 11: Definition affine algebraic set, vanishising ideal, Hilbert Nullstellen satz (Strong and weak) with proof, Noether normalization theorem, correspondence between affine algebraic set and radical ideals, morphisms and coordinate rings of affine algebraic sets, equivalence between category of affine algebraic sets and category of reduced k-algebras.

Oct 15: projective spaces, correspondence between projective algebraic sets and homogeneous ideals, motivating example for morphism of projective varieties (Mumford, p.11, example D), definition presheaf, sheaf.

Oct 18: More on sheafs, in particular sheafification, stalks, pushforward sheaf. Structure sheaf of (not necessarily irreducible) affine algebraic set, Definition affine variety, pre-variety (as in Mumford and equivalent condition that X is Noetherian irreducible with affine cover). Morphism of pre-variety.

Oct 22: Pset 1 due. Proof that for an affine variety the global ring of functions is the coordinate ring. Definition structure sheaf for projective algebraic sets. Definition projective variety.

Oct 25: Criterion for map to be morphism (Mumford, §5 Prop.6). Example of morphism for projective var (Mumford, p.30-32). Categorical product, product of affine varieties and pre-varieties.

Oct 29: Pset 2 due. Sheaf with respect to basis of topology and induced sheaf, product of projective variety, Separatedness of pre-varieties, definition varieties and basic properties.

Nov 1: No class: Allerheiligen

Nov 5: Pset 3 due. Definition and basic properties of the function field, rational maps, birational maps, statement (without proof) that two varieties are birational if and only if their function fields are isomorphic. Dimension

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[1]As of February 2, 2019

(as the transcendence degree of the function field), $\dim Y < \dim X$ if $Y \subset X$ proper, Krull principal ideal theorem, $\dim X = \text{topdim} X$, finite morphism and basic properties.

Nov 8: Proof Krull principal ideal theorem. Every codim r closed subvariety in affine variety is component of vanishing locus of r regular functions. Any two subvarieties of $\mathbb{P}^n$ of the correct dimension intersect.

Nov 12: Midterm 1. Dimension of fibers of morphisms, Mumford §8, Theorem 3, upper-semicontinouity of dimension, Chevalley theorem.

Nov 15: Definition $\text{Spec}(R)$, Zariski topology, value of function $f \in R$ at a point, basic open subsets are topology, closed/generic point, examples.

Nov 19: Pset 4 due. Topological properties of $\text{Spec}(R)$, structure sheaf.

Nov 22: Definition scheme

Nov 26: Pset 5 due. Fully faithful functor from pre-varieties to schemes, S valued points, Yoneda lemma, Fiber products in category of sets and schemes, intersections of schemes, fibers of morphisms.

Nov 29: Schemes over schemes, category of schemes over S, Properties of schemes: irreducible, connected, quasi-compact, reduced, integral, Noetherian, locally Noetherian. A property (here locally Noetherian) holds for an open affine covering if and only if it holds for every open affine. Finite type and quasi-compact morphism.

Dec 3: Pset 6 due. Closed subschemes, closed immersions, closed subscheme of affine is affine, zero-dimensional subschemes of $\mathbb{A}^2_{\mathbb{C}}$.

Dec 6: Reduced induced subscheme structure, Dimension of schemes, separated morphisms, connection to the definition in Mumford, definition abstract variety, equivalence to category of (classically defined) varieties.

Dec 10: Pset 7 due. Valuation rings, valuative criterion of separatedness including proof, base properties of separated morphisms.

Dec 13: Proper morphisms, valuative criterion of properness, proper normal curves.

Dec 17: Midterm 2. Equivalence of categories between category of proper normal curves over field k with non-constant morphisms and category of function fields of dimension 1 over k.

Dec 20: Graded ring, Proj of graded ring as set, as topological space and as scheme. Projective space. Weighted projected space.

Jan 7: $\mathcal{O}_X$-modules, category of $\mathcal{O}_X$-modules is abelian, $\tilde{M}$ on $\text{Spec } A$ for A-module M, quasi-coherent sheaves, quasi-coherent sheaves form an abelian category, ideal sheaf of closed subscheme if quasi-coherent.

Jan 10: Constructions of $\mathcal{O}_X$-modules (tensor product, hom sheaf, dual), locally free sheaves, invertible sheaves, Picard group, coherent sheaves, $\tilde{M}$ on Proj A for graded A-module M, Serre twisting sheaves.

Jan 14: Pset 8 due. Under generated in degree 1 assumption, $\mathcal{O}_X(n)$ are invertible and $\mathcal{O}_X(m) \otimes \mathcal{O}_X(n) = \mathcal{O}_X(m + n)$, Γ_*-functor, $\sim \circ \Gamma_* = \text{id}$ on

QCoh(Proj A). Extension of sections of quasi-coherent sheaves. Closed sub-schemes of $\mathbb{P}^n$ defined by homogeneous ideal. Locally free sheaves and vector bundles defined by transition functions, transition functions of $\mathcal{O}_{\mathbb{P}^1}(n)$.

Jan 17: Relative spec, equivalence of categories between quasi-coherent $\mathcal{O}_X$-algebras and affine morphisms to X, vector bundle $\mathbb{C}(\mathcal{E})$ associated to locally free sheaf $\mathcal{E}$, example of vector bundle sequence associated to $\mathcal{O}_{\mathbb{P}^n}(-1) \to \mathcal{O}_{\mathbb{P}^n}^{n+1}$, description of $\mathbb{V}(\mathcal{O}_X(-1))$ as tautological line bundle.

Jan 21: Pset 9 due. Regular ring, regular/non-singular scheme, curve C is regular if and only if normal, Jacobi criterion, issues with regularity if base field not algebraically closed.

Jan 24: Kähler differentials, first/second fundamental sequence, sheaf of relative differentials $\Omega_{X/Y}$

Jan 28: Pset 10 due. Genus of proper normal curve over field k. Computation of genus of $y^2z = (x+z)x(x-z)$ by constructing explicit global differential. Definition 'smooth over field k', tangent sheaf, canonical line bundle, smooth equivalent to regular over k perfect.

Jan 31: Blow-up of $\mathbb{A}^n$ and closed subvarieties. Desingularization of some plane curves and hypersurfaces. Quotient of affine scheme by finite group. Du-Val singularities.