UNIVERSALITY OF DESCENDENT INTEGRALS OVER MODULI SPACES OF STABLE SHEAVES ON K3 SURFACES

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Abstract. We interpret results of Markman on monodromy operators as a universality statement for descendent integrals over moduli spaces of stable sheaves on K3 surfaces. This reduces arbitrary descendent integrals on moduli space of stable sheaves on a K3 surface to integrals over the punctual Hilbert scheme. As an application we establish the higher rank Segre-Verlinde correspondence for K3 surfaces conjectured by Götsche and Kool.

1. Introduction

1.1. Descendent integrals. Let $M(v)$ be a proper moduli space of Gieseker stable sheaves $F$ on a K3 surface $S$ with Mukai vector

$$v(F) := \text{ch}(F)\sqrt{\text{td}_S} = v \in H^*(S, \mathbb{Z}).$$

We assume that $v$ is primitive and that there exists a universal family $\mathcal{F}$ on $M(v) \times S$. The $k$-th descendent of a class $\gamma \in H^*(S, \mathbb{Q})$ on the moduli space is defined by

$$\tau_k(\gamma) = \pi_{M(v)}^*(\pi_S^*(\gamma)\text{ch}_k(\mathcal{F})) \in H^*(M(v))$$

where $\pi_{M(v)}, \pi_S$ are the projections of $M(v) \times S$ to the factors. Consider an arbitrary integral of descendents and Chern classes of the tangent bundle over the moduli space:

$$\int_{M(v)} \tau_{k_1}(\gamma_1) \cdots \tau_{k_n}(\gamma_n)P(c_k(T_{M(v)}))$$

for an arbitrary polynomial $P(c_1, c_2, c_3, \ldots)$. The goal of this note is to explain the following application of Markman’s work on monodromy operators (see Section 2 for the precise form the reconstruction takes):

Theorem 1 (Markman, [6]). Any integral of the form (2) can be effectively reconstructed from the set of all integrals (2) where $M(v)$ is replaced by the Hilbert scheme of $n$ points of a K3 surface, for $n = \dim(M(v))/2$.

See Section 2 for the case where only a quasi-universal family exists.
1.2. An application. In [2] Göttsche and Kool conjectured a Segre-Verlinde correspondence for newly defined Segre and Verlinde numbers of moduli spaces of higher rank sheaves. Theorem 1 immediately gives a proof of this correspondence for K3 surfaces. More precisely, we establish Conjecture 5.1 in [2] which relates integrals over moduli spaces of higher rank on K3 surfaces to integrals over the punctual Hilbert schemes $S^{[n]}$:

**Theorem 2.** Let $M(v)$ be a 2n-dimensional proper moduli space of stable sheaves with Mukai vector $v$ on a K3 surface $S$, such that $\text{rk}(v) > 0$. For any K-theory class $\alpha \in K(S)$, class $L \in H^2(S)$ and $u \in \mathbb{C}$ we have

$$
\int_{M(v)} c(\alpha_M) e^{\mu(L) + u \mu(p)} = \int_{S^{[n]}} c(\beta^{[n]}) e^{\mu(L) + u \text{rk}(v) \mu(p)}
$$

for any K-theory class $\beta \in K(S)$ such that

$$
\text{rk}(\beta) = \frac{\text{rk}(\alpha)}{\text{rk}(v)}
$$

$$
v(\alpha)^2 = v(\beta)^2
$$

$$
c_1(\alpha)^2 = c_1(\beta)^2
$$

$$
c_1(\alpha) \cdot L = c_1(\beta) \cdot L.
$$

The inner products here are taken with respect to the Mukai pairing (see Section 2.1). We also refer to Section 3.1 for the definition of the descendent classes $\alpha_M := GK(\alpha)$ and $\mu(\sigma)$.

1.3. Higher-rank Segre/Verlinde correspondence. Let $\rho = \text{rk}(v)$, $s = \text{rk}(\alpha)$ and $n = \dim(M(v))/2$. As explained in [2, Cor.5.2] Theorem 2 implies the following closed evaluation of the higher Segree numbers of $M(v)$:

$$
\int_{M(v)} c(\alpha_M) = \rho^2 - \chi(\mathcal{O}_S) \text{Coeff}_{z^n} \left( V_s^c(\alpha) W_s^{c_1(\alpha)^2} X_s^{\chi(\mathcal{O}_S)} \right)
$$

where the functions $V_s, W_s, X_s$ were determined in [4] to be:

$$
V_s(z) = (1 + (1 - \frac{s}{\rho}) t)^{1-s} (1 + (2 - \frac{s}{\rho}) t)^{s} (1 + (1 - \frac{s}{\rho}) t)^{-\frac{1}{2}},
$$

$$
W_s(z) = (1 + (1 - \frac{s}{\rho}) t)^{\frac{s}{2}-1} (1 + (2 - \frac{s}{\rho}) t)^{\frac{1}{2}}(1 - s) (1 + (1 - \frac{s}{\rho}) t)^{\frac{1}{2}-\frac{1}{2}},
$$

$$
X_s(z) = (1 + (1 - \frac{s}{\rho}) t)^{\frac{s}{2}s-\frac{1}{2}} (1 + (2 - \frac{s}{\rho}) t)^{-\frac{1}{2}} (1 - \rho) t^{-\frac{(\rho-1)^2}{2\rho}}
$$

under the variable change $z = t(1 + (1 - \frac{s}{\rho}) t)^{1-\frac{s}{\rho}}$.

On the Verlinde side, the work [3] reduced the Verlinde numbers of $M(v)$ to those of $S^{[n]}$ using a result of Fujiki. One obtains (see [2] for notation and assumptions such as $\rho|\tau$) the following evaluation of Verlinde numbers:

$$
\chi(M(v), \mu(L) \otimes E^{\otimes r}) = \text{Coeff}_{w^n} \left( \rho^2 - \chi(\mathcal{O}_S) G_r^{\chi(L)} F_r^{\frac{1}{2}} \chi(\mathcal{O}_S) \right)
$$
where the universal functions $G_r, F_r$ were determined by [1] to be:

\[ G_r(w) = 1 + v, \]
\[ F_r(w) = (1 + v)^{\frac{\rho^2}{\rho^2 + r^2}} (1 + \frac{r^2}{\rho^2} v)^{-1}, \]

under the variable change $w = v(1 + v)^r/\rho^2 - 1$.

The universal functions above satisfy

\[ F_r(w) = V_s(z)^{\frac{r}{2}(\rho^2 + r^2)} W_s(z)^{-\frac{4t}{\rho}} X_s(z)^2, \]
\[ G_r(w) = V_s(z) W_s(z)^2, \]

where $s = \rho + r$ and $v = t(1 - \frac{\rho}{\rho^2})^{-1}$. This equality is called the higher-rank Segre-Verlinde correspondence for K3 surfaces.

**Corollary 1.** The higher rank Segre-Verlinde correspondence of [2] holds for K3 surfaces (and hence all K-trivial surfaces).

1.4. **Plan.** Section 2 can be viewed as an introduction to some of the ideas of Markman’s beautiful (but also intricate) article [6]. This leads to Theorem 1. For the application to Theorem 2 we reinterpret Markman’s result as a universality result for the descendent integrals in Theorem 4. In Section 3 we discuss the setting and proof of Theorem 2.

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2. **Markman’s Universality**

2.1. **Basic definitions.** Let $S$ be a K3 surface and consider the lattice $\Lambda = H^*(S, \mathbb{Z})$ endowed with the Mukai pairing

\[ (x \cdot y) := - \int_S x^\vee y, \]

where, if we decompose an element $x \in \Lambda$ according to degree as $(r, D, n)$, we have written $x^\vee = (r, -D, n)$. We will also write

\[ \text{rk}(x) = r, \quad c_1(x) = D, \quad v_2(x) = n. \]

Given a sheaf or complex $E$ on $S$ the Mukai vector of $E$ is defined by

\[ v(E) = \sqrt{\text{td}_S} \cdot \text{ch}(E) \in \Lambda. \]

Let $v \in \Lambda$ be an effective vector, $H$ be an ample divisor on $S$ and let $M_H(v)$ be a proper smooth moduli space of $H$-stable sheaves with Mukai
For simplicity we assume that there exists an universal sheaf $F$ on $M_H(v) \times S$, unique up to tensoring of a line bundle from the base.

The results we state below also hold in the general case where there exists only a universal twisted sheaf. By this we mean that all statements below can be formulated in terms of the Chern character $\text{ch}(F)$ alone and this class can be defined in the twisted case as well, see [5, Sec.3]. The proofs carry over likewise using that the ingredients hold in the twisted case as well.

Consider the morphism $\theta_F : \Lambda \to H^2(M_H(v), \mathbb{Z})$ defined by

$$\theta_F(x) = \left[ \pi_* \left( \text{ch}(F) \sqrt{\text{td}}_S \cdot x^\vee \right) \right]_{\text{deg}=1}.$$  

Then $\theta_F$ restricts to an isomorphism

$$\theta = \theta_F|_{v^\perp} : v^\perp \xrightarrow{\simeq} H^2(M_H(v), \mathbb{Z})$$

which does not depend on the choice of universal family (use that the degree 0 component of the pushforward [4] vanishes) and for which we hence have dropped the subscript $F$. The isomorphism $\theta$ is orthogonal with respect to the Mukai pairing on the left, and the pairing given by the Beauville-Bogomolov-Fujiki form on the right. We will identify $v^\perp \subset \Lambda$ with $H^2(M_H(v), \mathbb{Z})$ under this isomorphism.

The universal sheaf $F$ and hence its Chern character $\text{ch}(F)$ is unique only up to pullback of a line bundle from the base. We can pick a canonical normalization as follows:

$$u_v := \exp \left( \frac{\theta_F(v)}{(v, v)} \right) \cdot \text{ch}(F) \cdot \sqrt{\text{td}}_S$$

where we have suppressed the pullback morphisms from $M_H(v)$ and $S$ in the first and last term on the right. The invariance is a short check (replace $F$ by $F \otimes \mathcal{L}$ and calculate). The class $u_v$ is characterized among the classes $\text{ch}(F) \cdot \sqrt{\text{td}}_S$ by the property that $\theta_{u_v}(v) = 0$ (Use that $\pi_*(\text{ch}(F)\sqrt{\text{td}}_S \cdot v^\vee) = -(v \cdot v) + \theta_F(v) + \ldots$ for a universal family $F$).

**Example 1.** Let $M = \text{Hilb}_n(S)$ be the Hilbert scheme of $n$ points on $S$. We have $v = 1 - (n-1)p$, and we take $\mathcal{F} = I_Z$ the ideal sheaf of the universal subscheme. For $a \in H^2(S)$ we have

$$\theta(a) = \pi_*(\text{ch}_2(O_Z)\pi_Z^*(a)).$$

If $a$ is the class of a divisor $A$, then this is the class of subschemes incident to $A$. Similarly, define

$$\delta = -\frac{1}{2} \Delta_{\text{Hilb}_n(S)} = c_1(\pi_*O_Z) = \pi_*\text{ch}_3(O_Z).$$

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2 More generally, one can also work with $\sigma$-stable objects for a Bridgeland stability condition in the distinguished component.
Then under the identification (5) we have
\[ \delta = -(1 + (n - 1)c). \]
Since \( \theta_F(v) = -\delta \) the canonical normalization of \( \text{ch}(F) \) takes the form
\[ u_v = \exp \left( -\frac{\delta}{2n - 2} \right) \text{ch}(I_Z) \sqrt{\text{td}_S}. \]

2.2. **Markman’s operator.** For \( i = 1, 2 \) let \((S_i, H_i, v_i)\) be the data defining proper moduli space of stable sheaves \( M_i = M_{H_i}(S_i, v_i) \). Let \( F_i \) be the universal family on \( M_i \times S \).

Consider an isomorphism of Mukai lattices
\[ g : H^*(S_1, \mathbb{Z}) \rightarrow H^*(S_2, \mathbb{Z}) \]
such that \( g(v_1) = v_2 \). We will identify \( g \) also with an isomorphism of topological \( K \)-groups
\[ g : K_{\text{top}}(S_1) \rightarrow K_{\text{top}}(S_2) \]
using the lattice isomorphism \( K_{\text{top}}(S) \cong H^*(S, \mathbb{Z}) \) given by \( E \mapsto \nu(E) \). Hence the following diagram commutes
\[
\begin{array}{ccc}
K_{\text{top}}(S_1) & \xrightarrow{g} & K_{\text{top}}(S_2) \\
\downarrow v & & \downarrow v \\
H^*(S_1, \mathbb{Z}) & \xrightarrow{g} & H^*(S_2, \mathbb{Z}).
\end{array}
\]

Similar identification will apply to morphisms \( g \) defined over \( \mathbb{C} \). The Markman operator associated to \( g \) is given by the following result:

**Theorem 3. (Markman, [6])** For any isometry \( g : H^*(S_1, \mathbb{C}) \rightarrow H^*(S_2, \mathbb{C}) \) such that \( g(v_1) = v_2 \) there exists a unique operator
\[ \gamma(g) : H^*(M_1, \mathbb{C}) \rightarrow H^*(M_2, \mathbb{C}) \]
such that
\[(a) \quad \gamma(g) \text{ is degree-preserving orthogonal ring-isomorphism}
(b) \quad \gamma(g) \otimes g(u_{v_1}) = u_{v_2}.\]

The operator is called the Markman operator and given by
\[
\gamma(g) = c_m \left[ -\pi_\ast \left( (1 \otimes g)u_{v_1} \right)^\vee \cdot u_{v_2} \right].
\]
Moreover, we have
\[(c) \quad \gamma(g_1) \circ \gamma(g_2) = \gamma(g_1g_2) \text{ and } \gamma(g)^{-1} = \gamma(g^{-1}) \text{ if it makes sense.}
(d) \quad \gamma(g)c_k(T_{M_1}) = c_k(T_{M_2}).\]
Here the Chern class $c_m$ in (6) has the following definition: Let $\ell : \oplus_i H^{2i}(M, \mathbb{Q}) \rightarrow \oplus_i H^{2i}(M, \mathbb{Q})$ be the universal map that takes the exponential Chern character to Chern classes, so in particular $c(E) = \ell(ch(E))$ for any vector bundle. Then given $\alpha \in H^*(M)$ we write $c_m(\alpha)$ for $[\ell(\alpha)]_{\deg=m}$.

We can reinterpret the condition $(f \otimes g)(u_{v_1}) = u_{v_2}$ in terms of generators of the cohomology ring. Consider the canonical morphism

$$B : H^*(S, \mathbb{Q}) \rightarrow H^*(M, \mathbb{Q})$$

defined by

$$B(x) = \pi_*(u_v \cdot x^\vee).$$

We write $B_k(x)$ for its component in degree $2k$. In particular, $B_1(x) = \theta_F(x)$ for all $x \in v^\perp$.

**Lemma 1.** Let $f : H^*(M_1, \mathbb{Q}) \rightarrow H^*(M_2, \mathbb{Q})$ be a degree-preserving orthogonal ring isomorphism. Then the following are equivalent:

(a) $(f \otimes g)(u_{v_1}) = u_{v_2}$

(b) $f(B(x)) = B(gx)$ for all $x \in H^*(S_1, \mathbb{Q})$.

**Proof.** Since $g$ is an isometry of the Mukai lattice we have

$$\pi_*(u_{v_2} \cdot (gx)^\vee) = \pi_*((1 \otimes g^{-1})u_{v_2} \cdot x^\vee).$$

Indeed, if we write $ch(F) \sqrt{td_S} = \sum_i a_i \otimes b_i$ under the K"unneth decomposition, then

$$\pi_*((1 \otimes g^{-1})(ch(F)\sqrt{td_S}) \cdot x^\vee) = \sum_i a_i \int_S g^{-1}(b_i)x^\vee$$

$$= \sum_i -a_i \cdot (g^{-1}(b_i) \cdot x)$$

$$= \sum_i -a_i \cdot (b_i \cdot g(x))$$

$$= \sum_i a_i \int_S b_i g(x)^\vee$$

$$= \pi_*(ch(F')\sqrt{td_S} \cdot g(x)^\vee).$$

Hence we see that:

(b) $\iff \forall x \in H^*(S_1, \mathbb{Z}) : f \pi_*(u_{v_1} \cdot x^\vee) = \pi_*(u_{v_2} \cdot (gx)^\vee)$

$\iff \forall x \in H^*(S_1, \mathbb{Z}) : \pi_*((f \otimes 1)u_{v_1} \cdot x^\vee) = \pi_*((1 \otimes g^{-1})u_{v_2} \cdot x^\vee)$

$\iff (f \otimes 1)(u_{v_1}) = (1 \otimes g^{-1})(u_{v_2})$

$\iff (a).$

□

**Corollary 2.** In the setting of Theorem 3, $\gamma(g)B(x) = B(gx)$. 
2.3. **Universality.** We formulate what the above means for tautological integrals over the moduli spaces of stable sheaves $M(v)$ on a K3 surface $S$.

Let $P$ be a polynomial depending on the variables

$$t_{j,i}, \quad j = 1, \ldots, k, \quad i \geq 0, \quad u_j \geq 1.$$

Let also $A = (a_{ij})_{i,j=0}^k$ be a $(k+1) \times (k+1)$-matrix.

**Theorem 4.** *(Universality)* There exists $I(P, A) \in \mathbb{Q}$ (depending only on $P$ and $A$) such that for any proper moduli space of stable sheaves $M(v)$ on a K3 surface $S$ and for any $x_1, \ldots, x_k \in \Lambda$ with

$$\begin{pmatrix} v \cdot v \\ (x_i \cdot v)^k_{i=1} \end{pmatrix} = A$$

we have

$$\int_{M(v)} P(B_i(x_j), c_j(Tan)) = I(P, A).$$

In other words, the integral

$$\int_{M(v)} P(B_i(x_j), c_j(Tan))$$

depends upon the above data only through $P$, the dimension $\dim M(v) = 2n$, and the pairings $v \cdot x_i$ and $x_i \cdot x_j$ for all $i, j$.

**Proof.** Let $(M(v), x_i)$ and $(M(v'), x'_i)$ be two sets of classes with the same intersection matrix $A$. Then there exists an orthogonal matrix

$$g : \Lambda \rightarrow \Lambda$$

taking $(v, x_1, \ldots, x_k)$ to $(v', x'_1, \ldots, x'_k)$. Hence by Theorem 3(a) and Corollary 2,

$$\int_{M(v)} P(B_i(x_j), c_j(Tan)) = \int_{M(v')} P(B_i(x'_j), c_j(Tan)).$$

Theorem 4 clearly implies Theorem 1 since (a) any descendent $\tau_k(\gamma)$ defined as in [1] can been written in terms of the $B_k(x)$ ad Chern classes of the tangent bundle, and (b) after an orthogonal transformation of $\Lambda$ for any list of vectors $v, x_1, \ldots, x_k \in \Lambda$ we may assume that $v$ is the Mukai vector which defines the Hilbert scheme of $n$ points on a K3 surface.
2.4. A few words on the proof of Theorem 3. We briefly discuss what goes into the proof of Theorem 3 following [6]. This section is not relevant for the applications and can be skipped.

The main ingredient is the following uniqueness statement:

**Lemma 2.** Let \( f : H^*(M_1, \mathbb{Q}) \to H^*(M_2, \mathbb{Q}) \) be a morphism such that:

(i) \( f \) is a degree-preserving orthogonal ring isomorphism.

(ii) There exists universal families \( F \) on \( M_1 \times S_1 \) and \( F' \) on \( M_2 \times S_2 \) such that

\[
(f \otimes g) \left( \text{ch}(F) \sqrt{\text{td}S} \right) = \text{ch}(F') \sqrt{\text{td}S} \cdot \exp(\ell)
\]

for some \( \ell \in H^2(M_2, \mathbb{Q}) \).

Then we have

\[
f = c_m \left( -\text{Ext}_\pi((1 \otimes g)F, F') \right).
\]

Moreover, in (ii) it is enough to assume that \( F, F' \) are elements in \( K_{\text{top}}(M_i \times S_i)_\mathbb{Q} \), i.e. differ from a universal family by tensor product by a fractional line bundle from the base (see the proof). In particular, we have

\[
f = c_m \left[ -\pi_* \left( ((1 \otimes g)u_v) \cdot u_v \right) \right].
\]

The main input for the proof of the Lemma is the following theorem which we state for an arbitrary moduli space of stable sheaves \( M \) on a K3 surface:

**Theorem 5** (Markman [5]). For any universal families \( F, F' \) on \( M \times S \),

\[
\Delta_M = c_m \left( -\text{Ext}_\pi(F, F') \right).
\]

More generally, for any \( \gamma, \gamma' \in H^2(M, \mathbb{Q}) \) we have

\[
\Delta_M = c_m \left[ -\pi_* \left( (\exp(\gamma)\text{ch}(F) \sqrt{\text{td}S}) \cdot \exp(\gamma')\text{ch}(F) \sqrt{\text{td}S} \right) \right]
\]

**Proof of Lemma 2.** Assume that \( f \) satisfies (i) and (ii). Note that, since \( f \) is a ring isomorphism, the equality in (ii) is equivalent to the parallel equality where we replace \( \text{ch}(F) \) by \( \text{ch}(F) \exp(\mu) \) for any \( \mu \in H^2(M_1, \mathbb{Q}) \), and similarly for \( \text{ch}(F') \). Hence we may have also assumed (ii) with \( \text{ch}(F) \) replaced by \( \text{ch}(F) \exp(\mu) \) instead.

We will prove that for any \( \ell_i \in H^2(M_1, \mathbb{Q}) \) we have:

\[
f = c_m \left[ -\pi_* \left( ((1 \otimes g)(\exp(\ell_1))\text{ch}(F) \sqrt{\text{td}S}) \cdot (\exp(\ell_2)\text{ch}(F') \sqrt{\text{td}S}) \right) \right].
\]

Taking \( \ell_i \) both to be trivial then gives (7), and taking \( \ell_i \) to be as in the definition of \( u_v \) gives (8).

By Theorem 5, for any \( \gamma \in H^2(M_2, \mathbb{Q}) \) we have:

\[
\Delta_{M_2} = c_m \left[ -\pi_* \left( \left( \text{ch}(F') \sqrt{\text{td}S} \exp(\gamma) \right) \cdot \text{ch}(F') \sqrt{\text{td}S} \exp(\ell_2) \right) \right]
\]
Inserting
\[ \text{ch}(\mathcal{F}') \sqrt{\text{td} S} = (f \otimes g) \left( \exp(f^{-1}(\ell)) \text{ch}(\mathcal{F}) \sqrt{\text{td} S} \right) \]
in the first term, and then using that \( f \) is degree-preserving (so commutes with dualizing), and a ring isomorphism (so commutes with taking \( c_m \)), we get that \( \Delta_{M_2} \) is equal to
\[(f \otimes 1)c_m \left[ -\pi^* \left( ((1 \otimes g)(\text{ch}(\mathcal{F}) \sqrt{\text{td} S} \exp(\gamma + f^{-1}(\ell))))^\vee \text{ch}(\mathcal{F}') \sqrt{\text{td} \exp(\ell_2)} \right) \right] \]
Setting \( \gamma = -f^{-1}(\ell) + \ell_1 \), and taking \( Q \) to be the right hand side of (9) we find
\[ \text{id}_{H^*(M_2)} = \Delta_{M_2} = (f \otimes 1)(Q) = Q \circ f^t \]
where \( f \) is the transpose with respect to the standard cup product (or as a correspondence, identical with \( f \) up to swapping the factors). Since \( f \) is orthogonal, we conclude \( \text{id}_{H^*(M_2)} = Q \circ f^{-1} \), so \( f = Q \). \( \square \)

After the uniqueness, we prove a basic result on the operator satisfying the condition of the previous lemma.

**Lemma 3.** Assume \( f \) satisfies (i) and (ii) of Lemma 2. Then
\[(f \otimes g)(u_{v_1}) = u_{v_2}.\]

**Proof.** Assuming (i) and (ii) we have
\[(f \otimes g)(u_{v_1}) = \exp \left( f(\theta_{\mathcal{F}}(v_1)) \right) \text{ch}(\mathcal{F}') \sqrt{\text{td} S} \exp(\ell).\]

Hence the claim follows from the following calculation:
\[ f(\theta_{\mathcal{F}}(v_1)) = [\text{deg } 1] \pi_* \left( (1 \otimes g^{-1})(f \otimes g)(\text{ch}(\mathcal{F}) \sqrt{\text{td} S}) \cdot v_1^\vee \right) \]
\[ = [\text{deg } 1] \pi_* (1 \otimes g^{-1}) \left( \text{ch}(\mathcal{F}') \sqrt{\text{td} S} \cdot v_1^\vee \right) \exp(\ell) \]
\[ \overset{(*)}{=} [\text{deg } 1] \pi_* (\text{ch}(\mathcal{F}') \sqrt{\text{td} S} \cdot g(v_1)^\vee) \exp(\ell) \]
\[ = [\text{deg } 1] \left( -(v_2, v_2) + \theta_{\mathcal{F}'}(v_2) \right) \exp(\ell) \]
\[ = -(v_2, v_2) \ell + \theta_{\mathcal{F}'}(v_2), \]

where \((*)\) follows since \( g \) is an isometry of Mukai lattices, and \([\text{deg } k]\) stands for taking the (complex) degree \( k \) part. \( \square \)

**Sketch of Proof of Theorem 3.** The uniqueness part of the theorem is addressed by the two lemmas above. Hence we only need to show the existence of the operator in Theorem 3. One may be tempted to define the operators \( \gamma(g) \) directly using the closed formula (6) and then derive their properties from it. However, (6) is unfortunately very hard to work with in practice. It is even not clear how to use it to prove \( \gamma(g_1) \circ \gamma(g_2) = \gamma(g_1g_2) \). Nevertheless, it can be used for the following: if we know the statements of the theorem
for a Zariski dense subset of all operators $g$ (e.g. the integral isometries),
then we can define $\gamma(g)$ by (6) for arbitrary isometries and then conclude
Theorem 3 in general using Zariski density.
Hence it remains to consider the case of integral isometries. For this one
considers the set $S$ of triples
\[(S_1, H_1, v_1), (S_2, H_2, v_2), g : H^*(S_1, \mathbb{Z}) \to H^*(S_2, \mathbb{Z}),\]
where $g$ is a isometry such that $g(v_1) = v_2$, for which the result of the
theorem holds. Since the elements for which the statements of the theorem
hold are closed under composition and inverse, we can think about $S$ as
the set of arrows in a groupoid. Then elements of the groupoid can be
constructed in three different ways:

- For any deformation $(S_1, v_1, H_1) \rightsquigarrow (S_2, v_2, H_2)$ which keeps $v_1$ and $H_1$ of
  Hodge type, we have an associated deformation of moduli spaces
  \[M_{H_1}(S_1, v_1) \rightsquigarrow M_{H_2}(S_2, v_2).\]
The associated parallel transport operator $P$ is a degree-preserving
orthogonal ring isomorphism. Moreover, since $u_v$ is defined in terms of the
universal family which deforms along the family, the classes $u_{v_2}$ of the
individual fibers are the parallel transports of $u_{v_1}$. Hence if $g$ is the parallel
transport operator associated to $S_1 \rightsquigarrow S_2$, then $(P \otimes g)(u_{v_1}) = u_{v_2}$. We
conclude that $P$ satisfies the theorem.

- Assume that $\Phi : D^b(S_1) \to D^b(S_2)$ is a derived equivalence that takes $H_1$-
stable sheaves of Mukai vector $v_1$ to $H_2$-stable sheaves of Mukai vector
  $v_2$. Then $\Phi$ induces an isomorphism of moduli spaces
  \[\varphi : M_{H_1}(S_1, v_1) \to M_{H_2}(S_2, v_2)\]
such that by its construction we have
  \[(\text{Id} \boxtimes \Phi)(\mathcal{F}) = (\varphi \times \text{id})^*(\mathcal{F}').\]
This yields
  \[(\varphi_\ast \boxtimes \Phi_\ast)(\text{ch}(\mathcal{F})\sqrt{\text{td}_S}) = \text{ch}(\mathcal{F}')\sqrt{\text{td}_S}.\]
Hence with $g = \Phi_\ast$ we have $\gamma(g) = \varphi_\ast$ by the main lemma.

- Assume in a slight modification, that $(\Phi E)^\vee$ is $H_2$-stable of Mukai vector
  $v_2$ for any $H_1$-stable sheaf $E \in M_1$. Then we have an induced morphism
  \[\varphi : M_{H_1}(S_1, v_1) \to M_{H_2}(S_2, v_2)\]
such that by its definition we have
  \[(\text{Id} \boxtimes \Phi)(\mathcal{F})^\vee = (\varphi \times \text{id})^*(\mathcal{F}').\]
This yields
  \[(\varphi \times \text{id})_\ast(\text{Id} \boxtimes \Phi)(\mathcal{F})^\vee = \mathcal{F}'.\]
Since $\varphi$ commutes with dualizing, we get

$$(\varphi_* \otimes \Phi)(\mathcal{F})^\vee = \mathcal{F}'$$

and hence

$$\left( (\varphi_* \otimes \Phi_*)(\operatorname{ch}(\mathcal{F}) \sqrt{\operatorname{td}(S)}) \right)^\vee = \operatorname{ch}(\mathcal{F}') \sqrt{\operatorname{td}(S)}.$$

Going through the argument of the proof of Lemma 2 then shows

$$\varphi_* = D \circ \gamma(D\Phi_*)$$

where $D$ is the operator that acts on $H^{2i}$ by $(-1)^i$. We note that $\Phi_* v_1 = v_2$, so $g = D\Phi_*$ sends $v_1$ to $v_2$ as required. Since $D$ is a degree-preserving orthogonal ring isomorphism, we see that $\gamma(g)$ satisfies the statements of the Theorem.

This shows the existence of Markman operators for $g$ of these form. Markman then (roughly) shows that any integral $g$ can be written as a composition of these three operations. This concludes the proof.

The proof above also ties the operators $\gamma(g)$ directly to the monodromy action, the main application in [6]. □

3. The Göttsche-Kool conjecture

We specialize now to the setting of [2]. As before, we let $S$ be a K3 surface and $M(v)$ a moduli space of stable sheaves on $S$ of Mukai vector $v$. We set $2n := \dim M(v) = v \cdot v + 2$ and assume that $\operatorname{rk}(v) > 0$.

3.1. Normalization. Let $\alpha \in K(S)$. Following Göttsche and Kool [2] we define descendent classes on $M(v)$. If there exists a universal family $\mathcal{G}$ and a $\operatorname{rk}(v)$-th root of $\det(\mathcal{G})$, then we set

$$\operatorname{GK}(\alpha) := \operatorname{ch}(-\pi_*(\pi_S^*(\alpha) \otimes \mathcal{G} \otimes \det(\mathcal{G})^{-1/\operatorname{rk}(v)})).$$

where $\pi, \pi_S$ are the projections of $M(v) \times S$ to the factors. In the general case we use the Grothendieck-Riemann-Roch expression:

$$\operatorname{GK}(\alpha) := -\pi_* \left( v(\alpha) \operatorname{ch}(\mathcal{G}) \sqrt{\operatorname{td}(S)} \exp \left( -\frac{1}{\operatorname{rk}(v)} c_1(\mathcal{G}_m) \right) \right).$$

We let $\operatorname{GK}_k(\alpha)$ be the degree $2k$ component of $\operatorname{GK}(\alpha)$.

The $\operatorname{GK}(\alpha)$ are easily expressed in Markman’s normalization:

**Lemma 4.** We have

$$\operatorname{GK}(\alpha) = -B \left( v(\alpha^\vee) \exp \left( \frac{c_1(v)}{\operatorname{rk}(v)} \right) \right) \exp \left( B_1 \left( \frac{-p}{\operatorname{rk}(v)} - \frac{v}{\operatorname{rk}(v) \cdot v} \right) \right).$$

**Proof.** Using that $\operatorname{Pic}(M \times S) = \operatorname{Pic}(M) \otimes \operatorname{Pic}(S)$ we can write

$$c_1(\mathcal{G}) = \pi^*(\ell) + \pi_S^*(c_1(v))$$

where $\ell$ is the relative cotangent bundle.
for some $\ell \in H^2(M)$. By calculating $\theta_G(p)$ one finds $\ell = \theta_G(p)$. Hence

$$
\text{GK}(\alpha) = -\pi_* \left( v(\alpha) \text{ch}(G) \sqrt{\text{td}_S} \exp \left( -\frac{c_1(v)}{\text{rk}(v)} \right) \right) \exp \left( \theta_G(p)/(p \cdot v) \right)
$$

$$
= -B \left( v(\alpha) \exp \left( \frac{c_1(v)}{\text{rk}(v)} \right) \right) \exp \left( B_1 \left( \frac{-p}{\text{rk}(v)} - \frac{v}{v \cdot v} \right) \right)
\square
$$

For $\sigma \in H^*(S)$ Göttsche and Kool consider also the classes

$$
\mu(\sigma) = -\pi_* \left( \text{ch}_2(G \otimes \text{det}(G))^{-1/\text{rk}(v)} \pi_S^*(\sigma) \right).
$$

(defined by the GRR expression if only a semi-universal family exists).

**Lemma 5.** The class $\mu(\sigma)$ is the component of degree $\deg(\sigma)$ of

$$
- \exp \left( B_1 \left( \frac{p}{p \cdot v} - \frac{v}{v \cdot v} \right) \right) B \left( \sigma^\vee \exp \left( \frac{c_1(v)}{\text{rk}(v)} \right) \sqrt{\text{td}_S^{-1}} \right).
$$

**Proof.** We have that $\mu(\sigma)$ is the degree $\deg(\sigma)$ component of

$$
- \pi_* \left( \text{ch}(G) \otimes \text{det}(G)^{-1/\text{rk}(v)} \pi_S^*(\sigma) \right)
$$

$$
= -\pi_* \left( \text{ch}(G) \exp(-c_1(G)/\text{rk}(v)) \pi_S^*(\sigma) \right)
$$

$$
= -\exp \left( \frac{\theta_G(p)}{p} - \frac{\theta_G(v)}{v \cdot v} \right) \exp \left( \frac{\theta_G(v)}{v \cdot v} \right)
$$

$$
\cdot \pi_* \left( \text{ch}(G) \pi_S^* \left( \sigma^\vee e^{c_1(v)/\text{rk}(v)} \sqrt{\text{td}_S^{-1}} \right)^\vee \sqrt{\text{td}_S} \right)
$$

$$
= -\exp \left( B_1 \left( \frac{p}{p \cdot v} - \frac{v}{v \cdot v} \right) \right) B \left( \sigma^\vee \exp \left( \frac{c_1(v)}{\text{rk}(v)} \right) \sqrt{\text{td}_S^{-1}} \right),
$$

where we used again $c_1(G) = \pi^* \theta_G(p) + \pi_S^* c_1(V)$. \square

In particular, for $L \in H^2(S)$ we have that

$$
\mu(L) = B_1 \left( L \exp \left( \frac{c_1(v)}{\text{rk}(v)} \right) \right) - B_1 \left( \frac{p}{p \cdot v} - \frac{v}{v \cdot v} \right)
$$

and that $\mu(p)$ is a polynomial in $B_1 \left( \frac{p}{p \cdot v} - \frac{v}{v \cdot v} \right)$ and $B_i(p)$.

### 3.2. Dependence.

We conclude that any integral

$$
\int_{M(v)} P(\text{GK}(\alpha), \mu(L), \mu(u^p))
$$

(such as the Segre number) only depends upon $P$ and the intersection pairings in the Mukai lattice of the classes

$$
v, \ p/\text{rk}(v), \ v(\alpha)^\vee \exp \left( \frac{c_1(v)}{\text{rk}(v)} \right), \ L \exp \left( \frac{c_1(v)}{\text{rk}(v)} \right), \ u^p.
$$
Equating (i) - (iv) for $M$

The interesting intersections involving $L$

Moving to the Hilbert scheme.

3.3. Moving to the Hilbert scheme. Since \([10]\) only depends on the intersection pairings of (11) we have that

The pairings with $u$ from (11) by specializing to $v$ have the same intersection numbers as the list (11). (The list (12) is obtained from (11) by specializing to $v = 1 - (n-1)p$, the Mukai vector of $S^{[n]}$.)

The interesting parts of the intersections of (12) are

Equating (i) - (iv) for $M(v)$ and $S^{[n]}$ we hence get the system:

\[
-v_2(\alpha) \cdot \text{rk}(v) + \frac{1}{2} \text{rk}(v)(v \cdot v) = -v_2(\beta) + \frac{1}{2} \text{rk}(\beta)(2n - 2) \\
\frac{\text{rk}(\alpha)}{\text{rk}(v)} = -\text{rk}(\beta) \\
v(\alpha) \cdot v(\alpha) = v(\beta) \cdot v(\beta) \\
-c_1(\alpha) \cdot L = -c_1(\beta) \cdot L.
\]
Since \( v(\alpha)^2 = c_1(\alpha)^2 - 2 \text{rk}(\alpha)v_2(\alpha) \), this is equivalent to the system:

\[
\begin{align*}
\text{rk}(\beta) &= \frac{\text{rk}(\alpha)}{\text{rk}(v)} \\
v(\alpha)^2 &= v(\beta)^2 \\
c_1(\alpha)^2 &= c_1(\beta)^2 \\
c_1(\alpha) \cdot L &= c_1(\beta) \cdot L.
\end{align*}
\]

(13)

Moreover, we must have

\[-u' = u' p \cdot (1 - (n - 1)p) = up \cdot v = -\text{rk}(v)u.
\]

We have proven the following (which immediately implies Theorem 2):

**Theorem 6.** For any polynomial \( P \), we have

\[
\int_{M(v)} P(\text{GK}_k(\alpha), \mu(L), \mu(up)) = \int_{S[n]} P(\text{GK}_k(\beta), \mu(L), \mu(u \text{rk}(v)p))
\]

for any \( K \)-theory class \( \beta \in K(S) \) such that (13) is satisfied.

**References**


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