ON DESCENDENT INTEGRALS OVER MODULI SPACES OF STABLE SHEAVES ON K3 SURFACES

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Dedicated to Lothar Göttsche, on the occasion of his 60th birthday.

Abstract. We establish the higher rank Segre-Verlinde correspondence for K3 surfaces conjectured by Göttsche and Kool. For the proof we interprete results of Markman as a universality statement for integrals of descendents and Chern classes over moduli spaces of stable sheaves on K3 surfaces. This reduces arbitrary descendent integrals on moduli space of stable sheaves on a K3 surface to integrals over the punctual Hilbert scheme.

1. Introduction

1.1. Segre. Let \( S \) be a smooth projective surface and \( \alpha \in K_{\text{alg}}(S) \) an element in the Grothendieck group of coherent sheaves on \( S \). On the Hilbert scheme \( S^{[n]} \) of \( n \) points on \( S \) we have the tautological class

\[
\alpha^{[n]} = \pi^* (\pi^*_S(\alpha) \otimes \mathcal{O}_Z)
\]

where \( Z \subset S^{[n]} \times S \) is the universal subscheme, and \( \pi, \pi_S \) are the projections of \( S^{[n]} \times S \) to the factors. The generating series of Segre numbers is

\[
\sum_{n \geq 0} z^n \int_{S^{[n]}} c(\alpha^{[n]}).
\]

1.2. Verlinde. The Verlinde numbers of \( S \) are defined by holomorphic Euler characteristics of line bundles on the Hilbert scheme. Given a line bundle \( L \in \text{Pic}(S) \) define the line bundles on \( S^{[n]} \):

\[
\mu(L) = \det(L^{[n]}) \otimes \det(\mathcal{O}_S^{[n]})^{-1}
\]

\[
E := \det(\mathcal{O}_S^{[n]}).
\]

The generating series of Verlinde numbers is

\[
\sum_{n \geq 0} w^n \chi(S^{[n]}, \mu(L) \otimes E^\otimes r)
\]
1.3. **Segre/Verlinde.** The conjectural Segre-Verlinde correspondence of \[5\] relates the generating series of Segre numbers \[1\] to the one of Verlinde numbers \[2\] by an explicit variable change. The statement is proven for K3 surfaces by explicit computation, but the general case is open. A clear geometric link is missing. The correspondence is the surface analogue of strange duality for curves \[3\].

1.4. **Main result.** In \[2\] Göttsche and Kool conjectured a Segre-Verlinde correspondence for newly defined Segre and Verlinde numbers of moduli spaces of higher rank sheaves, see \[2\] for a precise form of the conjecture.

The main result here is a proof of this correspondence for K3 surfaces.

More precisely, we establish Conjecture 5.1 in \[2\] which relates integrals over moduli spaces of higher rank on K3 surfaces to integrals over the punctual Hilbert schemes:

**Theorem 1.** Let \(M(v)\) be a \(2n\)-dimensional proper moduli space of stable sheaves with Mukai vector \(v\) on a K3 surface \(S\), such that \(\text{rk}(v) > 0\). For any \(K\)-theory class \(\alpha \in K(S)\), class \(L \in H^2(S)\) and \(u \in \mathbb{C}\) we have

\[
\int_{M(v)} c(\alpha_M) e^{\mu(L) + u \mu(p)} = \int_{S[n]} c(\beta[n]) e^{\mu(L) + u \text{rk}(v) \mu(p)}
\]

for any \(K\)-theory class \(\beta \in K(S)\) such that

\[
\text{rk}(\beta) = \frac{\text{rk}(\alpha)}{\text{rk}(v)},
\]

\[
v(\alpha)^2 = v(\beta)^2,
\]

\[
c_1(\alpha)^2 = c_1(\beta)^2,
\]

\[
c_1(\alpha) \cdot L = c_1(\beta) \cdot L.
\]

Here \(v(\alpha) = \text{ch}(\alpha) \sqrt{\text{td}_S}\) is the Mukai vector and the inner products are taken with respect to the Mukai pairing, see Section 2.1. We also refer to Section 3.1 for the descendent classes \(\alpha_M := \mathbb{G}K(\alpha)\) and \(\mu(\sigma)\).

1.5. **Higher-rank Segre/Verlinde.** Let \(\rho = \text{rk}(v)\) and \(s = \text{rk}(\alpha)\). As explained in \[2\] Cor.5.2 Theorem 1 implies the closed evaluation:

\[
\int_{M(v)} c(\alpha_M) = \text{Coeff}_{z^n} \left( V_s^{2-e(S)} W_s c_1(\alpha)^2 X_s^{e(S)} \right)
\]

where the functions \(V_s, W_s, X_s\) were determined in \[5\] to be:

\[
V_s(z) = (1 + (1 - \frac{s}{\rho})t)^{1-s}(1 + (2 - \frac{s}{\rho})t)^s(1 + (1 - \frac{s}{\rho})t)^{\rho-1},
\]

\[
W_s(z) = (1 + (1 - \frac{s}{\rho})t)^{\frac{1}{2}s-1}(1 + (2 - \frac{s}{\rho})t)^{\frac{1}{2}(1-s)}(1 + (1 - \frac{s}{\rho})t)^{\frac{1}{2}-\frac{1}{2}\rho},
\]

\[
X_s(z) = (1 + (1 - \frac{s}{\rho})t)^{\frac{1}{4}s^{2-s}}(1 + (2 - \frac{s}{\rho})t)^{-\frac{1}{4}s^2+\frac{1}{4}}
\]

\[
\cdot (1 + (1 - \frac{s}{\rho})(2 - \frac{s}{\rho})t)^{-\frac{1}{4}}(1 + (1 - \frac{s}{\rho})t)^{-\frac{(\rho-1)^2}{2\rho^2}s}
\]
under the variable change $z = t(1 + (1 - \frac{r}{\rho})t)^{1 - \frac{s}{\rho}}$.

On the Verlinde side, the work [3] reduced the Verlinde numbers of $M(v)$ to those of $S[n]$ using a result of Fujiki. One obtains (see [2] for notation and assumptions such as $\rho | r$) that

$$\chi(M(v), \mu(L) \otimes E^{\otimes r}) = \text{Coeff}_{w^n} \left( 2^{\rho - 1} \rho^{2 - e(S)} G_{\rho}(L) F_{r}^{\frac{1}{2}e(S)} \right)$$

where the universal functions $G_{r}, F_{r}$ were determined by [1] to be:

$$G_{r}(w) = 1 + v,$$
$$F_{r}(w) = (1 + v)^{\frac{r^2}{\rho^2}} (1 + \frac{r^2}{\rho^2} v)^{-1},$$

under the variable change $w = v(1 + v)^{r^2/\rho^2-1}$.

The universal functions above satisfy

$$F_{r}(w) = V_{s}(z) \frac{2(\rho^2 - \rho^{-1} 2)}{\rho} W_{s}(z)^{-\frac{4}{\rho}} X_{s}(z)^{2},$$
$$G_{r}(w) = V_{s}(z) W_{s}(z)^{2},$$

where $s = \rho + r$ and $v = t(1 - \frac{r}{\rho} t)^{-1}$. This equality is called the higher-rank Segre-Verlinde correspondence for K3 surfaces.

**Corollary 1.** The higher rank Segre-Verlinde correspondence of [2] holds for K3 surfaces (and hence all $K$-trivial surfaces).

1.6. **Idea of proof and Markman’s universality.** The integrals appearing on both sides of (3) are typical instances of integrals over descendant classes. These are cohomology classes on $M(v)$ defined by the Künneth factors of the Chern character of a universal sheaf. For K3 surfaces, results of Markman [7] imply that the set descendant integrals over any two moduli spaces $M(v)$ and $M(v')$ are equivalent by an explicit correspondence given that $\dim M(v) = \dim M(v')$. This equivalence is most easily stated in form of a universality result (Theorem 3), which allows us to move any descendant integral over any moduli space of stable sheaves of arbitrary rank to the Hilbert scheme. The method also applies when integrating the descendents against arbitrary monomials in Chern classes of the tangent bundle.

Since this universality is of independent interest, we give a detailed exposition of it in Section 2 following [7]. Because the original reference is somewhat intricate, we also sketch the main ideas behind the proof.

Theorem [1] is then proven in Section 3 as a direct application.

1.7. **Acknowledgements.** I thank Martijn Kool for bringing Conjecture 5.1 of [2] to my attention and for useful discussions and comments.

The author is funded by the Deutsche Forschungsgemeinschaft (DFG) - OB 512/1-1.
2. Markman’s Universality

2.1. Basic definitions. Let $S$ be a K3 surface and consider the lattice $\Lambda = H^*(S, \mathbb{Z})$ endowed with the Mukai pairing

$$(x \cdot y) := - \int_S x^\vee y,$$

where, if we decompose an element $x \in \Lambda$ according to degree as $(r, D, n)$, we have written $x^\vee = (r, -D, n)$. We will also write

$$\text{rk}(x) = r, \quad c_1(x) = D, \quad v_2(x) = n.$$

Given a sheaf or complex $E$ on $S$ the Mukai vector of $E$ is defined by

$$v(E) = \sqrt{\text{td}_S} \cdot \text{ch}(E) \in \Lambda.$$

Let $v \in \Lambda$ be an effective vector, $H$ be an ample divisor on $S$ and let $M_H(v)$ be a proper smooth moduli space of $H$-stable sheaves with Mukai vector $v$.\footnote{More generally, one can also work with $\sigma$-stable objects for a Bridgeland stability condition in the distinguished component.} For simplicity we assume that there exists an universal sheaf $\mathcal{F}$ on $M_H(v) \times S$, unique up to tensoring of a line bundle from the base.

The results we state below also hold in the general case where there exists only a universal twisted sheaf. By this we mean that all statements below can be formulated in terms of the Chern character $\text{ch}(\mathcal{F})$ alone and this class can be defined in the twisted case as well, see \cite{6} Sec.3]. The proofs carry over likewise using that the ingredients hold in the twisted case as well.

Consider the morphism $\theta_{\mathcal{F}} : \Lambda \to H^2(M, \mathbb{Z})$ defined by

$$\theta_{\mathcal{F}}(x) = \left[ \pi_* \left( \text{ch}(\mathcal{F}) \sqrt{\text{td}_S} \cdot x^\vee \right) \right]_{\text{deg}=1}.$$

Then $\theta_{\mathcal{F}}$ restricts to an isomorphism

$$\theta = \theta_{\mathcal{F}}|_{v^\perp} : v^\perp \xrightarrow{\cong} H^2(M, \mathbb{Z})$$

which does not depend on the choice of universal family (use that the degree 0 component of the pushforward \cite{4} vanishes) and for which we hence have dropped the subscript $\mathcal{F}$. The isomorphism $\theta$ is orthogonal with respect to the Mukai pairing on the left, and the pairing given by the Beauville-Bogomolov-Fujiki form on the right. We will identify $v^\perp \subset \Lambda$ with $H^2(M_H(v), \mathbb{Z})$ under this isomorphism.

The universal sheaf $\mathcal{F}$ and hence its Chern character $\text{ch}(\mathcal{F})$ is unique only up to pullback of a line bundle from the base. We can pick a canonical normalization as follows:

$$u_v := \exp \left( \frac{\theta_{\mathcal{F}}(v)}{(v, v)} \right) \cdot \text{ch}(\mathcal{F}) \cdot \sqrt{\text{td}_S}$$

where we have suppressed the pullback morphisms from $M$ and $S$ in the first and last term on the right. The invariance is a short check (replace $\mathcal{F}$
by \( \mathcal{F} \otimes \mathcal{L} \) and calculate). The class \( u_v \) is characterized among the classes \( \text{ch}(\mathcal{F}) \cdot \sqrt{\text{td} S} \) by the property that \( \theta_u(v) = 0 \) (Use that \( \pi_* (\text{ch}(\mathcal{F}) \sqrt{\text{td} S} \cdot v^\vee) = -(v \cdot v) + \theta_F(v) + \ldots \) for a universal family \( \mathcal{F} \)).

**Example 1.** Let \( M = \text{Hilb}_n(S) \) be the Hilbert scheme of \( n \) points on \( S \). We have \( v = 1 - (n - 1)p \), and we take \( \mathcal{F} = I_Z \) the ideal sheaf of the universal subscheme. For \( \alpha \in H^2(S) \) we have

\[
\theta(\alpha) = \pi_* (\text{ch}_2(O_Z) \pi_* S(\alpha)).
\]

If \( \alpha \) is the class of a divisor \( A \), then this is the class of subschemes incident to \( A \). Similarly, define

\[
\delta = -\frac{1}{2} \Delta_{\text{Hilb}_n(S)} = c_1(\pi_* O_Z) = \pi_* \text{ch}_3(O_Z).
\]

Then under the identification (5) we have

\[
\delta = -(1 + (n - 1)c).
\]

Since \( \theta_F(v) = -\delta \) the canonical normalization of \( \text{ch}(\mathcal{F}) \) takes the form

\[
u_v = \exp \left(-\delta \frac{1}{2n - 2}\right) \text{ch}(I_Z) \sqrt{\text{td} S}.
\]

2.2. **Markman’s operator.** For \( i = 1, 2 \) let \( (S_i, H_i, v_i) \) be the data defining proper moduli space of stable sheaves \( M_i = M_{H_i}(S_i, v_i) \). Let \( \mathcal{F}_i \) be the universal family on \( M_i \times S \).

Consider an isomorphism of Mukai lattices

\[
g : H^*(S_1, \mathbb{Z}) \to H^*(S_2, \mathbb{Z})
\]

such that \( g(v_1) = v_2 \). We will identify \( g \) also with an isomorphism of topological \( K \)-groups

\[
g : K_{\text{top}}(S_1) \to K_{\text{top}}(S_2)
\]

using the lattice isomorphism \( K_{\text{top}}(S) \cong H^*(S, \mathbb{Z}) \) given by \( E \mapsto v(E) \).

Here \( K_{\text{top}}(S) \) carries the Euler pairing \( (E \cdot F) = -\chi(E^\vee \otimes F) \). Hence the following diagram commutes

\[
\begin{array}{ccc}
K_{\text{top}}(S_1) & \xrightarrow{g} & K_{\text{top}}(S_2) \\
\downarrow v & & \downarrow v \\
H^*(S_1, \mathbb{Z}) & \xrightarrow{g} & H^*(S_2, \mathbb{Z}).
\end{array}
\]

Similar identification will apply to morphisms \( g \) defined over \( \mathbb{C} \). The Markman operator associated to \( g \) is given by the following result:

**Theorem 2.** (Markman, [7]) For any isometry \( g : H^*(S_1, \mathbb{C}) \to H^*(S_2, \mathbb{C}) \) such that \( g(v_1) = v_2 \) there exists a unique operator

\[
\gamma(g) : H^*(M_1, \mathbb{C}) \to H^*(M_2, \mathbb{C})
\]

such that
(a) $\gamma(g)$ is degree-preserving orthogonal ring-isomorphism

(b) $\gamma(g) \otimes g(u_{v_1}) = u_{v_2}$.

The operator is called the Markman operator and given by

$\gamma(g) = c_m \left[ -\pi_* \left( (1 \otimes g) u_{v_1} \right)^\vee \cdot u_{v_2} \right]$.

Moreover, we have

(c) $\gamma(g_1) \circ \gamma(g_2) = \gamma(g_1 g_2)$ and $\gamma(g)^{-1} = \gamma(g^{-1})$ if it makes sense.

(d) $\gamma(g) c_k(T_{M_1}) = c_k(T_M)$.

We can reinterpret the condition $(f \otimes g)(u_{v_1}) = u_{v_2}$ in terms of generators of the cohomology ring. Consider the canonical morphism

$B : H^*(S, \mathbb{Q}) \to H^*(M, \mathbb{Q})$ defined by

$B(x) = \pi_* (u_v \cdot x^\vee)$.

We write $B_k(x)$ for its component in degree $2k$. In particular, $B_1(x) = \theta_{\mathcal{F}}(x)$ for all $x \in v^\perp$.

**Lemma 1.** Let $f : H^*(M_1, \mathbb{Q}) \to H^*(M_2, \mathbb{Q})$ be a degree-preserving orthogonal ring isomorphism. Then the following are equivalent:

(a) $(f \otimes g)(u_{v_1}) = u_{v_2}$

(b) $f(B(x)) = B(gx)$ for all $x \in H^*(S_1, \mathbb{Q})$.

**Proof.** Since $g$ is an isometry of the Mukai lattice we have

$\pi_*(u_{v_2} \cdot (gx)^\vee) = \pi_*((1 \otimes g^{-1}) u_{v_2} \cdot x^\vee)$.

Indeed, if we write $\text{ch}(\mathcal{F}_2) \sqrt{\text{td}_S} = \sum_i a_i \otimes b_i$ under the K"unneth decomposition, then

$\pi_*((1 \otimes g^{-1})(\text{ch}(\mathcal{F}_2) \sqrt{\text{td}_S} \cdot x^\vee)) = \sum_i a_i \int_S g^{-1}(b_i) x^\vee$

$= \sum_i -a_i \cdot (g^{-1}(b_i) \cdot x)$

$= \sum_i -a_i \cdot (b_i \cdot g(x))$

$= \sum_i a_i \int_S b_i g(x)^\vee$

$= \pi_*(\text{ch}(\mathcal{F}) \sqrt{\text{td}_S} \cdot g(x)^\vee)$. 

Hence we see that:

\[(b) \iff \forall x \in H^*(S_1, \mathbb{Z}) : f \pi_* (u_{v_1} \cdot x^\vee) = \pi_* (u_{v_2} \cdot (gx)^\vee) \]
\[\iff \forall x \in H^*(S_1, \mathbb{Z}) : \pi_* ((f \otimes 1) u_{v_1} \cdot x^\vee) = \pi_* ((1 \otimes g^{-1}) u_{v_2} \cdot x^\vee) \]
\[\iff (f \otimes 1)(u_{v_1}) = (1 \otimes g^{-1})(u_{v_2}) \]
\[\iff (a).\]

\[\square\]

**Corollary 2.** In the setting of Theorem, \( \gamma(g)B(x) = B(gx). \)

2.3. **Universality.** We formulate what the above means for tautological integrals over the moduli spaces of stable sheaves \( M(v) \) on a K3 surface \( S \).

Let \( P \) be a polynomial depending on the variables

\[ t_{j,i}, \quad j = 1, \ldots, k, \quad i \geq 0, \quad u_j \geq 1. \]

Let also \( A = (a_{ij})_{i,j=0}^k \) be a \( (k+1) \times (k+1) \)-matrix.

**Theorem 3.** (Universality) There exists \( I(P, A) \in \mathbb{Q} \) (depending only on \( P \) and \( A \)) such that for any proper moduli space of stable sheaves \( M(v) \) on a K3 surface \( S \) and for any \( x_1, \ldots, x_k \in \Lambda \) with

\[
\begin{pmatrix}
\sum v_i \cdot v & \sum (v_i x_j)_{i=1}^k \\
\sum (x_i v)_{i=1}^k & \sum (x_i x_j)_{i,j=1}^k
\end{pmatrix}
= A
\]

we have

\[
\int_{M(v)} P(B_i(x_j), c_j(Tan)) = I(P, A).
\]

In other words, the integral

\[
\int_{M(v)} P(B_i(x_j), c_j(Tan))
\]

depends upon the above data only through \( P \), the dimension \( \dim M(v) = 2n \), and the pairings \( v \cdot x_i \) and \( x_i \cdot x_j \) for all \( i, j \).

**Proof.** Let \( (M(v), x_i) \) and \( (M(v'), x'_i) \) be two sets of classes with the same intersection matrix \( A \). Then there exists an orthogonal matrix

\[ g : \Lambda_C \to \Lambda_C \]
taking \((v, x_1, \ldots, x_k)\) to \((v', x'_1, \ldots, x'_k)\). Hence by Theorem 2(a) and Corollary 2,

\[
\int_{M(v)} P(B_i(x_j), c_j(Tan)) = \int_{M(v')} \gamma(g) P(B_i(x_j), c_j(Tan))
\]

\[
= \int_{M(v')} P(\gamma(g)B_i(x_j), \gamma(g)c_j(Tan))
\]

\[
= \int_{M(v')} P(B_i(\gamma(g)x_j), c_j(Tan))
\]

\[
= \int_{M(v')} P(B_i(x'_j), c_j(Tan)).
\]

\[\square\]

2.4. A few words on the proof of Theorem 2. We briefly discuss what goes into the proof of Theorem 2 following [7]. This section is not relevant for the applications and can be skipped.

The main ingredient is the following uniqueness statement:

**Lemma 2.** Let \(f : H^*(M_1, \mathbb{Q}) \to H^*(M_2, \mathbb{Q})\) be a morphism such that:

(i) \(f\) is a degree-preserving orthogonal ring isomorphism.

(ii) There exists universal families \(\mathcal{F}\) on \(M_1 \times S_1\) and \(\mathcal{F}'\) on \(M_2 \times S_2\) such that

\[
(f \otimes g) \left( \text{ch}(\mathcal{F}) \sqrt{\text{td}(S)} \right) = \text{ch}(\mathcal{F}') \sqrt{\text{td}(S_2)} \cdot \exp(\ell)
\]

for some \(\ell \in H^2(M_2, \mathbb{Q})\).

Then we have

\[f = c_m \left( - \text{Ext}_\pi((1 \otimes g)\mathcal{F}, \mathcal{F}') \right).\]

Moreover, in (ii) it is enough to assume that \(\mathcal{F}, \mathcal{F}'\) are elements in \(K_{\text{top}}(M_1 \times S_1)\), i.e. differ from a universal family by tensor product by a fractional line bundle from the base (see the proof). In particular, we have

\[f = c_m \left[ -\pi_* \left( ((1 \otimes g)u_{v_1})' \cdot u_{v_2} \right) \right].\]

The main input for the proof of the Lemma is the following theorem which we state for an arbitrary moduli space of stable sheaves \(M\) on a K3 surface:

**Theorem 4** (Markman [6]). For any universal families \(\mathcal{F}, \mathcal{F}'\) on \(M \times S\),

\[\Delta_M = c_m(-\text{Ext}^*_\pi(\mathcal{F}, \mathcal{F}')).\]

More generally, for any \(\gamma, \gamma' \in H^2(M, \mathbb{Q})\) we have

\[\Delta_M = c_m \left[ -\pi_* \left( (\exp(\gamma)\text{ch}(\mathcal{F})\sqrt{\text{td}(S)})' \cdot \exp(\gamma')\text{ch}(\mathcal{F})\sqrt{\text{td}(S)} \right) \right].\]
Proof of Lemma 2. Assume that $f$ satisfies (i) and (ii). Note that, since $f$ is a ring isomorphism, the equality in (ii) is equivalent to the parallel equality where we replace $\text{ch}(F)$ by $\text{ch}(F) \exp(\mu)$ for any $\mu \in H^2(M_1, \mathbb{Q})$, and similarly for $\text{ch}(F')$. Hence we may have also assumed (ii) with $\text{ch}(F)$ replaced by $\text{ch}(F) \exp(\mu)$ instead.

We will prove that for any $\ell_i \in H^2(M_i, \mathbb{Q})$ we have:

$$f = c_m \left[ -\pi_* \left( ((1 \otimes g)(\exp(\ell_1)\text{ch}(F')\sqrt{\text{td}_S}))^\vee \cdot (\exp(\ell_2)\text{ch}(F')\sqrt{\text{td}_S}) \right) \right] .$$

Taking $\ell_i$ both to be trivial then gives (7), and taking $\ell_i$ to be as in the definition of $u_v$ gives (8).

By Theorem 4, for any $\gamma \in H^2(M_2, \mathbb{Q})$ we have:

$$\Delta_{M_2} = c_m \left[ -\pi_* \left( (\text{ch}(F')\sqrt{\text{td}_S} \exp(\gamma))^\vee \cdot \text{ch}(F')\sqrt{\text{td}_S} \exp(\ell_2) \right) \right]$$

Inserting

$$\text{ch}(F')\sqrt{\text{td}_S} = (f \otimes g) \left( \exp(f^{-1}(\ell))\text{ch}(F')\sqrt{\text{td}_{S_1}} \right)$$

in the first term, and then using that $f$ is degree-preserving (so commutes with dualizing), and a ring isomorphism (so commutes with taking $c_m$), we get that $\Delta_{M_2}$ is equal to

$$(f \otimes 1)c_m \left[ -\pi_* \left( ((1 \otimes g)(\text{ch}(F)\sqrt{\text{td}_S} \exp(\gamma + f^{-1}(\ell))))^\vee \text{ch}(F')\sqrt{\text{td} \exp(\ell_2)} \right) \right]$$

Setting $\gamma = -f^{-1}(\ell) + \ell_1$, and taking $Q$ to be the right hand side of (9) we find

$$\text{id}_{H^*(M_2)} = \Delta_{M_2} = (f \otimes 1)(Q) = Q \circ f^t$$

where $f$ is the transpose with respect to the standard cup product (or as a correspondence, identical with $f$ up to swapping the factors). Since $f$ is orthogonal, we conclude $\text{id}_{H^*(M_2)} = Q \circ f^{-1}$, so $f = Q$. □

After the uniqueness, we prove a basic result on the operator satisfying the condition of the previous lemma.

Lemma 3. Assume $f$ satisfies (i) and (ii) of Lemma 2. Then

$$(f \otimes g)(u_{v_1}) = u_{v_2}.$$
Hence the claim follows from the following calculation:

\[ f(\theta F(v_1)) = \deg_1 \pi_*( (1 \otimes g^{-1})(f \otimes g)(\text{ch}(F) \sqrt{td_S}) \cdot v_1^\vee) \]

\[ = \deg_1 \pi_*((1 \otimes g^{-1})(\text{ch}(F') \sqrt{td_S}) \cdot v_1^\vee) \exp(\ell) \]

\[(*) = \deg_1 \pi_*((1 \otimes g^{-1})(ch(F') \sqrt{td_S} \cdot g(v_1)^\vee) \exp(\ell) \]

\[ = [\deg 1](-(v_2, v_2) + \theta F(v_2)) \exp(\ell) \]

\[ = -(v_2, v_2)\ell + \theta F(v_2), \]

where (*) follows since \( g \) is an isometry of Mukai lattices, and \([\deg k]\) stands for taking the (complex) degree \(k\) part.

\[ \square \]

**Sketch of Proof of Theorem 2** The uniqueness part of the theorem is addressed by the two lemmas above. Hence we only need to show the existence of the operator in Theorem 2. One may be tempted to define the operators \( \gamma(g) \) directly using the closed formula (6) and then derive their properties from it. However, (6) is unfortunately very hard to work with in practice. It is even not clear how to use it to prove \( \gamma(g_1) \circ \gamma(g_2) = \gamma(g_1 g_2) \). Nevertheless, it can be used for the following: if we know the statements of the theorem for a Zariski dense subset of all operators \( g \) (e.g. the integral isometries), then we can define \( \gamma(g) \) by (6) for arbitrary isometries and then conclude Theorem 2 in general using Zariski density.

Hence it remains to consider the case of integral isometries. For this one considers the set \( S \) of triples

\[ ((S_1, H_1, v_1), (S_2, H_2, v_2), g : H^*(S_1, \mathbb{Z}) \to H^*(S_2, \mathbb{Z})) \]

where \( g \) is an isometry such that \( g(v_1) = v_2 \), for which the result of the theorem holds. Since the elements for which the statements of the theorem hold are closed under composition and inverse, we can think about \( S \) as the set of arrows in a groupoid. Then elements of the groupoid can be constructed in three different ways:

- For any deformation \((S_1, v_1, H_1) \leadsto (S_2, v_2, H_2)\) which keeps \( v_1 \) and \( H_1 \) of Hodge type, we have an associated deformation of moduli spaces

\[ M_{H_1}(S_1, v_1) \leadsto M_{H_2}(S_2, v_2). \]

The associated parallel transport operator \( P \) is a degree-preserving orthogonal ring isomorphism. Moreover, since \( u_v \) is defined in terms of the universal family which deforms along the family, the classes \( u_{v_2} \) of the individual fibers are the parallel transports of \( u_{v_1} \). Hence if \( g \) is the parallel transport operator associated to \( S_1 \leadsto S_2 \), then \( (P \otimes g)(u_{v_1}) = u_{v_2} \). We conclude that \( P \) satisfies the theorem.

- Assume that \( \Phi : D^b(S_1) \to D^b(S_2) \) is a derived equivalence that takes \( H_1 \)-stable sheaves of Mukai vector \( v_1 \) to \( H_2 \)-stable sheaves of Mukai vector
ON DESCENDENT INTEGRALS

\[ \varphi : M_{H_1}(S_1, v_1) \to M_{H_2}(S_2, v_2) \]
such that by its construction we have

\[ (\text{Id} \boxtimes \Phi)(F) = (\varphi \times \text{id})^*(F'). \]

This yields

\[ (\varphi_* \boxtimes \Phi_*)(\text{ch}(F) \sqrt{\text{td}}_S) = \text{ch}(F') \sqrt{\text{td}}_S. \]

Hence with \( g = \Phi_* \) we have \( \gamma(g) = \varphi_* \) by the main lemma.

• Assume in a slight modification, that \( (\Phi \ast E)^\vee \) is \( H_2 \)-stable of Mukai vector \( v_2 \) for any \( H_1 \)-stable sheaf \( E \in M_1 \). Then we have an induced morphism

\[ \varphi : M_{H_1}(S_1, v_1) \to M_{H_2}(S_2, v_2) \]
such that by its definition we have

\[ (\text{Id} \boxtimes \Phi)(F^\vee) = (\varphi \times \text{id})^*(F'). \]

This yields

\[ (\varphi \times \text{id})_* (\text{Id} \boxtimes \Phi)(F^\vee) = F'. \]

Since \( \varphi \) commutes with dualizing, we get

\[ (\varphi_* \boxtimes \Phi)(F)^\vee = F' \]

and hence

\[ \left( (\varphi_* \boxtimes \Phi_*)(\text{ch}(F) \sqrt{\text{td}}_S) \right)^\vee = \text{ch}(F') \sqrt{\text{td}}_S. \]

Going through the argument of the proof of Lemma [2] then shows

\[ \varphi_* = D \circ \gamma(D\Phi_*) \]

where \( D \) is the operator that acts on \( H^{2i} \) by \((-1)^i\). We note that \( \Phi_* v_1 = v_2^\vee \), so \( g = D\Phi_* \) sends \( v_1 \) to \( v_2 \) as required. Since \( D \) is a degree-preserving orthogonal ring isomorphism, we see that \( \gamma(g) \) satisfies the statements of the Theorem.

This shows the existence of Markman operators for \( g \) of these form. Markman then (roughly) shows that any integral \( g \) can be written as a composition of these three operations. This concludes the proof.

The proof above also ties the operators \( \gamma(g) \) directly to the monodromy action, the main application in [7].

3. THE GÖTTSCHE–KOOL CONJECTURE

We specialize now to the setting of [2]. As before, we let \( S \) be a K3 surface and \( M(v) \) a moduli space of stable sheaves on \( S \) of Mukai vector \( v \). We set \( 2n := \dim M(v) = v \cdot v + 2 \) and assume that \( \text{rk}(v) > 0 \).
3.1. **Normalization.** Let $\alpha \in K(S)$. Following Göttsche and Kool [2] we define descendent classes on $M(v)$. If there exists a universal family $G$ and a $\text{rk}(v)$-th root of $\det(G)$, then we set

$$
\text{GK}(\alpha) := \text{ch}(-\pi_s(\pi_S^*(\alpha) \otimes \mathcal{G} \otimes \det(G)^{-1/\text{rk}(v)})).
$$

where $\pi, \pi_S$ are the projections of $M(v) \times S$ to the factors. In the general case we use the Grothendieck-Riemann-Roch expression:

$$
\text{GK}(\alpha) := -\pi_s \left( v(\alpha) \text{ch}(\mathcal{G}) \sqrt{\text{td}_S} \exp \left( -\frac{1}{\text{rk}(v)} c_1(\mathbb{G}_m) \right) \right).
$$

We let $\text{GK}_k(\alpha)$ be the degree $2k$ component of $\text{GK}(\alpha)$.

The $\text{GK}(\alpha)$ are easily expressed in Markman’s normalization:

**Lemma 4.** We have

$$
\text{GK}(\alpha) = -B \left( v(\alpha^\vee) \exp \left( \frac{c_1(v)}{\text{rk}(v)} \right) \right) \exp \left( B_1 \left( -\frac{p}{\text{rk}(v)} - \frac{v}{v \cdot v} \right) \right).
$$

**Proof.** Using that $\text{Pic}(M \times S) = \text{Pic}(M) \otimes \text{Pic}(S)$ we can write

$$
c_1(\mathcal{G}) = \pi^*(\ell) + \pi_S^*(c_1(v))
$$

for some $\ell \in H^2(M)$. By calculating $\theta_G(p)$ one finds $\ell = \theta_G(p)$. Hence

$$
\text{GK}(\alpha) = -\pi_s \left( v(\alpha) \text{ch}(\mathcal{G}) \sqrt{\text{td}_S} \exp \left( -\frac{c_1(v)}{\text{rk}(v)} \right) \right) \exp(\theta_G(p)/(p \cdot v))
$$

$$
= -B \left( v(\alpha^\vee) \exp \left( \frac{c_1(v)}{\text{rk}(v)} \right) \right) \exp \left( B_1 \left( -\frac{p}{\text{rk}(v)} - \frac{v}{v \cdot v} \right) \right).
$$

□

For $\sigma \in H^*(S)$ Göttsche and Kool consider also the classes

$$
\mu(\sigma) = -\pi_s \left( \text{ch}_2(\mathcal{G} \otimes \det(G)^{-1/\text{rk}(v)}) \pi_S^*(\sigma) \right).
$$

(defined by the GRR expression if only a semi-universal family exists).

**Lemma 5.** The class $\mu(\sigma)$ is the component of degree $\deg(\sigma)$ of

$$
- \exp \left( B_1 \left( \frac{p}{p \cdot v} - \frac{v}{v \cdot v} \right) \right) B \left( \sigma^\vee \exp \left( \frac{c_1(v)}{\text{rk}(v)} \right) \sqrt{\text{td}_S^{-1}} \right).
$$
Proof. We have that \( \mu(\sigma) \) is the degree \( \deg(\sigma) \) component of
\[
- \pi_* \left( \text{ch}(G) \otimes \text{det}(G)^{-1/rk(v)}) \pi_S^*(\sigma) \right)
\]
\[
= - \pi_* \left( \text{ch}(G) \exp(-c_1(G)/rk(v)) \pi_S^*(\sigma) \right)
\]
\[
= - \exp \left( \frac{\theta_G(p)}{p \cdot v} - \frac{\theta_G(v)}{v \cdot v} \right) \exp \left( \frac{\theta_G(v)}{v \cdot v} \right)
\]
\[
\cdot \pi_* \left( \text{ch}(G) \pi_S^* \left( \sigma \exp^{c_1(v)/rk(v)} \sqrt{\text{td}_S^{-1}} \right) \right)
\]
\[
= - \exp \left( B_1 \left( \frac{p}{p \cdot v} - \frac{v}{v \cdot v} \right) \right) B(\sigma \exp^{c_1(v)/rk(v)} \sqrt{\text{td}_S^{-1}}).
\]
where we used again \( c_1(G) = \pi^* \theta_G(p) + \pi_S^* c_1(V) \).

In particular, for \( L \in H^2(S) \) we have that
\[
\mu(L) = B_1 \left( L \exp \left( \frac{c_1(v)}{rk(v)} \right) \right) - B_1 \left( \frac{p}{p \cdot v} - \frac{v}{v \cdot v} \right)
\]
and that \( \mu(p) \) is a polynomial in \( B_1 \left( \frac{p}{p \cdot v} - \frac{v}{v \cdot v} \right) \) and \( B_1(p) \).

3.2. Dependence. We conclude that any integral
\[
\int_{M(v)} P(GK_\mathcal{L}(\alpha), \mu(L), \mu(\mu(p)))
\]
(such as the Segre number) only depends upon \( P \) and the intersection pairings in the Mukai lattice of the classes
\[
v, \ p/rk(v), \ v(\alpha)^\vee \exp \left( \frac{c_1(v)}{rk(v)} \right), \ L \exp \left( \frac{c_1(v)}{rk(v)} \right), \ u \mu(p).
\]

Explicitly, the interesting pairings for the first three classes are
\[
(i) \quad v \cdot v(\alpha)^\vee \exp \left( \frac{c_1(v)}{rk(v)} \right) = -v_2(\alpha) \cdot rk(v) + \frac{1}{2} \frac{rk(\alpha)}{rk(v)}(v \cdot v)
\]
\[
(ii) \quad p/rk(v) \cdot v(\alpha)^\vee \exp \left( \frac{c_1(v)}{rk(v)} \right) = - \frac{rk(\alpha)}{rk(v)}
\]
\[
(iii) \quad \left( v(\alpha)^\vee \exp \left( \frac{c_1(v)}{rk(v)} \right) \right)^2 = v(\alpha) \cdot v(\alpha).
\]
The interesting intersections involving \( L \) are
\[
v \cdot L \exp \left( \frac{c_1(v)}{rk(v)} \right) = L \cdot c_1(v) - L \cdot c_1(v) = 0
\]
\[
(iv) \quad v(\alpha)^\vee \exp \left( \frac{c_1(v)}{rk(v)} \right) \cdot L \exp \left( \frac{c_1(v)}{rk(v)} \right) = v(\alpha)^\vee \cdot L = -c_1(\alpha) \cdot L
\]
\[
\left( L \exp \left( \frac{c_1(v)}{rk(v)} \right) \right)^2 = L^2.
\]
The pairings with \( u \mu(p) \) are \( u \cdot rk(v) \) times the pairings with \( p/rk(v) \).
3.3. Moving to the Hilbert scheme. Since \([10]\) only depends on the intersection pairings of \([11]\) we have that

\[
\int_{M(v)} P(GK_k(\alpha), \mu(L), \mu(up)) = \int_{S[n]} P(GK_k(\beta), \mu(L), \mu(u'p))
\]

for any \(K\)-theory class \(\beta \in K(S)\) and \(u' \in \mathbb{C}\) such that the list

\[
(12) \quad 1 - (n - 1)p, \quad p, \quad v(\beta)^\vee, \quad L, \quad u'p
\]

has the same intersection numbers as the list \([11]\). (The list \([12]\) is obtained from \([11]\) by specializing to \(v = 1 - (n - 1)p\), the Mukai vector of \(S[n]\)).

The interesting parts of the intersections of \([12]\) are

(i) \(v \cdot v(\beta)^\vee = -v_2(\beta) + \frac{1}{2} \text{rk}(\beta)(2n - 2)\)

(ii) \(p \cdot v(\beta)^\vee = -\text{rk}(\beta)\)

(iii) \(v(\beta)^\vee \cdot v(\beta)^\vee = v(\beta) \cdot v(\beta)\)

(iv) \(v(\beta)^\vee \cdot L = -c_1(\beta) \cdot L\)

Equating (i) - (iv) for \(M(v)\) and \(S[n]\) we hence get the system:

\[
-v_2(\alpha) \cdot \text{rk}(v) + \frac{1}{2} \frac{\text{rk}(\alpha)}{\text{rk}(v)} (v \cdot v) = -v_2(\beta) + \frac{1}{2} \text{rk}(\beta)(2n - 2)\]

\[
-\frac{\text{rk}(\alpha)}{\text{rk}(v)} = -\text{rk}(\beta)
\]

\[
v(\alpha) \cdot v(\alpha) = v(\beta) \cdot v(\beta)
\]

\[-c_1(\alpha) \cdot L = -c_1(\beta) \cdot L.
\]

Since \(v(\alpha)^2 = c_1(\alpha)^2 - 2\text{rk}(\alpha)v_2(\alpha)\), this is equivalent to the system:

\[
\text{rk}(\beta) = \frac{\text{rk}(\alpha)}{\text{rk}(v)}
\]

\[
(13) \quad v(\alpha)^2 = v(\beta)^2
\]

\[
c_1(\alpha)^2 = c_1(\beta)^2
\]

\[
c_1(\alpha) \cdot L = c_1(\beta) \cdot L.
\]

Moreover, we must have

\[-u' = u'p \cdot (1 - (n - 1)p) = up \cdot v = -\text{rk}(v)u.
\]

We have proven the following (which immediately implies Theorem 1):

**Theorem 5.** For any polynomial \(P\), we have

\[
\int_{M(v)} P(GK_k(\alpha), \mu(L), \mu(up)) = \int_{S[n]} P(GK_k(\beta), \mu(L), \mu(u \text{rk}(v)p))
\]

for any \(K\)-theory class \(\beta \in K(S)\) such that \([13]\) is satisfied.
References


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