Kaneko-Zagier equations for Jacobi forms and curve counting on CHL manifolds

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Based on the joint works:
arXiv:1811.06102 (with Jim Bryan),
arXiv:2007.03489 (with Jan-Willem van Ittersum and Aaron Pixton),
arXiv:1411.1514 (with Rahul Pandharipande)
Abstract

I will discuss counting curves on $K3 \times \mathbb{P}^1$ with relative conditions at $K3 \times 0$ and $K3 \times \infty$. Main features:

- Structure constants are quasi-Jacobi forms
- $g$-twisted traces $\leadsto$ CHL geometry
- Relations to Conway Moonshine VOAs (??)
Short Overview

Abstract

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- $g$-twisted traces $\rightsquigarrow$ CHL geometry
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Let $q = e^{2\pi i \tau}$ where $\tau \in \mathbb{H}$. Given $k > 2$ consider the Serre derivative

$$\vartheta_k = D_\tau - \frac{k}{12} E_2(\tau)$$

where $D_\tau = q \frac{d}{dq}$ and $E_2 = 1 - 24 \sum_{n \geq 1} \sigma(n) q^n$. 
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Lemma

\[ \vartheta_k : \text{Mod}_k \to \text{Mod}_{k+2} \]

Proof.

The transformation law \( E_2(-1/\tau) = \tau^2 E_2(\tau) - 6i \tau / \pi \) cancels the quasi-modularity arising by differentiating by \( \tau \). \( \square \)
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Definition (Kaneko-Zagier equation)

$$\vartheta_{k+2} \vartheta_k f_k(\tau) = \frac{k(k+2)}{144} E_4(\tau) f_k(\tau)$$
Theorem (Kaneko-Zagier)

For every \( k \not\equiv 2 \pmod{3} \), the solution \( f_k = 1 + \ldots \) to the KZ equation is a modular form of weight \( k \).
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For every $k \not\equiv 2 \pmod{3}$, the solution $f_k = 1 + \ldots$ to the KZ equation is a modular form of weight $k$.

Proof.

Let $\wp(x, \tau)$ be the Weierstraß function and define the formal power series

$$f(x) = \left( \frac{\wp'(x)}{-2} \right)^{-1/3}.$$

Let $\Phi(y)$ be formal inverse of $f$ in $x$, i.e. $\Phi(f(x)) = x$. Then the coefficients of the expansion

$$\Phi(y) = \sum_{k \geq 1} \frac{f_{k-1}}{k} y^k$$

are modular forms and the desired solutions (invert the diff eqn).

Remark. KZ equation is essentially unique second-order ODE with modular solution. Relations to characters of simple modules for rational vertex operator algebras (Kaneko, Nagatomo, Sakai, 2013).
Let $z \in \mathbb{C}$ elliptic variable, $p = e^{2\pi i z}$. Define the theta function

$$\vartheta_1(z, \tau) = \sum_{\nu \in \mathbb{Z} + \frac{1}{2}} (-1)^{\lfloor \nu \rfloor} p^{\nu} q^{\nu^2/2}$$

and the renormalization

$$\Theta(z, \tau) = \frac{\vartheta_1(z, \tau)}{\eta^3(\tau)}.$$

Let

$$F(z) := \frac{D_{\tau}^2 \Theta(z)}{\Theta(z)} = -\sum_{n \geq 1} \sum_{d|n} (n/d)^3 (p^{d/2} - p^{-d/2})^2 q^n,$$
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**Definition (vIOP, Kaneko-Zagier equation for Jacobi forms)**

For any $m \in \mathbb{Z}$ we consider:

$$\begin{cases} D_\tau^2 \varphi_m = m^2 F \varphi_m \\ \varphi_m = (p^{m/2} - p^{-m/2}) + O(q) \end{cases}$$
The solutions

With \( A = D_z \Theta/\Theta \) one has

\[
\begin{align*}
\varphi_1 &= \Theta \\
\varphi_2 &= 2 \Theta^2 A \\
\varphi_3 &= \frac{9}{2} \Theta^3 A^2 - \frac{3}{2} \Theta^3 \varphi \\
\varphi_4 &= \frac{32}{3} \Theta^4 A^3 - 8 \Theta^4 A \varphi - \frac{2}{3} \Theta^4 D_z(\varphi) \\
\varphi_5 &= \frac{625}{24} \Theta^5 A^4 - \frac{125}{4} \Theta^5 A^2 \varphi + \frac{15}{8} \Theta^5 \varphi^2 - \frac{25}{6} \Theta^5 AD_z(\varphi) + \frac{5}{2} \Theta^5 G_4 \\
&\ldots
\end{align*}
\]
Define the series

\[ f(x) = \frac{\Theta(x)}{\Theta(x + z)} = \frac{x}{\Theta(z)} + \ldots \]

Let \( \Phi(y) \) be formal inverse of \( f \) in \( x \), i.e. \( \Phi(f(x)) = x \). Consider the expansion

\[ \Phi(y) = \sum_{m=1}^{\infty} \frac{\varphi_m}{m} y^m. \]

**Theorem (vIOP)**

The functions \( \varphi_m \) defined above are solutions of the Jacobi KZ equation.

**Corollary**

Every \( \varphi_m \) is a quasi-Jacobi form of index \( |m|/2 \) and weight \(-1\).
Define a second set of functions \( \varphi_{m,n}(z, \tau) \) for \( m, n \in \mathbb{Z} \) as follows:

\[
\begin{cases}
D_\tau \varphi_{m,n} = mn\varphi_m\varphi_n F + (D_\tau \varphi_m)(D_\tau \varphi_n) \\
\varphi_{m,n} = O(q)
\end{cases}
\]

**Theorem (vIOP)**

For all \( m, n \in \mathbb{Z} \) the difference

\[
\varphi_{m,n} - |n| \delta_{m+n,0}
\]

is a quasi-Jacobi form of weight 0 and index \( \frac{1}{2}(|m| + |n|) \).

**Proof.**

For \( m \neq -n \) this is easy:

\[
\varphi_{m,n} = \frac{m}{m+n} \varphi_mD_\tau(\varphi_n) + \frac{n}{m+n} D_\tau(\varphi_m)\varphi_n
\]

For \( m = -n \) this becomes extremely subtle.
Let $g : S \to S$ be a symplectic automorphism of order $N$ of a K3 surface. The associated CHL Calabi-Yau threefold is:

$$X_g = (S \times E)/\langle g \times \tau_N \rangle.$$ 

String theory of $X_g$ studied by David, Jatkar, Sen, .. many others. Our strategy to evaluate its Gromov-Witten theory mathematically:

- Degenerate one period of the torus $E$:

  $$X_g \sim (S \times \mathbb{P}^1)/\sim, \quad (s, 0) \sim (gs, \infty)$$

- Express GW theory of $K3 \times \mathbb{P}^1$ as operator on Fock space
- $g$-twisted traces give topological string partition function
Consider the (negative) K3 lattice
\[ \Lambda = U^{\oplus 4} \oplus E_8^{\oplus 2} \]
and the associated Fock space
\[ \mathcal{F}_{K3} = \bigotimes_{m \geq 1} \text{Sym}^\bullet (\Lambda \otimes q^i). \]

It is acted on by the Heisenberg operators \( \alpha_i(\gamma) \) for \( i \in \mathbb{Z} \) and \( \gamma \in \Lambda \) subject to the commutation rule:
\[
[\alpha_m(\gamma), \alpha_n(\gamma')] = m\delta_{m+n,0}(\gamma, \gamma')\text{id}_\mathcal{F}.
\]

The Fock space \( \mathcal{F}_{K3} \) is freely generated from the action of the creation operators \( \alpha_i(\gamma), i < 0 \) on the vacuum vector \( |\emptyset\rangle \).
For every element $\beta \in \Lambda$ define the 'vertex operator'

$$\omega_\beta : \mathcal{F}_{K3} \to \mathcal{F}_{K3} \otimes \mathbb{Q}[p^{\pm 1/2}] \otimes \left( p^{1/2} - p^{-1/2} \right)^2$$

as the $q^{-\frac{1}{2}(\beta,\beta)}$ coefficient of

$$\Theta^2 \eta^{24} \exp \left( \sum_{m \neq 0} \alpha_m(\beta) \frac{\varphi_m}{m} \right) \exp \left( \frac{1}{2} \sum_{m,n \in \mathbb{Z}\setminus\{0\}} \alpha_m \alpha_n(\Delta) \frac{\varphi_{m,n}}{mn} \right) :$$

Here

- $\Delta \in \Lambda \otimes \Lambda$ is the class of the diagonal,
- $:\quad -$ stands is the normal ordered product,
- $\varphi_m, \varphi_{m,n}$ are the functions defined previously.
- dependence on $p, q$ is suppressed.
Example

- If \((\beta, \beta) > 2\), then \(\omega_\beta = 0\).
- If \((\beta, \beta) = 2\)

\[
\omega_\beta = : \left[ \frac{1}{(s - s^{-1})^2} \exp \left( \sum_{m \neq 0} \frac{(s^m - s^{-m})}{m} \alpha_m(\beta) \right) \right] : 
\]

where \(p = s^2\). This can be expressed in terms of action of \(\hat{gl}(2)\) (Maulik-Oblomkov).

- If \((\beta, \beta) = 0\),

\[
\omega_\beta = : \exp \left( \sum_{m \neq 0} \frac{s^m - s^{-m}}{m} \alpha_m(\beta) \right) \cdot \left[ \frac{(2s^{-2} + 20 + 2s^2)}{(s - s^{-1})^2} - \sum_{m \neq 0} m(s^m - s^{-m})\alpha_m(\beta) \right. \\
\left. - \frac{1}{2} \sum_{\ell, m \neq 0} (s^\ell - s^{-\ell})(s^m - s^{-m})\alpha_\ell \alpha_m(\Delta) \right] : 
\]
Conjecture

Assume $\beta$ primitive. For any $\mu, \nu \in F_n$ we have

$$\text{Coeff}_{p^k} \left( \mu \mid \omega \beta \nu \right)_{F_n} = (-1)^{k-1} \text{PT}^{K3 \times \mathbb{P}^1}_{k+n, (\beta, n)}(\mu, \nu)$$

under suitable identification of boundary conditions: $H^*(\text{Hilb}_n K3) \cong F_n$.

Conjecturally this yields full solution to Pandharipande-Thomas theory of $K3 \times \mathbb{P}^1$ relative to the divisors $K3 \times 0$ and $K3 \times \infty$. 

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**Upshot:** The PT theory of all CHL models becomes computable:

$$\text{PT}^{X_g}_{k, (\beta, n)} = \frac{1}{N} \sum_{\tilde{\beta} \in H_2(S, \mathbb{Z})} \text{PT}^{K3 \times \mathbb{P}^1}_{k+n, (\tilde{\beta}, n)} (\Gamma_g)$$

$$= \frac{1}{N} \sum_{P(\tilde{\beta}) = \beta} (-1)^{k+1} \text{Coeff}_{p^k} \left( (\Gamma_g)_1 \mid \omega_{\tilde{\beta}} (\Gamma_g)_2 \right)_{\mathcal{F}_n}$$

$$= (-1)^{k+1} \frac{1}{N} \text{Coeff}_{p^k} \sum_{P(\tilde{\beta}) = \beta} \text{Tr}_{\mathcal{F}_n} (g \omega_{\tilde{\beta}}).$$
Back to CHL models

Given $g : S \rightarrow S$ symplectic automorphism, let:

- $\Lambda = H^*(S, \mathbb{Z})$ as before.
- $L = H^2(S, \mathbb{Z})^g$ invariant lattice
- $P = \frac{1}{N} \sum_{i=0}^{N-1} g^i$

\[
H_2(X_g, \mathbb{Z})/\text{torsion} \cong \text{Im}(P : L \rightarrow L \otimes \mathbb{Q}) \oplus \mathbb{Z} \\
\cong (L^g)^\vee \oplus \mathbb{Z}
\]

Lemma

Assume $\gamma \in (L^g)^\vee$ primitive. Then $PT_{X_g}^{X_g} k, (\gamma, n)$ only depends on $(X_g, \gamma)$ via $s = \gamma^2$, $\alpha = [\gamma] \in (L^g)^\vee / L^g$, and $g : L \rightarrow L$ up to conjugation. We write $PT_{k,s,n,\alpha}$ instead.
CHL models II

**Definition**

Given $\alpha \in (L^g)^\vee / L^g$ define

$$Z_\alpha = \sum_{k,s,n} \text{PT}_{k,s,n,\alpha} (-1)^{k+1} p^k t^{n-1} q^{s/2}.$$ 

**Conjecture**

*Every* $Z_\alpha$ *is a Siegel modular form of genus 2 (for a congruence subgroup).*

**Computation scheme:**

$$Z_\alpha = \frac{1}{N} \sum_{s} \sum_{P(\bar{\beta}) = \beta_s} \text{Tr}_F (g \omega_{\bar{\beta}} t^{N-1}).$$

where $N$ is the energy operator: $N|_{F_n} = n \cdot \text{id}_{F_n}$, and $\beta_s \in (L^g)^\vee$ is a primitive class of square $\beta_s^2 = s$ and $[\beta_s] = \alpha$. 

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Example: $N = 1$

Simplest case: $g = \text{id}$, then $X_g = K3 \times E$. It is well-known:

$$Z^{K3 \times E} = \frac{1}{\chi_{10}(\tau, z)}$$

where $p = e^{2\pi iz}$, $q = e^{2\pi i\tau}$ and $t = e^{2\pi i\bar{\tau}}$. Hence on expects:

$$\text{Tr}_\mathcal{F} t^{N-1} \omega_\beta = \text{Coeff}_{q^{\beta^2/2}} \left[ \frac{1}{\chi_{10}(\Omega)} \right]$$

→ Strong evidence available (exact expressions later)
Example: $N = 2$

In order 2 the involution $g$ interchanging the $E_8$ factors.

$$L^g = E_8(-2) \oplus U^3$$

$$D(L^g) = (L^g)^{\vee} / L^g = D(E_8(2)) = \mathbb{Z}_2^8$$

The Weyl group acts on the discriminant group with three orbits $0, \alpha_1, \alpha_2$. If $[\gamma] \neq 0$, then its orbit is determined by $\gamma^2$. Hence we obtain two cases:

- If $[\gamma] = 0$ (untwisted case) $Z_{\text{untwisted}} = Z_{0}^{N=2}$
- If $[\gamma] \neq 0$ (twisted case) $Z_{\text{twisted}} = Z_{\alpha_1}^{N=2} + Z_{\alpha_2}^{N=2}$
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$$Z_{\text{twisted}} = \frac{1}{2^4} \frac{1}{\Phi_{6,3}(\Omega)}$$

(Ref: Jatkar and A. Sen, Dyon spectrum in CHL models; David, D. P. Jatkar and A. Sen Product Representation of Dyon Partition Function in CHL Models)
By joint work with Bryan one has:

\[
Z_{\text{untwisted}} = -\frac{1}{2} \left( \frac{1}{\Phi_{6,0}} + \frac{1}{2^4 \Phi_{6,3}} + \frac{1}{2^4 \Phi_{6,4}} \right)
\]

(the simplified formula here is from [Fischbach, Klemm, Nega - Lost Chapters in CHL Black Holes .. ])

\[
\frac{1}{\Phi_{6,0}} = \frac{\theta_0^2 \theta_{0001}^2 \theta_{0010}^2 \theta_{0011}^2}{\chi_{10}} \tag{1}
\]

\[
\frac{1}{\Phi_{6,3}} = \frac{\theta_0^2 \theta_{0010}^2 \theta_{0100}^2 \theta_{0110}^2}{\chi_{10}} \tag{2}
\]

\[
\frac{1}{\Phi_{6,4}} = -\frac{\theta_0^2 \theta_{0011}^2 \theta_{0100}^2 \theta_{0110}^2}{\chi_{10}}. \tag{3}
\]
The computation scheme

\[ Z^N_{\alpha} = \frac{1}{N} \sum_s \sum_{P(\beta) = \beta_s} \text{Tr}_F(g \omega_\beta t^{N-1}). \]

yields conjectural identities relating Siegel modular forms with the functions \( \varphi_{mn} \). Already \( g = \text{id} \) is very interesting, it gives the following:

Define functions \( L(z, \tau, \tilde{\tau}) \) and \( M(z, \tau, \tilde{\tau}) \) by

\[
L = \sum_{r \geq 1} (-1)^{r-1} \sum_{d_1, \ldots, d_r \in \mathbb{Z} \setminus 0} \frac{1}{|d_1| \cdots |d_r|} \frac{\tilde{q}^{d_1}}{1 - \tilde{q}^{d_1}} \cdots \frac{\tilde{q}^{d_r}}{1 - \tilde{q}^{d_r}} \varphi_{d_1} \varphi_{-d_1,d_2} \cdots \varphi_{-d_r-1,d_r} \varphi_{-d_r}
\]

\[
M = \sum_{r \geq 1} \frac{(-1)^{r-1}}{r} \sum_{d_1, \ldots, d_r \in \mathbb{Z} \setminus 0} \frac{1}{|d_1| \cdots |d_r|} \frac{\tilde{q}^{d_1}}{1 - \tilde{q}^{d_1}} \cdots \frac{\tilde{q}^{d_r}}{1 - \tilde{q}^{d_r}} \varphi_{d_1,-d_2} \cdots \varphi_{d_r,-d_1}
\]

**Conjecture**

\[
\sum_{k \geq 0} \frac{1}{k!} D^k (\exp(M) \Theta(z, \tau)^2 \Delta(\tau) \Delta(\tilde{\tau})) = \frac{1}{\chi_{10} \left( \frac{\tau}{z} \frac{z}{\tilde{\tau}} \right)}
\]
Main open questions:

- What is the algebra structure of the operators $\omega_\beta$?
- Are there relations to the Conway Moonshine VOA?
- Mathematical proofs
- Non-commutative CHL models
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Thank you! (the end)

(Kummer K3 surface, source:mo-labs.com)