1. An enumerative problem

Let $S$ be a smooth complex projective surface which we assume here for simplicity to be Fano (in particular, $p_g = q = 0$). Let $L$ be a line bundle with no higher cohomology. We are interested in counting curves in the linear system $|L|$ of given geometric genus and gonality.

**Definition 1.** A smooth proper connected curve $C$ is $n$-gonal if there exists a morphism $C \to \mathbb{P}^1$ of degree $n$.

**Definition 2.** Let $N_{g,n}(L)$ be the number of irreducible curves $C \in |L|$ such that:

(i) the normalization $\tilde{C}$ is $n$-gonal of genus $g$

(ii) $C$ passes through $\ell(g,n)$ generic points.

Here $\ell(g,n)$ is the number of points which makes the problem of expected dimension 0. To find it recall first that because the Brill-Noether number reads $\rho(g,a,d) = g - (a + 1)(g - d + a)$, in a given family of genus $g$ curves the loci of $n$-gonal curves has expected codimension $-\rho(g,1,n) = g + 2 - 2n$. Second, the locus of geometric genus $g$ curves in a family arithmetic genus $p_a$ curves is of expected codimension $p_a - g$. Let $p_a(L)$ be the arithmetic genus of a curve in $|L|$. Hence

\[
\ell = \ell(g,n) = \dim |L| - (p_a(L) - g) + \rho(g,1,n)
\]

\[
= \frac{1}{2}L \cdot (L - K) - \left( \frac{1}{2}L \cdot (K + L) + 1 - g \right) - (g + 2 - 2n)
\]

\[
= c_1(S) \cdot L - 1 + 2n - 2.
\]

2. Hilbert schemes

By a classical idea of Graber, the Hilbert scheme of $n$ points $S^{[n]}$ can be used to approach the count $N_{g,n}(L)$. By definition a morphism $T \to S^{[n]}$ from a Noetherian scheme $T$ corresponds to a closed subscheme $C \subset T \times S$ flat over $T$ of degree $n$. Hence we find the natural bijection:

\[
\left\{ \text{maps } f : \mathbb{P}^1 \to S^{[n]} \right\} \cong \left\{ \text{subcurves } C \subset \mathbb{P}^1 \times S \text{ flat over } \mathbb{P}^1 \text{ of degree } n \right\}.
\]

Moreover, as explained in [3 Sec. 1] the map $f$ has class $\beta + kA$ under the natural isomorphism $H_2(S^{[n]},\mathbb{Z}) \cong H_2(S,\mathbb{Z}) \oplus \mathbb{Z}A$ if and only if we have $[C] = \beta + n[\mathbb{P}^1] \in H_2(S \times \mathbb{P}^1,\mathbb{Z})$ and $\chi(O_C) = k + n$. Similarly, the projection of $C$ to $S$ is incident to a point $P \in S$ if and only if $f(\mathbb{P}^1)$ is incident to the cycle $I(P) = \{ \xi \in S^{[n]} | P \in \xi \}$.

Define the genus $g$ Gromov-Witten invariant of the Hilbert scheme:

\[
\langle \alpha; \gamma_1, \ldots, \gamma_N \rangle_{g,\beta+kA}^{S^{[n]}} := \int_{[\mathbb{P}^1,\beta+kA]}^{\text{vir}} \ev_1^*(\gamma_1) \cdots \ev_N^*(\gamma_N) \tau^*(\alpha)
\]
where $\alpha$ is a tautological class on $\overline{M}_{g,N}$, which is the target of the forgetful morphism $\tau$. A virtual count $H_{g,n}(\beta)$ of $n$-gonal genus $g$ curves on $S$ in class $\beta$ passing through $\ell$ points is then defined by

$$
\sum_{k \in \mathbb{Z}} \langle 1(P)^k \rangle_{0,\beta+kA}^{|S^{[n]}|} = \sum_{g} H_{g,n}(\beta)(p^{-1/2} + p^{1/2})^{2n+2g-2}.
$$

The justification for this is that for an isolated genus $g$ curves $C \subset S \times \mathbb{P}^1$, the corresponding map $f: \mathbb{P}^1 \to S^{[n]}$ meets the diagonal $\Delta_{S^{[n]}}$ in $2n + 2g - 2$ points, and by Graber each of these intersection points should contribute $p^{-1/2} + p^{1/2}$ to the left hand side. In particular Graber proves:

**Theorem 1** ([1]). For $S = \mathbb{P}^2$ the count $H_{g,2}(\beta)$ is enumerative, or in other words equal to $N_{g,2}(\beta)$. For an explicit recursion see [1].

### 3. K3 surfaces

The above discussion motivates the study of the Gromov-Witten theory of the Hilbert scheme of points of a K3 surface. We state a triality of conjectures which governs the structure of the theory. Let $S \to \mathbb{P}^1$ be an elliptic K3 surface with section $B$ and fiber class $F$. We define potential of reduced Gromov-Witten invariants:

$$
F_{g,m}(\alpha; \gamma_1, \ldots, \gamma_N) = \sum_{d=-m}^{\infty} \sum_{r \in \mathbb{Z}} \langle \alpha; \gamma_1, \ldots, \gamma_N \rangle_{g,m(B+F)+dF+kA} q^d (-p)^k.
$$

By deformation invariance these series determine all Gromov-Witten invariants of hyper-Kähler varieties of $K3^{[n]}$ type [4]. By convention, we assume $k = 0$ for $n = 1$. Recall the algebra $QJac$ of quasi-Jacobi forms [3].

**Conjecture A.** $F_{g,m}(\alpha; \gamma_1, \ldots, \gamma_N)$ is a quasi-Jacobi form of index $n - 1$ and weight $n(2g-2) + \sum \deg(\gamma_i) - 10$ of the form

$$
F_{g,m}(\alpha; \gamma_1, \ldots, \gamma_N) \in \frac{1}{\Delta(q)} QJac.
$$

Here, if $\gamma \in H^*(S^{[n]})$ is written in terms of the action of Nakajima operators

$$
\gamma = \prod_i q_{a_i}(\delta_i)1, \quad 1 \in H^*(S^{[0]})
$$

where $\delta_i$ are elements of a fixed basis $\{W := B + F, F, p, 1, e_3, \ldots, e_{22}\}$ with $e_i \in H^2(S)$ orthogonal to $W, F$, then the modified degree function $\deg$ is defined by

$$
\deg(\gamma) = \deg(\gamma) + w(\gamma) - f(\gamma)
$$

where $w(\gamma)$ and $f(\gamma)$ are the number of classes $\delta_i$ equal to $W$ and $F$ respectively.

**Conjecture B.** We have the multiple cover conjecture:

$$
F_{g,m}(\alpha; \gamma_1, \ldots, \gamma_N) = m \sum_i \deg(\gamma_i) - \deg(\gamma_i) \cdot T_{m,\ell} F_{g,1}(\alpha; \gamma_1, \ldots, \gamma_N)
$$

where $\ell = n(2g-2) + \sum \deg(\gamma_i)$ and $T_{m,\ell}$ is the formal Hecke operator on Jacobi forms, see [4, 2.6].

Conjecture [3] implies that every $F_{g,m}$ is a quasi-Jacobi form (with poles at $q = 0$) of index $m(n-1)$ for the congruence subgroup $\Gamma_0(n) \times \mathbb{Z}^2$. The weight is as before.
Conjecture C. We have the holomorphic anomaly equation:

\[ \frac{d}{dG_2} F_{g,m}(\alpha; \gamma_1, \ldots, \gamma_N) = F_{g-1,m}(\alpha; \gamma_1, \ldots, \gamma_N, U) \]

\[ + 2 \sum_{g=g_1+g_2, \{1,\ldots,N\}=A\sqcup B} F_{g_1,m}(\alpha; \gamma_A, U_1) F_{g_2}^{\text{vir}}(\alpha_2; \gamma_B, U_2) \]

\[ - 2 \sum_{i=1}^N F_{g,m}(\alpha; q^i(\psi_1); \gamma_1, \ldots, \gamma_{i-1}, U \gamma_i, \gamma_{i+1}, \ldots, \gamma_N) \]

\[ - \frac{1}{m} \sum_{a,b} (G^{-1})_{ab} T_{e_a, T_{e_b}} F_{g,m}(\alpha; \gamma_1, \ldots, \gamma_N) \]

with the following notations:

- by convention the last term vanishes in case \( m = 0 \),
- the intersection matrix \( G \) of the \( e_a \) is defined by \( G_{ab} = \langle e_a, e_b \rangle \),
- we let \( \rho : \wedge^2 H^2(X) \cong so(\langle H^2(X) \rangle) \to \text{End} H^*(X) \) be the Looijenga-Lunts-Verbitsky algebra action for \( X = S^{[n]} \) with the conventions of [2],
- \( U = \hat{f}_F = \rho(-f \wedge F) \),
- \( T_\lambda F_{g,m}(\alpha; \gamma_1, \ldots, \gamma_N) = \sum_{i=1}^N F_{g,m}(\alpha; \gamma_1, \ldots, \gamma_{i-1}, \rho(\lambda \wedge F) \gamma_i, \gamma_{i+1}, \ldots, \gamma_N) \),
- \( q : \overline{M}_{g,N}(S^{[n]}, \beta) \to \overline{M}_{g,N}(\mathbb{P}^n, \pi_+ \beta) \) is induced by the Lagrangian fibration \( \pi : S^{[n]} \to \mathbb{P}^1 \),
- \( F^{\text{vir}}_g \) is the potential of ordinary (non-reduced) Gromov-Witten invariants.

The first two conjectures can be found in [3] and [4]. The last one generalizes the K3 surface case [5]. Example calculations will be discussed elsewhere.

References


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