Algebra II, First lecture

October 20, 2021

Topics of the lecture in this term:

Algebra II, First lecture

October 20, 2021

Topics of the lecture in this term:

- Hilbert polymomials

Algebra II, First lecture

October 20, 2021

Topics of the lecture in this term:

- Hilbert polymomials
- Dimension theory

Algebra II, First lecture

October 20, 2021

Topics of the lecture in this term:

- Hilbert polymomials
- Dimension theory
- The principal ideal theorem.

Algebra II, First lecture

October 20, 2021

Topics of the lecture in this term:

- Hilbert polymomials
- Dimension theory
- The principal ideal theorem.
- Possibly proof of Bezout's theorem.

Algebra II, First lecture

October 20, 2021

Topics of the lecture in this term:

- Hilbert polymomials
- Dimension theory
- The principal ideal theorem.
- Possibly proof of Bezout's theorem.
- Possibly an introduction to cohomological methods in commutative algebra.

Algebra II, First lecture

October 20, 2021

Topics of the lecture in this term:

- Hilbert polymomials
- Dimension theory
- The principal ideal theorem.
- Possibly proof of Bezout's theorem.
- Possibly an introduction to cohomological methods in commutative algebra.
- Regular rings.

Topics of this lecture

- The length of a module.

Topics of this lecture

- The length of a module.
- Hilbert polynomials.

Topics of this lecture

- The length of a module.
- Hilbert polynomials.
- The notion of an Artinian ring.

Topics of this lecture

- The length of a module.
- Hilbert polynomials.
- The notion of an Artinian ring.

Unless otherwise specified, $\mathfrak{k}$ will always denote a field and R a ring.

Topics of this lecture

- The length of a module.
- Hilbert polynomials.
- The notion of an Artinian ring.

Unless otherwise specified, $\mathfrak{k}$ will always denote a field and R a ring. I will use the abbreviations PID for "principal ideal domain" and UFD for "factorial domain".

The length of a module

Definition

The length of an R-module M is the largest natural number n for which there exists a strictly ascending sequence $0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$ of submodules of M,

The length of a module

Definition

The length of an R-module M is the largest natural number n for which there exists a strictly ascending sequence $0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$ of submodules of M, or ∞ if such sequences exist for arbitrary natural numbers n.

The length of a module

Definition

The length of an R-module M is the largest natural number n for which there exists a strictly ascending sequence $0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$ of submodules of M, or ∞ if such sequences exist for arbitrary natural numbers n. In the first case, we say that M has finite length, otherwise it has infinite length.

The length of a module

Definition

The length of an R-module M is the largest natural number n for which there exists a strictly ascending sequence $0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$ of submodules of M, or ∞ if such sequences exist for arbitrary natural numbers n. In the first case, we say that M has finite length, otherwise it has infinite length. The length of M is written $\ell(M) = \ell_R(M)$.

The length of a module

Definition

The length of an R-module M is the largest natural number n for which there exists a strictly ascending sequence $0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$ of submodules of M, or ∞ if such sequences exist for arbitrary natural numbers n. In the first case, we say that M has finite length, otherwise it has infinite length. The length of M is written $\ell(M) = \ell_R(M)$. We say that M is simple if $\ell(M) = 1$,

The length of a module

Definition

The length of an R-module M is the largest natural number n for which there exists a strictly ascending sequence $0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$ of submodules of M, or ∞ if such sequences exist for arbitrary natural numbers n. In the first case, we say that M has finite length, otherwise it has infinite length. The length of M is written $\ell(M) = \ell_R(M)$. We say that M is simple if $\ell(M) = 1$, in other words, if it has precisely the two submodules $\{0\}$ and M.

The length of a module

Definition

The length of an R-module M is the largest natural number n for which there exists a strictly ascending sequence $0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$ of submodules of M, or ∞ if such sequences exist for arbitrary natural numbers n. In the first case, we say that M has finite length, otherwise it has infinite length. The length of M is written $\ell(M) = \ell_R(M)$. We say that M is simple if $\ell(M) = 1$, in other words, if it has precisely the two submodules $\{0\}$ and M.

Example

If R is a field, then an R-vector space has finite length if and only if it is finite dimensional, and its length is equal to its dimension.

The length of a module

Definition

The length of an R-module M is the largest natural number n for which there exists a strictly ascending sequence $0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$ of submodules of M, or ∞ if such sequences exist for arbitrary natural numbers n. In the first case, we say that M has finite length, otherwise it has infinite length. The length of M is written $\ell(M) = \ell_R(M)$. We say that M is simple if $\ell(M) = 1$, in other words, if it has precisely the two submodules $\{0\}$ and M.

Example

If R is a field, then an R-vector space has finite length if and only if it is finite dimensional, and its length is equal to its dimension.

Example

The rings $R = \mathbb{Z}$ or $R = \mathfrak{k}[T]$ have infinite length as modules over themselves.

The length of a module

Definition

The length of an R-module M is the largest natural number n for which there exists a strictly ascending sequence $0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$ of submodules of M, or ∞ if such sequences exist for arbitrary natural numbers n. In the first case, we say that M has finite length, otherwise it has infinite length. The length of M is written $\ell(M) = \ell_R(M)$. We say that M is simple if $\ell(M) = 1$, in other words, if it has precisely the two submodules $\{0\}$ and M.

Example

If R is a field, then an R-vector space has finite length if and only if it is finite dimensional, and its length is equal to its dimension.

Example

The rings $R = \mathbb{Z}$ or $R = \mathfrak{k}[T]$ have infinite length as modules over themselves. The same holds for any UFD which is not a field,

The length of a module

Definition

The length of an R-module M is the largest natural number n for which there exists a strictly ascending sequence $0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$ of submodules of M, or ∞ if such sequences exist for arbitrary natural numbers n. In the first case, we say that M has finite length, otherwise it has infinite length. The length of M is written $\ell(M) = \ell_R(M)$. We say that M is simple if $\ell(M) = 1$, in other words, if it has precisely the two submodules $\{0\}$ and M.

Example

If R is a field, then an R-vector space has finite length if and only if it is finite dimensional, and its length is equal to its dimension.

Example

The rings $R = \mathbb{Z}$ or $R = \mathfrak{k}[T]$ have infinite length as modules over themselves. The same holds for any UFD which is not a field, because when $p \in R$ is a prime element then $R \supset pR \supset p^2R \supset \cdots \supset p^kR \supset p^{k+1}R \supset \ldots$ is an infinite strictly descending chain of ideals in R.

The length of a module

Definition

The length of an R-module M is the largest natural number n for which there exists a strictly ascending sequence $0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$ of submodules of M, or ∞ if such sequences exist for arbitrary natural numbers n. In the first case, we say that M has finite length, otherwise it has infinite length. The length of M is written $\ell(M) = \ell_R(M)$. We say that M is simple if $\ell(M) = 1$, in other words, if it has precisely the two submodules $\{0\}$ and M.

Example

If R is a field, then an R-vector space has finite length if and only if it is finite dimensional, and its length is equal to its dimension.

Example

The rings $R = \mathbb{Z}$ or $R = \mathfrak{k}[T]$ have infinite length as modules over themselves. The same holds for any UFD which is not a field, because when $p \in R$ is a prime element then $R \supset pR \supset p^2R \supset \cdots \supset p^kR \supset p^{k+1}R \supset \ldots$ is an infinite strictly descending chain of ideals in R. In fact, the same can be shown for any domain which is not a field.

The length of a module

Definition

The length of an R-module M is the largest natural number n for which there exists a strictly ascending sequence $0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$ of submodules of M, or ∞ if such sequences exist for arbitrary natural numbers n. In the first case, we say that M has finite length, otherwise it has infinite length. The length of M is written $\ell(M) = \ell_R(M)$. We say that M is simple if $\ell(M) = 1$, in other words, if it has precisely the two submodules $\{0\}$ and M.

Example

If R is a field, then an R-vector space has finite length if and only if it is finite dimensional, and its length is equal to its dimension.

Example

The rings $R = \mathbb{Z}$ or $R = \mathfrak{k}[T]$ have infinite length as modules over themselves. The same holds for any UFD which is not a field, because when $p \in R$ is a prime element then $R \supset pR \supset p^2R \supset \cdots \supset p^kR \supset p^{k+1}R \supset \ldots$ is an infinite strictly descending chain of ideals in R. In fact, the same can be shown for any domain which is not a field.

Example

We have $\ell(M) = 0$ if and only if $M = \{0\}$.

More examples on ℓ

Fact

If I is an ideal in R and M an R-module with $I \cdot M = 0$, then $\ell_R(M) = \ell_{R/I}(M)$.

More examples on ℓ

Fact

If I is an ideal in R and M an R-module with $I \cdot M = 0$, then $\ell_R(M) = \ell_{R/I}(M)$.

This is so because the sets of R-submodules and of R/I-submodules of M conicide.

More examples on ℓ

Fact

If I is an ideal in R and M an R-module with $I \cdot M = 0$, then $\ell_R(M) = \ell_{R/I}(M)$.

This is so because the sets of R-submodules and of R/I-submodules of M conicide.

Example

When R is a PID and $r \in R \setminus \{0\}$ then the R/rR has finite length as an R-module and $\ell(R/rR)$ is equal to the number of prime factors of r counted by multiplicity.

More examples on ℓ

Fact

If I is an ideal in R and M an R-module with $I \cdot M = 0$, then $\ell_R(M) = \ell_{R/I}(M)$.

This is so because the sets of R-submodules and of R/I-submodules of M conicide.

Example

When R is a PID and $r \in R \setminus \{0\}$ then the R/rR has finite length as an R-module and $\ell(R/rR)$ is equal to the number of prime factors of r counted by multiplicity. For instance, the length of $\mathbb{Z}/2021\mathbb{Z}$ as a module over $\mathbb{Z}$ or over itself is 2.

More examples on ℓ

Fact

If I is an ideal in R and M an R-module with $I \cdot M = 0$, then $\ell_R(M) = \ell_{R/I}(M)$.

This is so because the sets of R-submodules and of R/I-submodules of M conicide.

Example

When R is a PID and $r \in R \setminus \{0\}$ then the R/rR has finite length as an R-module and $\ell(R/rR)$ is equal to the number of prime factors of r counted by multiplicity. For instance, the length of $\mathbb{Z}/2021\mathbb{Z}$ as a module over $\mathbb{Z}$ or over itself is 2.

We will see that the following holds.

Proposition (Akizuki-Hopkins)

For a ring R, the following properties are equivalent:

More examples on ℓ

Fact

If I is an ideal in R and M an R-module with $I \cdot M = 0$, then $\ell_R(M) = \ell_{R/I}(M)$.

This is so because the sets of R-submodules and of R/I-submodules of M conicide.

Example

When R is a PID and $r \in R \setminus \{0\}$ then the R/rR has finite length as an R-module and $\ell(R/rR)$ is equal to the number of prime factors of r counted by multiplicity. For instance, the length of $\mathbb{Z}/2021\mathbb{Z}$ as a module over $\mathbb{Z}$ or over itself is 2.

We will see that the following holds.

Proposition (Akizuki-Hopkins)

For a ring R, the following properties are equivalent:

A *R is Noetherian and has Krull dimension zero.*

More examples on ℓ

Fact

If I is an ideal in R and M an R-module with $I \cdot M = 0$, then $\ell_R(M) = \ell_{R/I}(M)$.

This is so because the sets of R-submodules and of R/I-submodules of M conicide.

Example

When R is a PID and $r \in R \setminus \{0\}$ then the R/rR has finite length as an R-module and $\ell(R/rR)$ is equal to the number of prime factors of r counted by multiplicity. For instance, the length of $\mathbb{Z}/2021\mathbb{Z}$ as a module over $\mathbb{Z}$ or over itself is 2.

We will see that the following holds.

Proposition (Akizuki-Hopkins)

For a ring R, the following properties are equivalent:

A *R is Noetherian and has Krull dimension zero.*

B *R has finite length as a module over itself.*

More examples on ℓ

Fact

If I is an ideal in R and M an R-module with $I \cdot M = 0$, then $\ell_R(M) = \ell_{R/I}(M)$.

This is so because the sets of R-submodules and of R/I-submodules of M conicide.

Example

When R is a PID and $r \in R \setminus \{0\}$ then the R/rR has finite length as an R-module and $\ell(R/rR)$ is equal to the number of prime factors of r counted by multiplicity. For instance, the length of $\mathbb{Z}/2021\mathbb{Z}$ as a module over $\mathbb{Z}$ or over itself is 2.

We will see that the following holds.

Proposition (Akizuki-Hopkins)

For a ring R, the following properties are equivalent:

- A *R is Noetherian and has Krull dimension zero.*
- B *R has finite length as a module over itself.*
- C *There is no infinite descending chain $R = I_0 \supset I_1 \supset \cdots \supset I_k \supset I_{k+1} \supset \ldots$ of ideals in R.*

More examples on ℓ

Fact

If I is an ideal in R and M an R-module with $I \cdot M = 0$, then $\ell_R(M) = \ell_{R/I}(M)$.

This is so because the sets of R-submodules and of R/I-submodules of M conicide.

Example

When R is a PID and $r \in R \setminus \{0\}$ then the R/rR has finite length as an R-module and $\ell(R/rR)$ is equal to the number of prime factors of r counted by multiplicity. For instance, the length of $\mathbb{Z}/2021\mathbb{Z}$ as a module over $\mathbb{Z}$ or over itself is 2.

We will see that the following holds.

Proposition (Akizuki-Hopkins)

For a ring R, the following properties are equivalent:

- A *R is Noetherian and has Krull dimension zero.*
- B *R has finite length as a module over itself.*
- C *There is no infinite descending chain $R = I_0 \supset I_1 \supset \cdots \supset I_k \supset I_{k+1} \supset \ldots$ of ideals in R.*
- B' *For every finitely generated R-module M, $\ell_R(M)$ is finite.*

More examples on ℓ

Fact

If I is an ideal in R and M an R-module with $I \cdot M = 0$, then $\ell_R(M) = \ell_{R/I}(M)$.

This is so because the sets of R-submodules and of R/I-submodules of M conicide.

Example

When R is a PID and $r \in R \setminus \{0\}$ then the R/rR has finite length as an R-module and $\ell(R/rR)$ is equal to the number of prime factors of r counted by multiplicity. For instance, the length of $\mathbb{Z}/2021\mathbb{Z}$ as a module over $\mathbb{Z}$ or over itself is 2.

We will see that the following holds.

Proposition (Akizuki-Hopkins)

For a ring R, the following properties are equivalent:

- A *R is Noetherian and has Krull dimension zero.*
- B *R has finite length as a module over itself.*
- C *There is no infinite descending chain $R = I_0 \supset I_1 \supset \cdots \supset I_k \supset I_{k+1} \supset \dots$ of ideals in R.*
- B' *For every finitely generated R-module M, $\ell_R(M)$ is finite.*
- C' *For every finitely generated R-module M, There is no infinite descending chain $M = M_0 \supset M_1 \supset \cdots \supset M_k \supset M_{k+1} \supset \dots$ of submodules of M.*

More examples on ℓ

Fact

If I is an ideal in R and M an R-module with $I \cdot M = 0$, then $\ell_R(M) = \ell_{R/I}(M)$.

This is so because the sets of R-submodules and of R/I-submodules of M conicide.

Example

When R is a PID and $r \in R \setminus \{0\}$ then the R/rR has finite length as an R-module and $\ell(R/rR)$ is equal to the number of prime factors of r counted by multiplicity. For instance, the length of $\mathbb{Z}/2021\mathbb{Z}$ as a module over $\mathbb{Z}$ or over itself is 2.

We will see that the following holds.

Proposition (Akizuki-Hopkins)

For a ring R, the following properties are equivalent:

- A *R is Noetherian and has Krull dimension zero.*
- B *R has finite length as a module over itself.*
- C *There is no infinite descending chain $R = I_0 \supset I_1 \supset \cdots \supset I_k \supset I_{k+1} \supset \dots$ of ideals in R.*
- B' *For every finitely generated R-module M, $\ell_R(M)$ is finite.*
- C' *For every finitely generated R-module M, There is no infinite descending chain $M = M_0 \supset M_1 \supset \cdots \supset M_k \supset M_{k+1} \supset \dots$ of submodules of M.*

I postpone the proof, of which only the relatively easy part will be relevant for the exam.

More examples on ℓ

Fact

If I is an ideal in R and M an R-module with $I \cdot M = 0$, then $\ell_R(M) = \ell_{R/I}(M)$.

This is so because the sets of R-submodules and of R/I-submodules of M conicide.

Example

When R is a PID and $r \in R \setminus \{0\}$ then the R/rR has finite length as an R-module and $\ell(R/rR)$ is equal to the number of prime factors of r counted by multiplicity. For instance, the length of $\mathbb{Z}/2021\mathbb{Z}$ as a module over $\mathbb{Z}$ or over itself is 2.

We will see that the following holds.

Proposition (Akizuki-Hopkins)

For a ring R, the following properties are equivalent:

- A *R is Noetherian and has Krull dimension zero.*
- B *R has finite length as a module over itself.*
- C *There is no infinite descending chain $R = I_0 \supset I_1 \supset \cdots \supset I_k \supset I_{k+1} \supset \dots$ of ideals in R.*
- B' *For every finitely generated R-module M, $\ell_R(M)$ is finite.*
- C' *For every finitely generated R-module M, There is no infinite descending chain $M = M_0 \supset M_1 \supset \cdots \supset M_k \supset M_{k+1} \supset \dots$ of submodules of M.*

I postpone the proof, of which only the relatively easy part will be relevant for the exam.

Definition

A ring is called Artinian if it has these equivalent properties.

Additivity of length

Proposition

Let $0 \to M' \xrightarrow{\iota} M \xrightarrow{\pi} M'' \to 0$ *be a short exact sequence of* R*-modules and* $\ell = \ell_R$*. Then* $\ell(M) = \ell(M') + \ell(M'')$*, where the equality is between elements of* $\mathbb{N} \cup \{\infty\}$*.*

Additivity of length

Proposition

Let $0 \to M' \xrightarrow{\iota} M \xrightarrow{\pi} M'' \to 0$ *be a short exact sequence of R-modules and* $\ell = \ell_R$*. Then* $\ell(M) = \ell(M') + \ell(M'')$*, where the equality is between elements of* $\mathbb{N} \cup \{\infty\}$*.*

This again confirms the suspicion that length of modules has properties similar to the dimension of vector spaces.

Additivity of length

Proposition

Let $0 \to M' \xrightarrow{\iota} M \xrightarrow{\pi} M'' \to 0$ *be a short exact sequence of R-modules and* $\ell = \ell_R$*. Then* $\ell(M) = \ell(M') + \ell(M'')$*, where the equality is between elements of* $\mathbb{N} \cup \{\infty\}$*.*

This again confirms the suspicion that length of modules has properties similar to the dimension of vector spaces. The rest of this page and the next page are devoted to the proof.

Additivity of length

Proposition

Let $0 \to M' \xrightarrow{\iota} M \xrightarrow{\pi} M'' \to 0$ *be a short exact sequence of R-modules and* $\ell = \ell_R$*.*
Then $\ell(M) = \ell(M') + \ell(M'')$*, where the equality is between elements of* $\mathbb{N} \cup \{\infty\}$*.*

This again confirms the suspicion that length of modules has properties similar to the dimension of vector spaces. The rest of this page and the next page are devoted to the proof.

It is easy to see that $\ell(M') \leq \ell(M)$, hence the case where $\ell(M') = \infty$ is trivial.

Additivity of length

Proposition

Let $0 \to M' \xrightarrow{\iota} M \xrightarrow{\pi} M'' \to 0$ *be a short exact sequence of R-modules and* $\ell = \ell_R$*.* *Then* $\ell(M) = \ell(M') + \ell(M'')$*, where the equality is between elements of* $\mathbb{N} \cup \{\infty\}$*.*

This again confirms the suspicion that length of modules has properties similar to the dimension of vector spaces. The rest of this page and the next page are devoted to the proof.
It is easy to see that $\ell(M') \leq \ell(M)$, hence the case where $\ell(M') = \infty$ is trivial. It is also clear that the strict inequality $\ell(M') < \ell(M)$ holds when $M' \subset M$ and $\ell(M') < \infty$.

Additivity of length

Proposition

Let $0 \to M' \xrightarrow{\iota} M \xrightarrow{\pi} M'' \to 0$ *be a short exact sequence of* R*-modules and* $\ell = \ell_R$*. Then* $\ell(M) = \ell(M') + \ell(M'')$*, where the equality is between elements of* $\mathbb{N} \cup \{\infty\}$*.*

This again confirms the suspicion that length of modules has properties similar to the dimension of vector spaces. The rest of this page and the next page are devoted to the proof.
It is easy to see that $\ell(M') \leq \ell(M)$, hence the case where $\ell(M') = \infty$ is trivial. It is also clear that the strict inequality $\ell(M') < \ell(M)$ holds when $M' \subset M$ and $\ell(M') < \infty$. Let us assume for the moment that the assertion holds when $\ell(M') = 1$.

Additivity of length

Proposition

Let $0 \to M' \xrightarrow{\iota} M \xrightarrow{\pi} M'' \to 0$ *be a short exact sequence of* R*-modules and* $\ell = \ell_R$*. Then* $\ell(M) = \ell(M') + \ell(M'')$*, where the equality is between elements of* $\mathbb{N} \cup \{\infty\}$*.*

This again confirms the suspicion that length of modules has properties similar to the dimension of vector spaces. The rest of this page and the next page are devoted to the proof.

It is easy to see that $\ell(M') \leq \ell(M)$, hence the case where $\ell(M') = \infty$ is trivial. It is also clear that the strict inequality $\ell(M') < \ell(M)$ holds when $M' \subset M$ and $\ell(M') < \infty$. Let us assume for the moment that the assertion holds when $\ell(M') = 1$. For $\ell(M') < \infty$, we may then use induction on $\ell(M')$ as follows:

Additivity of length

Proposition

Let $0 \to M' \xrightarrow{\iota} M \xrightarrow{\pi} M'' \to 0$ be a short exact sequence of R-modules and $\ell = \ell_R$. Then $\ell(M) = \ell(M') + \ell(M'')$, where the equality is between elements of $\mathbb{N} \cup \{\infty\}$.

This again confirms the suspicion that length of modules has properties similar to the dimension of vector spaces. The rest of this page and the next page are devoted to the proof.

It is easy to see that $\ell(M') \leq \ell(M)$, hence the case where $\ell(M') = \infty$ is trivial. It is also clear that the strict inequality $\ell(M') < \ell(M)$ holds when $M' \subset M$ and $\ell(M') < \infty$. Let us assume for the moment that the assertion holds when $\ell(M') = 1$. For $\ell(M') < \infty$, we may then use induction on $\ell(M')$ as follows: If $\ell(M') = 0$, then $M' = \{0\}$ and $M \cong M''$, establishing our claim.

Additivity of length

Proposition

Let $0 \to M' \xrightarrow{\iota} M \xrightarrow{\pi} M'' \to 0$ *be a short exact sequence of* R*-modules and* $\ell = \ell_R$*. Then* $\ell(M) = \ell(M') + \ell(M'')$*, where the equality is between elements of* $\mathbb{N} \cup \{\infty\}$*.*

This again confirms the suspicion that length of modules has properties similar to the dimension of vector spaces. The rest of this page and the next page are devoted to the proof.

It is easy to see that $\ell(M') \leq \ell(M)$, hence the case where $\ell(M') = \infty$ is trivial. It is also clear that the strict inequality $\ell(M') < \ell(M)$ holds when $M' \subset M$ and $\ell(M') < \infty$. Let us assume for the moment that the assertion holds when $\ell(M') = 1$. For $\ell(M') < \infty$, we may then use induction on $\ell(M')$ as follows: If $\ell(M') = 0$, then $M' = \{0\}$ and $M \cong M''$, establishing our claim. Let $\ell(M') \neq 0$ and the assertion hold when for short exact sequences starting with a module of smaller length than M'.

Additivity of length

Proposition

Let $0 \to M' \xrightarrow{\iota} M \xrightarrow{\pi} M'' \to 0$ *be a short exact sequence of R-modules and* $\ell = \ell_R$*. Then* $\ell(M) = \ell(M') + \ell(M'')$*, where the equality is between elements of* $\mathbb{N} \cup \{\infty\}$*.*

This again confirms the suspicion that length of modules has properties similar to the dimension of vector spaces. The rest of this page and the next page are devoted to the proof.
It is easy to see that $\ell(M') \le \ell(M)$, hence the case where $\ell(M') = \infty$ is trivial. It is also clear that the strict inequality $\ell(M') < \ell(M)$ holds when $M' \subset M$ and $\ell(M') < \infty$. Let us assume for the moment that the assertion holds when $\ell(M') = 1$. For $\ell(M') < \infty$, we may then use induction on $\ell(M')$ as follows: If $\ell(M') = 0$, then $M' = \{0\}$ and $M \cong M''$, establishing our claim. Let $\ell(M') \neq 0$ and the assertion hold when for short exact sequences starting with a module of smaller length than M'. If M' is simple, the assertion was assumed to hold,

Additivity of length

Proposition

Let $0 \to M' \xrightarrow{\iota} M \xrightarrow{\pi} M'' \to 0$ *be a short exact sequence of* R*-modules and* $\ell = \ell_R$*. Then* $\ell(M) = \ell(M') + \ell(M'')$*, where the equality is between elements of* $\mathbb{N} \cup \{\infty\}$*.*

This again confirms the suspicion that length of modules has properties similar to the dimension of vector spaces. The rest of this page and the next page are devoted to the proof.
It is easy to see that $\ell(M') \leq \ell(M)$, hence the case where $\ell(M') = \infty$ is trivial. It is also clear that the strict inequality $\ell(M') < \ell(M)$ holds when $M' \subset M$ and $\ell(M') < \infty$. Let us assume for the moment that the assertion holds when $\ell(M') = 1$. For $\ell(M') < \infty$, we may then use induction on $\ell(M')$ as follows: If $\ell(M') = 0$, then $M' = \{0\}$ and $M \cong M''$, establishing our claim. Let $\ell(M') \neq 0$ and the assertion hold when for short exact sequences starting with a module of smaller length than M'. If M' is simple, the assertion was assumed to hold, otherwise if $M' \neq 0$ there is a proper submodule $0 \subset N \subset M'$ of M'.

Additivity of length

Proposition

Let $0 \to M' \xrightarrow{\iota} M \xrightarrow{\pi} M'' \to 0$ *be a short exact sequence of* R*-modules and* $\ell = \ell_R$*. Then* $\ell(M) = \ell(M') + \ell(M'')$*, where the equality is between elements of* $\mathbb{N} \cup \{\infty\}$*.*

This again confirms the suspicion that length of modules has properties similar to the dimension of vector spaces. The rest of this page and the next page are devoted to the proof.
It is easy to see that $\ell(M') \leq \ell(M)$, hence the case where $\ell(M') = \infty$ is trivial. It is also clear that the strict inequality $\ell(M') < \ell(M)$ holds when $M' \subset M$ and $\ell(M') < \infty$. Let us assume for the moment that the assertion holds when $\ell(M') = 1$. For $\ell(M') < \infty$, we may then use induction on $\ell(M')$ as follows: If $\ell(M') = 0$, then $M' = \{0\}$ and $M \cong M''$, establishing our claim. Let $\ell(M') \neq 0$ and the assertion hold when for short exact sequences starting with a module of smaller length than M'. If M' is simple, the assertion was assumed to hold, otherwise if $M' \neq 0$ there is a proper submodule $0 \subset N \subset M'$ of M'. Then $\ell(N) < \ell(M')$ as was seen above,

Additivity of length

Proposition

Let $0 \to M' \xrightarrow{\iota} M \xrightarrow{\pi} M'' \to 0$ *be a short exact sequence of* R*-modules and* $\ell = \ell_R$*. Then* $\ell(M) = \ell(M') + \ell(M'')$*, where the equality is between elements of* $\mathbb{N} \cup \{\infty\}$*.*

This again confirms the suspicion that length of modules has properties similar to the dimension of vector spaces. The rest of this page and the next page are devoted to the proof.
It is easy to see that $\ell(M') \leq \ell(M)$, hence the case where $\ell(M') = \infty$ is trivial. It is also clear that the strict inequality $\ell(M') < \ell(M)$ holds when $M' \subset M$ and $\ell(M') < \infty$. Let us assume for the moment that the assertion holds when $\ell(M') = 1$. For $\ell(M') < \infty$, we may then use induction on $\ell(M')$ as follows: If $\ell(M') = 0$, then $M' = \{0\}$ and $M \cong M''$, establishing our claim. Let $\ell(M') \neq 0$ and the assertion hold when for short exact sequences starting with a module of smaller length than M'. If M' is simple, the assertion was assumed to hold, otherwise if $M' \neq 0$ there is a proper submodule $0 \subset N \subset M'$ of M'. Then $\ell(N) < \ell(M')$ as was seen above, and by the induction assumption $\ell(M') = \ell(N) + \ell(M'/N)$,

Additivity of length

Proposition

Let $0 \to M' \xrightarrow{\iota} M \xrightarrow{\pi} M'' \to 0$ be a short exact sequence of R-modules and $\ell = \ell_R$. Then $\ell(M) = \ell(M') + \ell(M'')$, where the equality is between elements of $\mathbb{N} \cup \{\infty\}$.

This again confirms the suspicion that length of modules has properties similar to the dimension of vector spaces. The rest of this page and the next page are devoted to the proof.
It is easy to see that $\ell(M') \leq \ell(M)$, hence the case where $\ell(M') = \infty$ is trivial. It is also clear that the strict inequality $\ell(M') < \ell(M)$ holds when $M' \subset M$ and $\ell(M') < \infty$. Let us assume for the moment that the assertion holds when $\ell(M') = 1$. For $\ell(M') < \infty$, we may then use induction on $\ell(M')$ as follows: If $\ell(M') = 0$, then $M' = \{0\}$ and $M \cong M''$, establishing our claim. Let $\ell(M') \neq 0$ and the assertion hold when for short exact sequences starting with a module of smaller length than M'. If M' is simple, the assertion was assumed to hold, otherwise if $M' \neq 0$ there is a proper submodule $0 \subset N \subset M'$ of M'. Then $\ell(N) < \ell(M')$ as was seen above, and by the induction assumption $\ell(M') = \ell(N) + \ell(M'/N)$, $\ell(M) = \ell(N) + \ell\big(M/\iota(N)\big)$

Additivity of length

Proposition

Let $0 \to M' \xrightarrow{\iota} M \xrightarrow{\pi} M'' \to 0$ *be a short exact sequence of* R*-modules and* $\ell = \ell_R$*. Then* $\ell(M) = \ell(M') + \ell(M'')$*, where the equality is between elements of* $\mathbb{N} \cup \{\infty\}$*.*

This again confirms the suspicion that length of modules has properties similar to the dimension of vector spaces. The rest of this page and the next page are devoted to the proof.
It is easy to see that $\ell(M') \leq \ell(M)$, hence the case where $\ell(M') = \infty$ is trivial. It is also clear that the strict inequality $\ell(M') < \ell(M)$ holds when $M' \subset M$ and $\ell(M') < \infty$. Let us assume for the moment that the assertion holds when $\ell(M') = 1$. For $\ell(M') < \infty$, we may then use induction on $\ell(M')$ as follows: If $\ell(M') = 0$, then $M' = \{0\}$ and $M \cong M''$, establishing our claim. Let $\ell(M') \neq 0$ and the assertion hold when for short exact sequences starting with a module of smaller length than M'. If M' is simple, the assertion was assumed to hold, otherwise if $M' \neq 0$ there is a proper submodule $0 \subset N \subset M'$ of M'. Then $\ell(N) < \ell(M')$ as was seen above, and by the induction assumption $\ell(M') = \ell(N) + \ell(M'/N)$, $\ell(M) = \ell(N) + \ell\big(M/\iota(N)\big)$ and $\ell\big(M/\iota(N)\big) = \ell(M'/N) + \ell(M'')$

Additivity of length

Proposition

Let $0 \to M' \xrightarrow{\iota} M \xrightarrow{\pi} M'' \to 0$ *be a short exact sequence of* R*-modules and* $\ell = \ell_R$*. Then* $\ell(M) = \ell(M') + \ell(M'')$*, where the equality is between elements of* $\mathbb{N} \cup \{\infty\}$*.*

This again confirms the suspicion that length of modules has properties similar to the dimension of vector spaces. The rest of this page and the next page are devoted to the proof.
It is easy to see that $\ell(M') \leq \ell(M)$, hence the case where $\ell(M') = \infty$ is trivial. It is also clear that the strict inequality $\ell(M') < \ell(M)$ holds when $M' \subset M$ and $\ell(M') < \infty$. Let us assume for the moment that the assertion holds when $\ell(M') = 1$. For $\ell(M') < \infty$, we may then use induction on $\ell(M')$ as follows: If $\ell(M') = 0$, then $M' = \{0\}$ and $M \cong M''$, establishing our claim. Let $\ell(M') \neq 0$ and the assertion hold when for short exact sequences starting with a module of smaller length than M'. If M' is simple, the assertion was assumed to hold, otherwise if $M' \neq 0$ there is a proper submodule $0 \subset N \subset M'$ of M'. Then $\ell(N) < \ell(M')$ as was seen above, and by the induction assumption $\ell(M') = \ell(N) + \ell(M'/N)$, $\ell(M) = \ell(N) + \ell\big(M/\iota(N)\big)$ and $\ell\big(M/\iota(N)\big) = \ell(M'/N) + \ell(M'')$ because there is a short exact sequence $0 \to M'/N \to M/\iota(N) \to M'' \to 0$.

Additivity of length

Proposition

Let $0 \to M' \xrightarrow{\iota} M \xrightarrow{\pi} M'' \to 0$ *be a short exact sequence of* R*-modules and* $\ell = \ell_R$*. Then* $\ell(M) = \ell(M') + \ell(M'')$*, where the equality is between elements of* $\mathbb{N} \cup \{\infty\}$*.*

This again confirms the suspicion that length of modules has properties similar to the dimension of vector spaces. The rest of this page and the next page are devoted to the proof.

It is easy to see that $\ell(M') \leq \ell(M)$, hence the case where $\ell(M') = \infty$ is trivial. It is also clear that the strict inequality $\ell(M') < \ell(M)$ holds when $M' \subset M$ and $\ell(M') < \infty$. Let us assume for the moment that the assertion holds when $\ell(M') = 1$. For $\ell(M') < \infty$, we may then use induction on $\ell(M')$ as follows: If $\ell(M') = 0$, then $M' = \{0\}$ and $M \cong M''$, establishing our claim. Let $\ell(M') \neq 0$ and the assertion hold when for short exact sequences starting with a module of smaller length than M'. If M' is simple, the assertion was assumed to hold, otherwise if $M' \neq 0$ there is a proper submodule $0 \subset N \subset M'$ of M'. Then $\ell(N) < \ell(M')$ as was seen above, and by the induction assumption $\ell(M') = \ell(N) + \ell(M'/N)$, $\ell(M) = \ell(N) + \ell\big(M/\iota(N)\big)$ and $\ell\big(M/\iota(N)\big) = \ell(M'/N) + \ell(M'')$ because there is a short exact sequence $0 \to M'/N \to M/\iota(N) \to M'' \to 0$. It follows that

$$\ell(M') + \ell(M'') =$$

Additivity of length

Proposition

Let $0 \to M' \xrightarrow{\iota} M \xrightarrow{\pi} M'' \to 0$ be a short exact sequence of R-modules and $\ell = \ell_R$. Then $\ell(M) = \ell(M') + \ell(M'')$, where the equality is between elements of $\mathbb{N} \cup \{\infty\}$.

This again confirms the suspicion that length of modules has properties similar to the dimension of vector spaces. The rest of this page and the next page are devoted to the proof.

It is easy to see that $\ell(M') \leq \ell(M)$, hence the case where $\ell(M') = \infty$ is trivial. It is also clear that the strict inequality $\ell(M') < \ell(M)$ holds when $M' \subset M$ and $\ell(M') < \infty$. Let us assume for the moment that the assertion holds when $\ell(M') = 1$. For $\ell(M') < \infty$, we may then use induction on $\ell(M')$ as follows: If $\ell(M') = 0$, then $M' = \{0\}$ and $M \cong M''$, establishing our claim. Let $\ell(M') \neq 0$ and the assertion hold when for short exact sequences starting with a module of smaller length than M'. If M' is simple, the assertion was assumed to hold, otherwise if $M' \neq 0$ there is a proper submodule $0 \subset N \subset M'$ of M'. Then $\ell(N) < \ell(M')$ as was seen above, and by the induction assumption $\ell(M') = \ell(N) + \ell(M'/N)$, $\ell(M) = \ell(N) + \ell\big(M/\iota(N)\big)$ and $\ell\big(M/\iota(N)\big) = \ell(M'/N) + \ell(M'')$ because there is a short exact sequence $0 \to M'/N \to M/\iota(N) \to M'' \to 0$. It follows that

$$\ell(M') + \ell(M'') = \ell(N) + \ell(M'/N) + \ell(M'')$$

Additivity of length

Proposition

Let $0 \to M' \xrightarrow{\iota} M \xrightarrow{\pi} M'' \to 0$ *be a short exact sequence of* R*-modules and* $\ell = \ell_R$*. Then* $\ell(M) = \ell(M') + \ell(M'')$*, where the equality is between elements of* $\mathbb{N} \cup \{\infty\}$*.*

This again confirms the suspicion that length of modules has properties similar to the dimension of vector spaces. The rest of this page and the next page are devoted to the proof.

It is easy to see that $\ell(M') \leq \ell(M)$, hence the case where $\ell(M') = \infty$ is trivial. It is also clear that the strict inequality $\ell(M') < \ell(M)$ holds when $M' \subset M$ and $\ell(M') < \infty$. Let us assume for the moment that the assertion holds when $\ell(M') = 1$. For $\ell(M') < \infty$, we may then use induction on $\ell(M')$ as follows: If $\ell(M') = 0$, then $M' = \{0\}$ and $M \cong M''$, establishing our claim. Let $\ell(M') \neq 0$ and the assertion hold when for short exact sequences starting with a module of smaller length than M'. If M' is simple, the assertion was assumed to hold, otherwise if $M' \neq 0$ there is a proper submodule $0 \subset N \subset M'$ of M'. Then $\ell(N) < \ell(M')$ as was seen above, and by the induction assumption $\ell(M') = \ell(N) + \ell(M'/N)$, $\ell(M) = \ell(N) + \ell\big(M/\iota(N)\big)$ and $\ell\big(M/\iota(N)\big) = \ell(M'/N) + \ell(M'')$ because there is a short exact sequence $0 \to M'/N \to M/\iota(N) \to M'' \to 0$. It follows that

$$\ell(M') + \ell(M'') = \ell(N) + \ell(M'/N) + \ell(M'') = \ell(N) + \ell\big(M/\iota(N)\big)$$

Additivity of length

Proposition

Let $0 \to M' \xrightarrow{\iota} M \xrightarrow{\pi} M'' \to 0$ *be a short exact sequence of R-modules and* $\ell = \ell_R$*. Then* $\ell(M) = \ell(M') + \ell(M'')$*, where the equality is between elements of* $\mathbb{N} \cup \{\infty\}$*.*

This again confirms the suspicion that length of modules has properties similar to the dimension of vector spaces. The rest of this page and the next page are devoted to the proof.

It is easy to see that $\ell(M') \leq \ell(M)$, hence the case where $\ell(M') = \infty$ is trivial. It is also clear that the strict inequality $\ell(M') < \ell(M)$ holds when $M' \subset M$ and $\ell(M') < \infty$. Let us assume for the moment that the assertion holds when $\ell(M') = 1$. For $\ell(M') < \infty$, we may then use induction on $\ell(M')$ as follows: If $\ell(M') = 0$, then $M' = \{0\}$ and $M \cong M''$, establishing our claim. Let $\ell(M') \neq 0$ and the assertion hold when for short exact sequences starting with a module of smaller length than M'. If M' is simple, the assertion was assumed to hold, otherwise if $M' \neq 0$ there is a proper submodule $0 \subset N \subset M'$ of M'. Then $\ell(N) < \ell(M')$ as was seen above, and by the induction assumption $\ell(M') = \ell(N) + \ell(M'/N)$, $\ell(M) = \ell(N) + \ell\big(M/\iota(N)\big)$ and $\ell\big(M/\iota(N)\big) = \ell(M'/N) + \ell(M'')$ because there is a short exact sequence $0 \to M'/N \to M/\iota(N) \to M'' \to 0$. It follows that

$$\ell(M') + \ell(M'') = \ell(N) + \ell(M'/N) + \ell(M'') = \ell(N) + \ell\big(M/\iota(N)\big) = \ell(M)$$

Additivity of length

Proposition

Let $0 \to M' \xrightarrow{\iota} M \xrightarrow{\pi} M'' \to 0$ *be a short exact sequence of R-modules and* $\ell = \ell_R$. *Then* $\ell(M) = \ell(M') + \ell(M'')$, *where the equality is between elements of* $\mathbb{N} \cup \{\infty\}$.

This again confirms the suspicion that length of modules has properties similar to the dimension of vector spaces. The rest of this page and the next page are devoted to the proof.

It is easy to see that $\ell(M') \leq \ell(M)$, hence the case where $\ell(M') = \infty$ is trivial. It is also clear that the strict inequality $\ell(M') < \ell(M)$ holds when $M' \subset M$ and $\ell(M') < \infty$. Let us assume for the moment that the assertion holds when $\ell(M') = 1$. For $\ell(M') < \infty$, we may then use induction on $\ell(M')$ as follows: If $\ell(M') = 0$, then $M' = \{0\}$ and $M \cong M''$, establishing our claim. Let $\ell(M') \neq 0$ and the assertion hold when for short exact sequences starting with a module of smaller length than M'. If M' is simple, the assertion was assumed to hold, otherwise if $M' \neq 0$ there is a proper submodule $0 \subset N \subset M'$ of M'. Then $\ell(N) < \ell(M')$ as was seen above, and by the induction assumption $\ell(M') = \ell(N) + \ell(M'/N)$, $\ell(M) = \ell(N) + \ell\big(M/\iota(N)\big)$ and $\ell\big(M/\iota(N)\big) = \ell(M'/N) + \ell(M'')$ because there is a short exact sequence $0 \to M'/N \to M/\iota(N) \to M'' \to 0$. It follows that

$$\ell(M') + \ell(M'') = \ell(N) + \ell(M'/N) + \ell(M'') = \ell(N) + \ell\big(M/\iota(N)\big) = \ell(M)$$

as claimed.

End of the proof

We want to show $\ell(M) = \ell(M'') + 1$ for a short exact sequence $0 \to M' \xrightarrow{\iota} M \xrightarrow{\pi} M'' \to 0$ with $\ell(M') = 1$,

End of the proof

We want to show $\ell(M) = \ell(M'') + 1$ for a short exact sequence $0 \to M' \xrightarrow{\iota} M \xrightarrow{\pi} M'' \to 0$ with $\ell(M') = 1$, i. e., when M' is simple.

End of the proof

We want to show $\ell(M) = \ell(M'') + 1$ for a short exact sequence $0 \to M' \xrightarrow{\iota} M \xrightarrow{\pi} M'' \to 0$ with $\ell(M') = 1$, i. e., when M' is simple. If $\{0\} = M''_0 \subset M''_1 \subset \cdots \subset M''_k = M''$ is a strictly ascending sequence of submodules of M'', then $\{0\} = M_0 \subset M_1 \subset \cdots \subset M_{k+1} = M$,

End of the proof

We want to show $\ell(M) = \ell(M'') + 1$ for a short exact sequence $0 \to M' \xrightarrow{\iota} M \xrightarrow{\pi} M'' \to 0$ with $\ell(M') = 1$, i. e., when M' is simple. If $\{0\} = M''_0 \subset M''_1 \subset \cdots \subset M''_k = M''$ is a strictly ascending sequence of submodules of M'', then $\{0\} = M_0 \subset M_1 \subset \cdots \subset M_{k+1} = M$, whith $M_i = \pi^{-1}(M''_{i-1})$ when $i > 0$

End of the proof

We want to show $\ell(M) = \ell(M'') + 1$ for a short exact sequence $0 \to M' \xrightarrow{\iota} M \xrightarrow{\pi} M'' \to 0$ with $\ell(M') = 1$, i. e., when M' is simple. If $\{0\} = M''_0 \subset M''_1 \subset \cdots \subset M''_k = M''$ is a strictly ascending sequence of submodules of M'', then $\{0\} = M_0 \subset M_1 \subset \cdots \subset M_{k+1} = M$, whith $M_i = \pi^{-1}\big(M''_{i-1}\big)$ when $i > 0$ (thus $M_1 = M'$)

End of the proof

We want to show $\ell(M) = \ell(M'') + 1$ for a short exact sequence $0 \to M' \xrightarrow{\iota} M \xrightarrow{\pi} M'' \to 0$ with $\ell(M') = 1$, i. e., when M' is simple. If $\{0\} = M''_0 \subset M''_1 \subset \cdots \subset M''_k = M''$ is a strictly ascending sequence of submodules of M'', then $\{0\} = M_0 \subset M_1 \subset \cdots \subset M_{k+1} = M$, whith $M_i = \pi^{-1}\big(M''_{i-1}\big)$ when $i > 0$ (thus $M_1 = M'$) is a similar sequence for M,

End of the proof

We want to show $\ell(M) = \ell(M'') + 1$ for a short exact sequence $0 \to M' \xrightarrow{\iota} M \xrightarrow{\pi} M'' \to 0$ with $\ell(M') = 1$, i. e., when M' is simple. If $\{0\} = M''_0 \subset M''_1 \subset \cdots \subset M''_k = M''$ is a strictly ascending sequence of submodules of M'', then $\{0\} = M_0 \subset M_1 \subset \cdots \subset M_{k+1} = M$, whith $M_i = \pi^{-1}(M''_{i-1})$ when $i > 0$ (thus $M_1 = M'$) is a similar sequence for M, showing that $\ell(M) \geq \ell(M'') + 1$.

End of the proof

We want to show $\ell(M) = \ell(M'') + 1$ for a short exact sequence $0 \to M' \xrightarrow{\iota} M \xrightarrow{\pi} M'' \to 0$ with $\ell(M') = 1$, i. e., when M' is simple. If $\{0\} = M_0'' \subset M_1'' \subset \cdots \subset M_k'' = M''$ is a strictly ascending sequence of submodules of M'', then $\{0\} = M_0 \subset M_1 \subset \cdots \subset M_{k+1} = M$, whith $M_i = \pi^{-1}(M_{i-1}'')$ when $i > 0$ (thus $M_1 = M'$) is a similar sequence for M, showing that $\ell(M) \geq \ell(M'') + 1$. Conversely, let $0 = M_0 \subset M_1 \subset \cdots \subset M_k = M$ be a strictly ascending sequence of submodules of M.

End of the proof

We want to show $\ell(M) = \ell(M'') + 1$ for a short exact sequence $0 \to M' \xrightarrow{\iota} M \xrightarrow{\pi} M'' \to 0$ with $\ell(M') = 1$, i. e., when M' is simple. If $\{0\} = M''_0 \subset M''_1 \subset \cdots \subset M''_k = M''$ is a strictly ascending sequence of submodules of M'', then $\{0\} = M_0 \subset M_1 \subset \cdots \subset M_{k+1} = M$, whith $M_i = \pi^{-1}\big(M''_{i-1}\big)$ when $i > 0$ (thus $M_1 = M'$) is a similar sequence for M, showing that $\ell(M) \geq \ell(M'') + 1$.
Conversely, let $0 = M_0 \subset M_1 \subset \cdots \subset M_k = M$ be a strictly ascending sequence of submodules of M. Because $M_k \cap M' = M' \neq 0$, there is a smallest natural number $j \leq k$ with $M_j \cap M' \neq 0$,

End of the proof

We want to show $\ell(M) = \ell(M'') + 1$ for a short exact sequence $0 \to M' \xrightarrow{\iota} M \xrightarrow{\pi} M'' \to 0$ with $\ell(M') = 1$, i. e., when M' is simple. If $\{0\} = M''_0 \subset M''_1 \subset \cdots \subset M''_k = M''$ is a strictly ascending sequence of submodules of M'', then $\{0\} = M_0 \subset M_1 \subset \cdots \subset M_{k+1} = M$, whith $M_i = \pi^{-1}(M''_{i-1})$ when $i > 0$ (thus $M_1 = M'$) is a similar sequence for M, showing that $\ell(M) \geq \ell(M'') + 1$. Conversely, let $0 = M_0 \subset M_1 \subset \cdots \subset M_k = M$ be a strictly ascending sequence of submodules of M. Because $M_k \cap M' = M' \neq 0$, there is a smallest natural number $j \leq k$ with $M_j \cap M' \neq 0$, and obviously $j > 0$.

End of the proof

We want to show $\ell(M) = \ell(M'') + 1$ for a short exact sequence $0 \to M' \xrightarrow{\iota} M \xrightarrow{\pi} M'' \to 0$ with $\ell(M') = 1$, i. e., when M' is simple. If $\{0\} = M''_0 \subset M''_1 \subset \cdots \subset M''_k = M''$ is a strictly ascending sequence of submodules of M'', then $\{0\} = M_0 \subset M_1 \subset \cdots \subset M_{k+1} = M$, whith $M_i = \pi^{-1}(M''_{i-1})$ when $i > 0$ (thus $M_1 = M'$) is a similar sequence for M, showing that $\ell(M) \geq \ell(M'') + 1$. Conversely, let $0 = M_0 \subset M_1 \subset \cdots \subset M_k = M$ be a strictly ascending sequence of submodules of M. Because $M_k \cap M' = M' \neq 0$, there is a smallest natural number $j \leq k$ with $M_j \cap M' \neq 0$, and obviously $j > 0$. As M' is simple, $M' \cap M_j = M'$

End of the proof

We want to show $\ell(M) = \ell(M'') + 1$ for a short exact sequence $0 \to M' \xrightarrow{\iota} M \xrightarrow{\pi} M'' \to 0$ with $\ell(M') = 1$, i. e., when M' is simple. If $\{0\} = M''_0 \subset M''_1 \subset \cdots \subset M''_k = M''$ is a strictly ascending sequence of submodules of M'', then $\{0\} = M_0 \subset M_1 \subset \cdots \subset M_{k+1} = M$, whith $M_i = \pi^{-1}(M''_{i-1})$ when $i > 0$ (thus $M_1 = M'$) is a similar sequence for M, showing that $\ell(M) \geq \ell(M'') + 1$. Conversely, let $0 = M_0 \subset M_1 \subset \cdots \subset M_k = M$ be a strictly ascending sequence of submodules of M. Because $M_k \cap M' = M' \neq 0$, there is a smallest natural number $j \leq k$ with $M_j \cap M' \neq 0$, and obviously $j > 0$. As M' is simple, $M' \cap M_j = M'$ or $M' \subseteq M_j$.

End of the proof

We want to show $\ell(M) = \ell(M'') + 1$ for a short exact sequence $0 \to M' \xrightarrow{\iota} M \xrightarrow{\pi} M'' \to 0$ with $\ell(M') = 1$, i. e., when M' is simple. If $\{0\} = M''_0 \subset M''_1 \subset \cdots \subset M''_k = M''$ is a strictly ascending sequence of submodules of M'', then $\{0\} = M_0 \subset M_1 \subset \cdots \subset M_{k+1} = M$, whith $M_i = \pi^{-1}(M''_{i-1})$ when $i > 0$ (thus $M_1 = M'$) is a similar sequence for M, showing that $\ell(M) \geq \ell(M'') + 1$.
Conversely, let $0 = M_0 \subset M_1 \subset \cdots \subset M_k = M$ be a strictly ascending sequence of submodules of M. Because $M_k \cap M' = M' \neq 0$, there is a smallest natural number $j \leq k$ with $M_j \cap M' \neq 0$, and obviously $j > 0$. As M' is simple, $M' \cap M_j = M'$ or $M' \subseteq M_j$. Now

$$M''_j = \begin{cases} \pi(M_j) & j < i \\ & \end{cases}$$

End of the proof

We want to show $\ell(M) = \ell(M'') + 1$ for a short exact sequence $0 \to M' \xrightarrow{\iota} M \xrightarrow{\pi} M'' \to 0$ with $\ell(M') = 1$, i. e., when M' is simple. If $\{0\} = M''_0 \subset M''_1 \subset \cdots \subset M''_k = M''$ is a strictly ascending sequence of submodules of M'', then $\{0\} = M_0 \subset M_1 \subset \cdots \subset M_{k+1} = M$, whith $M_i = \pi^{-1}(M''_{i-1})$ when $i > 0$ (thus $M_1 = M'$) is a similar sequence for M, showing that $\ell(M) \geq \ell(M'') + 1$. Conversely, let $0 = M_0 \subset M_1 \subset \cdots \subset M_k = M$ be a strictly ascending sequence of submodules of M. Because $M_k \cap M' = M' \neq 0$, there is a smallest natural number $j \leq k$ with $M_j \cap M' \neq 0$, and obviously $j > 0$. As M' is simple, $M' \cap M_j = M'$ or $M' \subseteq M_j$. Now

$$M''_j = \begin{cases} \pi(M_j) & j < i \\ \pi(M_{j+1}) & i \leq j < k \end{cases}$$

End of the proof

We want to show $\ell(M) = \ell(M'') + 1$ for a short exact sequence $0 \to M' \xrightarrow{\iota} M \xrightarrow{\pi} M'' \to 0$ with $\ell(M') = 1$, i. e., when M' is simple. If $\{0\} = M''_0 \subset M''_1 \subset \cdots \subset M''_k = M''$ is a strictly ascending sequence of submodules of M'', then $\{0\} = M_0 \subset M_1 \subset \cdots \subset M_{k+1} = M$, whith $M_i = \pi^{-1}(M''_{i-1})$ when $i > 0$ (thus $M_1 = M'$) is a similar sequence for M, showing that $\ell(M) \geq \ell(M'') + 1$. Conversely, let $0 = M_0 \subset M_1 \subset \cdots \subset M_k = M$ be a strictly ascending sequence of submodules of M. Because $M_k \cap M' = M' \neq 0$, there is a smallest natural number $j \leq k$ with $M_j \cap M' \neq 0$, and obviously $j > 0$. As M' is simple, $M' \cap M_j = M'$ or $M' \subseteq M_j$. Now

$$M''_j = \begin{cases} \pi(M_j) & j < i \\ \pi(M_{j+1}) & i \leq j < k \end{cases}$$

defines a strictly ascending sequence of submodules of M'' of length $k-1$, showing $\ell(M'') \geq \ell(M) - 1$.

End of the proof

We want to show $\ell(M) = \ell(M'') + 1$ for a short exact sequence $0 \to M' \xrightarrow{\iota} M \xrightarrow{\pi} M'' \to 0$ with $\ell(M') = 1$, i. e., when M' is simple. If $\{0\} = M''_0 \subset M''_1 \subset \cdots \subset M''_k = M''$ is a strictly ascending sequence of submodules of M'', then $\{0\} = M_0 \subset M_1 \subset \cdots \subset M_{k+1} = M$, whith $M_i = \pi^{-1}(M''_{i-1})$ when $i > 0$ (thus $M_1 = M'$) is a similar sequence for M, showing that $\ell(M) \geq \ell(M'') + 1$. Conversely, let $0 = M_0 \subset M_1 \subset \cdots \subset M_k = M$ be a strictly ascending sequence of submodules of M. Because $M_k \cap M' = M' \neq 0$, there is a smallest natural number $j \leq k$ with $M_j \cap M' \neq 0$, and obviously $j > 0$. As M' is simple, $M' \cap M_j = M'$ or $M' \subseteq M_j$. Now

$$M''_j = \begin{cases} \pi(M_j) & j < i \\ \pi(M_{j+1}) & i \leq j < k \end{cases}$$

defines a strictly ascending sequence of submodules of M'' of length $k-1$, showing $\ell(M'') \geq \ell(M) - 1$. To see that it is strictly ascending, we use that π maps M_i isomorphically to M''_i when $i < j$.

End of the proof

We want to show $\ell(M) = \ell(M'') + 1$ for a short exact sequence $0 \to M' \xrightarrow{\iota} M \xrightarrow{\pi} M'' \to 0$ with $\ell(M') = 1$, i. e., when M' is simple. If $\{0\} = M''_0 \subset M''_1 \subset \cdots \subset M''_k = M''$ is a strictly ascending sequence of submodules of M'', then $\{0\} = M_0 \subset M_1 \subset \cdots \subset M_{k+1} = M$, whith $M_i = \pi^{-1}(M''_{i-1})$ when $i > 0$ (thus $M_1 = M'$) is a similar sequence for M, showing that $\ell(M) \geq \ell(M'') + 1$. Conversely, let $0 = M_0 \subset M_1 \subset \cdots \subset M_k = M$ be a strictly ascending sequence of submodules of M. Because $M_k \cap M' = M' \neq 0$, there is a smallest natural number $j \leq k$ with $M_j \cap M' \neq 0$, and obviously $j > 0$. As M' is simple, $M' \cap M_j = M'$ or $M' \subseteq M_j$. Now

$$M''_j = \begin{cases} \pi(M_j) & j < i \\ \pi(M_{j+1}) & i \leq j < k \end{cases}$$

defines a strictly ascending sequence of submodules of M'' of length $k-1$, showing $\ell(M'') \geq \ell(M) - 1$. To see that it is strictly ascending, we use that π maps M_i isomorphically to M''_i when $i < j$. When $i \geq j$, $M''_{i-1} \subseteq \pi(M_i)$ with equality when $i > j$,

End of the proof

We want to show $\ell(M) = \ell(M'') + 1$ for a short exact sequence $0 \to M' \xrightarrow{\iota} M \xrightarrow{\pi} M'' \to 0$ with $\ell(M') = 1$, i. e., when M' is simple. If $\{0\} = M''_0 \subset M''_1 \subset \cdots \subset M''_k = M''$ is a strictly ascending sequence of submodules of M'', then $\{0\} = M_0 \subset M_1 \subset \cdots \subset M_{k+1} = M$, whith $M_i = \pi^{-1}(M''_{i-1})$ when $i > 0$ (thus $M_1 = M'$) is a similar sequence for M, showing that $\ell(M) \geq \ell(M'') + 1$. Conversely, let $0 = M_0 \subset M_1 \subset \cdots \subset M_k = M$ be a strictly ascending sequence of submodules of M. Because $M_k \cap M' = M' \neq 0$, there is a smallest natural number $j \leq k$ with $M_j \cap M' \neq 0$, and obviously $j > 0$. As M' is simple, $M' \cap M_j = M'$ or $M' \subseteq M_j$. Now

$$M''_j = \begin{cases} \pi(M_j) & j < i \\ \pi(M_{j+1}) & i \leq j < k \end{cases}$$

defines a strictly ascending sequence of submodules of M'' of length $k-1$, showing $\ell(M'') \geq \ell(M) - 1$. To see that it is strictly ascending, we use that π maps M_i isomorphically to M''_i when $i < j$. When $i \geq j$, $M''_{i-1} \subseteq \pi(M_i)$ with equality when $i > j$, and

$$\pi(M_{i+1}) \;/\; \pi(M_i)$$

End of the proof

We want to show $\ell(M) = \ell(M'') + 1$ for a short exact sequence $0 \to M' \xrightarrow{\iota} M \xrightarrow{\pi} M'' \to 0$ with $\ell(M') = 1$, i. e., when M' is simple. If $\{0\} = M''_0 \subset M''_1 \subset \cdots \subset M''_k = M''$ is a strictly ascending sequence of submodules of M'', then $\{0\} = M_0 \subset M_1 \subset \cdots \subset M_{k+1} = M$, whith $M_i = \pi^{-1}(M''_{i-1})$ when $i > 0$ (thus $M_1 = M'$) is a similar sequence for M, showing that $\ell(M) \geq \ell(M'') + 1$. Conversely, let $0 = M_0 \subset M_1 \subset \cdots \subset M_k = M$ be a strictly ascending sequence of submodules of M. Because $M_k \cap M' = M' \neq 0$, there is a smallest natural number $j \leq k$ with $M_j \cap M' \neq 0$, and obviously $j > 0$. As M' is simple, $M' \cap M_j = M'$ or $M' \subseteq M_j$. Now

$$M''_j = \begin{cases} \pi(M_j) & j < i \\ \pi(M_{j+1}) & i \leq j < k \end{cases}$$

defines a strictly ascending sequence of submodules of M'' of length $k-1$, showing $\ell(M'') \geq \ell(M) - 1$. To see that it is strictly ascending, we use that π maps M_i isomorphically to M''_i when $i < j$. When $i \geq j$, $M''_{i-1} \subseteq \pi(M_i)$ with equality when $i > j$, and

$$\pi(M_{i+1}) \,/\, \pi(M_i) \cong (M_{i+1} \,/\, \iota(M')) \,/\, (M_i \,/\, \iota(M'))$$

End of the proof

We want to show $\ell(M) = \ell(M'') + 1$ for a short exact sequence $0 \to M' \xrightarrow{\iota} M \xrightarrow{\pi} M'' \to 0$ with $\ell(M') = 1$, i. e., when M' is simple. If $\{0\} = M''_0 \subset M''_1 \subset \cdots \subset M''_k = M''$ is a strictly ascending sequence of submodules of M'', then $\{0\} = M_0 \subset M_1 \subset \cdots \subset M_{k+1} = M$, whith $M_i = \pi^{-1}\left(M''_{i-1}\right)$ when $i > 0$ (thus $M_1 = M'$) is a similar sequence for M, showing that $\ell(M) \geq \ell(M'') + 1$. Conversely, let $0 = M_0 \subset M_1 \subset \cdots \subset M_k = M$ be a strictly ascending sequence of submodules of M. Because $M_k \cap M' = M' \neq 0$, there is a smallest natural number $j \leq k$ with $M_j \cap M' \neq 0$, and obviously $j > 0$. As M' is simple, $M' \cap M_j = M'$ or $M' \subseteq M_j$. Now

$$M''_j = \begin{cases} \pi(M_j) & j < i \\ \pi(M_{j+1}) & i \leq j < k \end{cases}$$

defines a strictly ascending sequence of submodules of M'' of length $k-1$, showing $\ell(M'') \geq \ell(M) - 1$. To see that it is strictly ascending, we use that π maps M_i isomorphically to M''_i when $i < j$. When $i \geq j$, $M''_{i-1} \subseteq \pi(M_i)$ with equality when $i > j$, and

$$\pi(M_{i+1}) \,/\, \pi(M_i) \cong \left(M_{i+1} \,/\, \iota(M')\right) \,/\, \left(M_i \,/\, \iota(M')\right) \cong M_{i+1}/M_i$$

End of the proof

We want to show $\ell(M) = \ell(M'') + 1$ for a short exact sequence $0 \to M' \xrightarrow{\iota} M \xrightarrow{\pi} M'' \to 0$ with $\ell(M') = 1$, i. e., when M' is simple. If $\{0\} = M''_0 \subset M''_1 \subset \cdots \subset M''_k = M''$ is a strictly ascending sequence of submodules of M'', then $\{0\} = M_0 \subset M_1 \subset \cdots \subset M_{k+1} = M$, whith $M_i = \pi^{-1}(M''_{i-1})$ when $i > 0$ (thus $M_1 = M'$) is a similar sequence for M, showing that $\ell(M) \geq \ell(M'') + 1$. Conversely, let $0 = M_0 \subset M_1 \subset \cdots \subset M_k = M$ be a strictly ascending sequence of submodules of M. Because $M_k \cap M' = M' \neq 0$, there is a smallest natural number $j \leq k$ with $M_j \cap M' \neq 0$, and obviously $j > 0$. As M' is simple, $M' \cap M_j = M'$ or $M' \subseteq M_j$. Now

$$M''_j = \begin{cases} \pi(M_j) & j < i \\ \pi(M_{j+1}) & i \leq j < k \end{cases}$$

defines a strictly ascending sequence of submodules of M'' of length $k-1$, showing $\ell(M'') \geq \ell(M) - 1$. To see that it is strictly ascending, we use that π maps M_i isomorphically to M''_i when $i < j$. When $i \geq j$, $M''_{i-1} \subseteq \pi(M_i)$ with equality when $i > j$, and

$$\pi(M_{i+1}) \,/\, \pi(M_i) \cong (M_{i+1} \,/\, \iota(M')) \,/\, (M_i \,/\, \iota(M')) \cong M_{i+1}/M_i \neq \{0\},$$

End of the proof

We want to show $\ell(M) = \ell(M'') + 1$ for a short exact sequence $0 \to M' \xrightarrow{\iota} M \xrightarrow{\pi} M'' \to 0$ with $\ell(M') = 1$, i. e., when M' is simple. If $\{0\} = M''_0 \subset M''_1 \subset \cdots \subset M''_k = M''$ is a strictly ascending sequence of submodules of M'', then $\{0\} = M_0 \subset M_1 \subset \cdots \subset M_{k+1} = M$, whith $M_i = \pi^{-1}(M''_{i-1})$ when $i > 0$ (thus $M_1 = M'$) is a similar sequence for M, showing that $\ell(M) \geq \ell(M'') + 1$. Conversely, let $0 = M_0 \subset M_1 \subset \cdots \subset M_k = M$ be a strictly ascending sequence of submodules of M. Because $M_k \cap M' = M' \neq 0$, there is a smallest natural number $j \leq k$ with $M_j \cap M' \neq 0$, and obviously $j > 0$. As M' is simple, $M' \cap M_j = M'$ or $M' \subseteq M_j$. Now

$$M''_j = \begin{cases} \pi(M_j) & j < i \\ \pi(M_{j+1}) & i \leq j < k \end{cases}$$

defines a strictly ascending sequence of submodules of M'' of length $k-1$, showing $\ell(M'') \geq \ell(M) - 1$. To see that it is strictly ascending, we use that π maps M_i isomorphically to M''_i when $i < j$. When $i \geq j$, $M''_{i-1} \subseteq \pi(M_i)$ with equality when $i > j$, and

$$\pi(M_{i+1}) \,/\, \pi(M_i) \cong \left(M_{i+1} \,/\, \iota(M')\right) \,/\, \left(M_i \,/\, \iota(M')\right) \cong M_{i+1}/M_i \neq \{0\},$$

showing that the filtration M''_i is indeed strictly ascending.

End of the proof

We want to show $\ell(M) = \ell(M'') + 1$ for a short exact sequence $0 \to M' \xrightarrow{\iota} M \xrightarrow{\pi} M'' \to 0$ with $\ell(M') = 1$, i. e., when M' is simple. If $\{0\} = M''_0 \subset M''_1 \subset \cdots \subset M''_k = M''$ is a strictly ascending sequence of submodules of M'', then $\{0\} = M_0 \subset M_1 \subset \cdots \subset M_{k+1} = M$, whith $M_i = \pi^{-1}(M''_{i-1})$ when $i > 0$ (thus $M_1 = M'$) is a similar sequence for M, showing that $\ell(M) \geq \ell(M'') + 1$. Conversely, let $0 = M_0 \subset M_1 \subset \cdots \subset M_k = M$ be a strictly ascending sequence of submodules of M. Because $M_k \cap M' = M' \neq 0$, there is a smallest natural number $j \leq k$ with $M_j \cap M' \neq 0$, and obviously $j > 0$. As M' is simple, $M' \cap M_j = M'$ or $M' \subseteq M_j$. Now

$$M''_j = \begin{cases} \pi(M_j) & j < i \\ \pi(M_{j+1}) & i \leq j < k \end{cases}$$

defines a strictly ascending sequence of submodules of M'' of length $k-1$, showing $\ell(M'') \geq \ell(M) - 1$. To see that it is strictly ascending, we use that π maps M_i isomorphically to M''_i when $i < j$. When $i \geq j$, $M''_{i-1} \subseteq \pi(M_i)$ with equality when $i > j$, and

$$\pi(M_{i+1}) \,/\, \pi(M_i) \cong \left(M_{i+1} \,/\, \iota(M')\right) \,/\, \left(M_i \,/\, \iota(M')\right) \cong M_{i+1}/M_i \neq \{0\},$$

showing that the filtration M''_i is indeed strictly ascending.

Remark

When worked out more carefully, the argument shows that for a module of finite length the length and, up to permutation, the sequence of isomorphism classes of M_{i+1}/M_i for a longest possible strictly ascending filtration of a finite length module M by submodules M_i is independent of the choice of such a filtration.

End of the proof

We want to show $\ell(M) = \ell(M'') + 1$ for a short exact sequence $0 \to M' \xrightarrow{\iota} M \xrightarrow{\pi} M'' \to 0$ with $\ell(M') = 1$, i. e., when M' is simple. If $\{0\} = M''_0 \subset M''_1 \subset \cdots \subset M''_k = M''$ is a strictly ascending sequence of submodules of M'', then $\{0\} = M_0 \subset M_1 \subset \cdots \subset M_{k+1} = M$, whith $M_i = \pi^{-1}(M''_{i-1})$ when $i > 0$ (thus $M_1 = M'$) is a similar sequence for M, showing that $\ell(M) \geq \ell(M'') + 1$. Conversely, let $0 = M_0 \subset M_1 \subset \cdots \subset M_k = M$ be a strictly ascending sequence of submodules of M. Because $M_k \cap M' = M' \neq 0$, there is a smallest natural number $j \leq k$ with $M_j \cap M' \neq 0$, and obviously $j > 0$. As M' is simple, $M' \cap M_j = M'$ or $M' \subseteq M_j$. Now

$$M''_j = \begin{cases} \pi(M_j) & j < i \\ \pi(M_{j+1}) & i \leq j < k \end{cases}$$

defines a strictly ascending sequence of submodules of M'' of length $k - 1$, showing $\ell(M'') \geq \ell(M) - 1$. To see that it is strictly ascending, we use that π maps M_i isomorphically to M''_i when $i < j$. When $i \geq j$, $M''_{i-1} \subseteq \pi(M_i)$ with equality when $i > j$, and

$$\pi(M_{i+1}) \,/\, \pi(M_i) \cong \big(M_{i+1} \,/\, \iota(M')\big) \,/\, \big(M_i \,/\, \iota(M')\big) \cong M_{i+1}/M_i \neq \{0\},$$

showing that the filtration M''_i is indeed strictly ascending.

Remark

When worked out more carefully, the argument shows that for a module of finite length the length and, up to permutation, the sequence of isomorphism classes of M_{i+1}/M_i for a longest possible strictly ascending filtration of a finite length module M by submodules M_i is independent of the choice of such a filtration. This should be compared with the Jordan-Hölder theorem in Group theory, which has a similar but slightly different proof.

The Hilbert polynomial

Let $R_\bullet$ be a $\mathbb{N}$- or $\mathbb{Z}$-graded ring and M an ordinary module over the ring R obtained by forgetting the grading.

The Hilbert polynomial

Let $R_\bullet$ be a $\mathbb{N}$- or $\mathbb{Z}$-graded ring and M an ordinary module over the ring R obtained by forgetting the grading. By a grading $M_\bullet$ on M we understand a sequence $(M_i)_{i=-\infty}^{\infty}$ of subgroups of the abelian group $(M, +)$

The Hilbert polynomial

Let $R_\bullet$ be a $\mathbb{N}$- or $\mathbb{Z}$-graded ring and M an ordinary module over the ring R obtained by forgetting the grading. By a grading $M_\bullet$ on M we understand a sequence $(M_i)_{i=-\infty}^{\infty}$ of subgroups of the abelian group $(M, +)$ such that every $m \in M$ has a unique decomposition as $m = \sum_{i=-\infty}^{\infty} m_i$

The Hilbert polynomial

Let $R_\bullet$ be a $\mathbb{N}$- or $\mathbb{Z}$-graded ring and M an ordinary module over the ring R obtained by forgetting the grading. By a grading $M_\bullet$ on M we understand a sequence $(M_i)_{i=-\infty}^{\infty}$ of subgroups of the abelian group $(M,+)$ such that every $m \in M$ has a unique decomposition as $m = \sum_{i=-\infty}^{\infty} m_i$ with $m_i \in M_i$ and only finitely many $m_i \neq 0$,

The Hilbert polynomial

Let $R_\bullet$ be a $\mathbb{N}$- or $\mathbb{Z}$-graded ring and M an ordinary module over the ring R obtained by forgetting the grading. By a grading $M_\bullet$ on M we understand a sequence $(M_i)_{i=-\infty}^{\infty}$ of subgroups of the abelian group $(M,+)$ such that every $m \in M$ has a unique decomposition as $m = \sum_{i=-\infty}^{\infty} m_i$ with $m_i \in M_i$ and only finitely many $m_i \neq 0$, and such that $R_i \cdot M_j \subseteq M_{i+j}$.

The Hilbert polynomial

Let $R_\bullet$ be a $\mathbb{N}$- or $\mathbb{Z}$-graded ring and M an ordinary module over the ring R obtained by forgetting the grading. By a grading $M_\bullet$ on M we understand a sequence $(M_i)_{i=-\infty}^{\infty}$ of subgroups of the abelian group $(M,+)$ such that every $m \in M$ has a unique decomposition as $m = \sum_{i=-\infty}^{\infty} m_i$ with $m_i \in M_i$ and only finitely many $m_i \neq 0$, and such that $R_i \cdot M_j \subseteq M_{i+j}$. By a graded $R_\bullet$-module we understand a $R_\bullet$-module with a grading.

The Hilbert polynomial

Let $R_\bullet$ be a $\mathbb{N}$- or $\mathbb{Z}$-graded ring and M an ordinary module over the ring R obtained by forgetting the grading. By a grading $M_\bullet$ on M we understand a sequence $(M_i)_{i=-\infty}^{\infty}$ of subgroups of the abelian group $(M,+)$ such that every $m \in M$ has a unique decomposition as $m = \sum_{i=-\infty}^{\infty} m_i$ with $m_i \in M_i$ and only finitely many $m_i \neq 0$, and such that $R_i \cdot M_j \subseteq M_{i+j}$. By a graded $R_\bullet$-module we understand a $R_\bullet$-module with a grading. Unless otherwise specified, all graded rings in this lecture are assumed to be $\mathbb{N}$-graded.

The Hilbert polynomial

Let $R_\bullet$ be a $\mathbb{N}$- or $\mathbb{Z}$-graded ring and M an ordinary module over the ring R obtained by forgetting the grading. By a grading $M_\bullet$ on M we understand a sequence $(M_i)_{i=-\infty}^{\infty}$ of subgroups of the abelian group $(M, +)$ such that every $m \in M$ has a unique decomposition as $m = \sum_{i=-\infty}^{\infty} m_i$ with $m_i \in M_i$ and only finitely many $m_i \neq 0$, and such that $R_i \cdot M_j \subseteq M_{i+j}$. By a graded $R_\bullet$-module we understand a $R_\bullet$-module with a grading. Unless otherwise specified, all graded rings in this lecture are assumed to be $\mathbb{N}$-graded. We will only consider $\mathbb{Z}$-graded modules.

The Hilbert polynomial

Let $R_\bullet$ be a $\mathbb{N}$- or $\mathbb{Z}$-graded ring and M an ordinary module over the ring R obtained by forgetting the grading. By a grading $M_\bullet$ on M we understand a sequence $(M_i)_{i=-\infty}^{\infty}$ of subgroups of the abelian group $(M,+)$ such that every $m \in M$ has a unique decomposition as $m = \sum_{i=-\infty}^{\infty} m_i$ with $m_i \in M_i$ and only finitely many $m_i \neq 0$, and such that $R_i \cdot M_j \subseteq M_{i+j}$. By a graded $R_\bullet$-module we understand a $R_\bullet$-module with a grading. Unless otherwise specified, all graded rings in this lecture are assumed to be $\mathbb{N}$-graded. We will only consider $\mathbb{Z}$-graded modules. A graded module is said to be finitely generated if the underlying ungraded module is.

The Hilbert polynomial

Let $R_\bullet$ be a $\mathbb{N}$- or $\mathbb{Z}$-graded ring and M an ordinary module over the ring R obtained by forgetting the grading. By a grading $M_\bullet$ on M we understand a sequence $(M_i)_{i=-\infty}^{\infty}$ of subgroups of the abelian group $(M,+)$ such that every $m \in M$ has a unique decomposition as $m = \sum_{i=-\infty}^{\infty} m_i$ with $m_i \in M_i$ and only finitely many $m_i \neq 0$, and such that $R_i \cdot M_j \subseteq M_{i+j}$. By a graded $R_\bullet$-module we understand a $R_\bullet$-module with a grading. Unless otherwise specified, all graded rings in this lecture are assumed to be $\mathbb{N}$-graded. We will only consider $\mathbb{Z}$-graded modules. A graded module is said to be finitely generated if the underlying ungraded module is. By our assumptions, the finite generating set can always be chosen to consist of homogenouos elements of M.

The Hilbert polynomial

Let $R_\bullet$ be a $\mathbb{N}$- or $\mathbb{Z}$-graded ring and M an ordinary module over the ring R obtained by forgetting the grading. By a grading $M_\bullet$ on M we understand a sequence $(M_i)_{i=-\infty}^{\infty}$ of subgroups of the abelian group $(M,+)$ such that every $m \in M$ has a unique decomposition as $m = \sum_{i=-\infty}^{\infty} m_i$ with $m_i \in M_i$ and only finitely many $m_i \neq 0$, and such that $R_i \cdot M_j \subseteq M_{i+j}$. By a graded $R_\bullet$-module we understand a $R_\bullet$-module with a grading. Unless otherwise specified, all graded rings in this lecture are assumed to be $\mathbb{N}$-graded. We will only consider $\mathbb{Z}$-graded modules. A graded module is said to be finitely generated if the underlying ungraded module is. By our assumptions, the finite generating set can always be chosen to consist of homogenouos elements of M.

Proposition

Let $R_\bullet$ be a graded ring such that R_0 is Artinian,

The Hilbert polynomial

Let $R_\bullet$ be a $\mathbb{N}$- or $\mathbb{Z}$-graded ring and M an ordinary module over the ring R obtained by forgetting the grading. By a grading $M_\bullet$ on M we understand a sequence $(M_i)_{i=-\infty}^{\infty}$ of subgroups of the abelian group $(M,+)$ such that every $m \in M$ has a unique decomposition as $m = \sum_{i=-\infty}^{\infty} m_i$ with $m_i \in M_i$ and only finitely many $m_i \neq 0$, and such that $R_i \cdot M_j \subseteq M_{i+j}$. By a graded $R_\bullet$-module we understand a $R_\bullet$-module with a grading. Unless otherwise specified, all graded rings in this lecture are assumed to be $\mathbb{N}$-graded. We will only consider $\mathbb{Z}$-graded modules. A graded module is said to be finitely generated if the underlying ungraded module is. By our assumptions, the finite generating set can always be chosen to consist of homogenouos elements of M.

Proposition

Let $R_\bullet$ be a graded ring such that R_0 is Artinian, such that $R_\bullet$ of finite type over R_0

The Hilbert polynomial

Let $R_\bullet$ be a $\mathbb{N}$- or $\mathbb{Z}$-graded ring and M an ordinary module over the ring R obtained by forgetting the grading. By a grading $M_\bullet$ on M we understand a sequence $(M_i)_{i=-\infty}^{\infty}$ of subgroups of the abelian group $(M,+)$ such that every $m \in M$ has a unique decomposition as $m = \sum_{i=-\infty}^{\infty} m_i$ with $m_i \in M_i$ and only finitely many $m_i \neq 0$, and such that $R_i \cdot M_j \subseteq M_{i+j}$. By a graded $R_\bullet$-module we understand a $R_\bullet$-module with a grading. Unless otherwise specified, all graded rings in this lecture are assumed to be $\mathbb{N}$-graded. We will only consider $\mathbb{Z}$-graded modules. A graded module is said to be finitely generated if the underlying ungraded module is. By our assumptions, the finite generating set can always be chosen to consist of homogenouos elements of M.

Proposition

Let $R_\bullet$ be a graded ring such that R_0 is Artinian, such that $R_\bullet$ of finite type over R_0 and such that $R_\bullet$ is generated by R_1 as an R_0-algebra.

The Hilbert polynomial

Let $R_\bullet$ be a $\mathbb{N}$- or $\mathbb{Z}$-graded ring and M an ordinary module over the ring R obtained by forgetting the grading. By a grading $M_\bullet$ on M we understand a sequence $(M_i)_{i=-\infty}^{\infty}$ of subgroups of the abelian group $(M,+)$ such that every $m \in M$ has a unique decomposition as $m = \sum_{i=-\infty}^{\infty} m_i$ with $m_i \in M_i$ and only finitely many $m_i \neq 0$, and such that $R_i \cdot M_j \subseteq M_{i+j}$. By a graded $R_\bullet$-module we understand a $R_\bullet$-module with a grading. Unless otherwise specified, all graded rings in this lecture are assumed to be $\mathbb{N}$-graded. We will only consider $\mathbb{Z}$-graded modules. A graded module is said to be finitely generated if the underlying ungraded module is. By our assumptions, the finite generating set can always be chosen to consist of homogenouos elements of M.

Proposition

Let $R_\bullet$ be a graded ring such that R_0 is Artinian, such that $R_\bullet$ of finite type over R_0 and such that $R_\bullet$ is generated by R_1 as an R_0-algebra. If $M_\bullet$ is a finitely generated graded $R_\bullet$-module, then there exists a polynomial $P_{M_\bullet} \in \mathbb{Q}[T]$ such that $\ell_{R_0}(M_k) = P(k)$ for all sufficiently large natural numbers k.

The Hilbert polynomial

Let $R_\bullet$ be a $\mathbb{N}$- or $\mathbb{Z}$-graded ring and M an ordinary module over the ring R obtained by forgetting the grading. By a grading $M_\bullet$ on M we understand a sequence $(M_i)_{i=-\infty}^{\infty}$ of subgroups of the abelian group $(M,+)$ such that every $m \in M$ has a unique decomposition as $m = \sum_{i=-\infty}^{\infty} m_i$ with $m_i \in M_i$ and only finitely many $m_i \neq 0$, and such that $R_i \cdot M_j \subseteq M_{i+j}$. By a graded $R_\bullet$-module we understand a $R_\bullet$-module with a grading. Unless otherwise specified, all graded rings in this lecture are assumed to be $\mathbb{N}$-graded. We will only consider $\mathbb{Z}$-graded modules. A graded module is said to be finitely generated if the underlying ungraded module is. By our assumptions, the finite generating set can always be chosen to consist of homogenouos elements of M.

Proposition

Let $R_\bullet$ be a graded ring such that R_0 is Artinian, such that $R_\bullet$ of finite type over R_0 and such that $R_\bullet$ is generated by R_1 as an R_0-algebra. If $M_\bullet$ is a finitely generated graded $R_\bullet$-module, then there exists a polynomial $P_{M_\bullet} \in \mathbb{Q}[T]$ such that $\ell_{R_0}(M_k) = P(k)$ for all sufficiently large natural numbers k. The degree of this polynomial is $< d$, where d is the number of elements of R_1 needed to generate $R_\bullet$ as a R_0-algebra.

The Hilbert polynomial

Let $R_\bullet$ be a $\mathbb{N}$- or $\mathbb{Z}$-graded ring and M an ordinary module over the ring R obtained by forgetting the grading. By a grading $M_\bullet$ on M we understand a sequence $(M_i)_{i=-\infty}^{\infty}$ of subgroups of the abelian group $(M,+)$ such that every $m \in M$ has a unique decomposition as $m = \sum_{i=-\infty}^{\infty} m_i$ with $m_i \in M_i$ and only finitely many $m_i \neq 0$, and such that $R_i \cdot M_j \subseteq M_{i+j}$. By a graded $R_\bullet$-module we understand a $R_\bullet$-module with a grading. Unless otherwise specified, all graded rings in this lecture are assumed to be $\mathbb{N}$-graded. We will only consider $\mathbb{Z}$-graded modules. A graded module is said to be finitely generated if the underlying ungraded module is. By our assumptions, the finite generating set can always be chosen to consist of homogenouos elements of M.

Proposition

Let $R_\bullet$ be a graded ring such that R_0 is Artinian, such that $R_\bullet$ of finite type over R_0 and such that $R_\bullet$ is generated by R_1 as an R_0-algebra. If $M_\bullet$ is a finitely generated graded $R_\bullet$-module, then there exists a polynomial $P_{M_\bullet} \in \mathbb{Q}[T]$ such that $\ell_{R_0}(M_k) = P(k)$ for all sufficiently large natural numbers k. The degree of this polynomial is $< d$, where d is the number of elements of R_1 needed to generate $R_\bullet$ as a R_0-algebra.

Definition

The polynomial $P_{M_\bullet}$ is called the Hilbert polynomial of $M_\bullet$.

The Hilbert polynomial

Let $R_\bullet$ be a $\mathbb{N}$- or $\mathbb{Z}$-graded ring and M an ordinary module over the ring R obtained by forgetting the grading. By a grading $M_\bullet$ on M we understand a sequence $(M_i)_{i=-\infty}^{\infty}$ of subgroups of the abelian group $(M,+)$ such that every $m \in M$ has a unique decomposition as $m = \sum_{i=-\infty}^{\infty} m_i$ with $m_i \in M_i$ and only finitely many $m_i \neq 0$, and such that $R_i \cdot M_j \subseteq M_{i+j}$. By a graded $R_\bullet$-module we understand a $R_\bullet$-module with a grading. Unless otherwise specified, all graded rings in this lecture are assumed to be $\mathbb{N}$-graded. We will only consider $\mathbb{Z}$-graded modules. A graded module is said to be finitely generated if the underlying ungraded module is. By our assumptions, the finite generating set can always be chosen to consist of homogenouos elements of M.

Proposition

Let $R_\bullet$ be a graded ring such that R_0 is Artinian, such that $R_\bullet$ of finite type over R_0 and such that $R_\bullet$ is generated by R_1 as an R_0-algebra. If $M_\bullet$ is a finitely generated graded $R_\bullet$-module, then there exists a polynomial $P_{M_\bullet} \in \mathbb{Q}[T]$ such that $\ell_{R_0}(M_k) = P(k)$ for all sufficiently large natural numbers k. The degree of this polynomial is $< d$, where d is the number of elements of R_1 needed to generate $R_\bullet$ as a R_0-algebra.

Definition

The polynomial $P_{M_\bullet}$ is called the Hilbert polynomial of $M_\bullet$.

Remark

Obviously, the Hilbert polynomial is uniquely determined by the above property.

The Hilbert polynomial

Let $R_\bullet$ be a $\mathbb{N}$- or $\mathbb{Z}$-graded ring and M an ordinary module over the ring R obtained by forgetting the grading. By a grading $M_\bullet$ on M we understand a sequence $(M_i)_{i=-\infty}^{\infty}$ of subgroups of the abelian group $(M,+)$ such that every $m \in M$ has a unique decomposition as $m = \sum_{i=-\infty}^{\infty} m_i$ with $m_i \in M_i$ and only finitely many $m_i \neq 0$, and such that $R_i \cdot M_j \subseteq M_{i+j}$. By a graded $R_\bullet$-module we understand a $R_\bullet$-module with a grading. Unless otherwise specified, all graded rings in this lecture are assumed to be $\mathbb{N}$-graded. We will only consider $\mathbb{Z}$-graded modules. A graded module is said to be finitely generated if the underlying ungraded module is. By our assumptions, the finite generating set can always be chosen to consist of homogenouos elements of M.

Proposition

Let $R_\bullet$ be a graded ring such that R_0 is Artinian, such that $R_\bullet$ of finite type over R_0 and such that $R_\bullet$ is generated by R_1 as an R_0-algebra. If $M_\bullet$ is a finitely generated graded $R_\bullet$-module, then there exists a polynomial $P_{M_\bullet} \in \mathbb{Q}[T]$ such that $\ell_{R_0}(M_k) = P(k)$ for all sufficiently large natural numbers k. The degree of this polynomial is $< d$, where d is the number of elements of R_1 needed to generate $R_\bullet$ as a R_0-algebra.

Definition

The polynomial $P_{M_\bullet}$ is called the Hilbert polynomial of $M_\bullet$.

Remark

Obviously, the Hilbert polynomial is uniquely determined by the above property. Despite of having integral values at integral arguments, the polynomial coefficients may be non-integral rational numbers.

Remarks and examples

The assumptions of the proposition must be understood in the light of the following proposition of the 11-th lecture of Algebra I, frame “Noetherianness of graded rings”:

Remarks and examples

The assumptions of the proposition must be understood in the light of the following proposition of the 11-th lecture of Algebra I, frame "Noetherianness of graded rings":

Proposition

For a graded ring $R_\cdot$, the following conditions are equivalent:

Remarks and examples

The assumptions of the proposition must be understood in the light of the following proposition of the 11-th lecture of Algebra I, frame "Noetherianness of graded rings":

Proposition

For a graded ring $R_\cdot$, the following conditions are equivalent:

A R *is Noetherian.*

Remarks and examples

The assumptions of the proposition must be understood in the light of the following proposition of the 11-th lecture of Algebra I, frame "Noetherianness of graded rings":

Proposition

For a graded ring $R_\cdot$, the following conditions are equivalent:

A *R is Noetherian.*

B *Every homogeneous ideal of $R_\cdot$ is finitely generated.*

Remarks and examples

The assumptions of the proposition must be understood in the light of the following proposition of the 11-th lecture of Algebra I, frame "Noetherianness of graded rings":

Proposition

For a graded ring $R_.$, the following conditions are equivalent:

- A *R is Noetherian.*
- B *Every homogeneous ideal of $R_.$ is finitely generated.*
- C *For every chain $I_0 \subseteq I_1 \subseteq \ldots$ of homogeneous ideals there is $k \in \mathbb{N}$ with $I_l = I_k$ for $l \geq k$.*

Remarks and examples

The assumptions of the proposition must be understood in the light of the following proposition of the 11-th lecture of Algebra I, frame "Noetherianness of graded rings":

Proposition

For a graded ring $R.$, the following conditions are equivalent:

- A *R is Noetherian.*
- B *Every homogeneous ideal of $R.$ is finitely generated.*
- C *For every chain $I_0 \subseteq I_1 \subseteq \ldots$ of homogeneous ideals there is $k \in \mathbb{N}$ with $I_l = I_k$ for $l \geq k$.*
- D *Every set $\mathfrak{M}$ of homogeneous ideals has a $\subseteq$-maximal element.*

Remarks and examples

The assumptions of the proposition must be understood in the light of the following proposition of the 11-th lecture of Algebra I, frame "Noetherianness of graded rings":

Proposition

For a graded ring $R.$, the following conditions are equivalent:

- A *R is Noetherian.*
- B *Every homogeneous ideal of $R.$ is finitely generated.*
- C *For every chain $I_0 \subseteq I_1 \subseteq \ldots$ of homogeneous ideals there is $k \in \mathbb{N}$ with $I_l = I_k$ for $l \geq k$.*
- D *Every set $\mathfrak{M}$ of homogeneous ideals has a $\subseteq$-maximal element.*
- E *R_0 is Noetherian and the ideal R_+ is finitely generated.*

Remarks and examples

The assumptions of the proposition must be understood in the light of the following proposition of the 11-th lecture of Algebra I, frame "Noetherianness of graded rings":

Proposition

For a graded ring $R.$, the following conditions are equivalent:

- A *R is Noetherian.*
- B *Every homogeneous ideal of $R.$ is finitely generated.*
- C *For every chain $I_0 \subseteq I_1 \subseteq \dots$ of homogeneous ideals there is $k \in \mathbb{N}$ with $I_l = I_k$ for $l \geq k$.*
- D *Every set $\mathfrak{M}$ of homogeneous ideals has a $\subseteq$-maximal element.*
- E *R_0 is Noetherian and the ideal R_+ is finitely generated.*
- F *R_0 is Noetherian, and R/R_0 is of finite type.*

Remarks and examples

The assumptions of the proposition must be understood in the light of the following proposition of the 11-th lecture of Algebra I, frame “Noetherianness of graded rings”:

Proposition

For a graded ring $R.$, the following conditions are equivalent:

- A *R is Noetherian.*
- B *Every homogeneous ideal of $R.$ is finitely generated.*
- C *For every chain $I_0 \subseteq I_1 \subseteq \dots$ of homogeneous ideals there is $k \in \mathbb{N}$ with $I_l = I_k$ for $l \geq k$.*
- D *Every set $\mathfrak{M}$ of homogeneous ideals has a $\subseteq$-maximal element.*
- E *R_0 is Noetherian and the ideal R_+ is finitely generated.*
- F *R_0 is Noetherian, and R/R_0 is of finite type.*

Example

Let $R_\bullet = \mathfrak{k}[X_0, \dots, X_n]$ be equipped with its usual grading.

Remarks and examples

The assumptions of the proposition must be understood in the light of the following proposition of the 11-th lecture of Algebra I, frame “Noetherianness of graded rings”:

Proposition

For a graded ring $R_\cdot$, the following conditions are equivalent:

A *R is Noetherian.*

B *Every homogeneous ideal of $R_\cdot$ is finitely generated.*

C *For every chain $I_0 \subseteq I_1 \subseteq \dots$ of homogeneous ideals there is $k \in \mathbb{N}$ with $I_l = I_k$ for $l \geq k$.*

D *Every set $\mathfrak{M}$ of homogeneous ideals has a $\subseteq$-maximal element.*

E *R_0 is Noetherian and the ideal R_+ is finitely generated.*

F *R_0 is Noetherian, and R/R_0 is of finite type.*

Example

Let $R_\bullet = \mathfrak{k}[X_0, \dots, X_n]$ be equipped with its usual grading. Then the Hilbert polynomial of $R_\bullet$ as a module over itself is given by

Remarks and examples

The assumptions of the proposition must be understood in the light of the following proposition of the 11-th lecture of Algebra I, frame "Noetherianness of graded rings":

Proposition

For a graded ring $R.$, the following conditions are equivalent:

- A *R is Noetherian.*
- B *Every homogeneous ideal of $R.$ is finitely generated.*
- C *For every chain $I_0 \subseteq I_1 \subseteq \dots$ of homogeneous ideals there is $k \in \mathbb{N}$ with $I_l = I_k$ for $l \geq k$.*
- D *Every set $\mathfrak{M}$ of homogeneous ideals has a $\subseteq$-maximal element.*
- E *R_0 is Noetherian and the ideal R_+ is finitely generated.*
- F *R_0 is Noetherian, and R/R_0 is of finite type.*

Example

Let $R_\bullet = \mathfrak{k}[X_0, \dots, X_n]$ be equipped with its usual grading. Then the Hilbert polynomial of $R_\bullet$ as a module over itself is given by

$$P_{R_\bullet}(T) = \binom{T+n}{n} = \frac{(T+n)\cdot \dots \cdot (T+1)}{n!}$$

Proof of the formula for the Hilbert Polynomial of $\mathfrak{k}[X_0, \ldots, X_n]$.

For $R_\bullet = \mathfrak{k}[X_0, \ldots, X_n]$, we have $R_0 = \mathfrak{k}$ and

Proof of the formula for the Hilbert Polynomial of $\mathfrak{k}[X_0, \ldots, X_n]$.

For $R_\bullet = \mathfrak{k}[X_0, \ldots, X_n]$, we have $R_0 = \mathfrak{k}$ and

$$p_{n,j} := \ell_{R_0}(R_j) = \dim_{\mathfrak{k}}(\mathfrak{k}[X_0, \ldots, X_n]_j),$$

Proof of the formula for the Hilbert Polynomial of $\mathfrak{k}[X_0, \ldots, X_n]$.

For $R_\bullet = \mathfrak{k}[X_0, \ldots, X_n]$, we have $R_0 = \mathfrak{k}$ and

$$p_{n,j} := \ell_{R_0}(R_j) = \dim_{\mathfrak{k}}(\mathfrak{k}[X_0, \ldots, X_n]_j),$$

the dimension of the space of homogeneous polynomials of degree j in $n + 1$ variables.

Proof of the formula for the Hilbert Polynomial of $\mathfrak{k}[X_0, \ldots, X_n]$.

For $R_\bullet = \mathfrak{k}[X_0, \ldots, X_n]$, we have $R_0 = \mathfrak{k}$ and

$$p_{n,j} := \ell_{R_0}(R_j) = \dim_{\mathfrak{k}}(\mathfrak{k}[X_0, \ldots, X_n]_j),$$

the dimension of the space of homogeneous polynomials of degree j in $n+1$ variables. This is equal to the number of $(n+1)$-indices α with $|\alpha| = j$,

Proof of the formula for the Hilbert Polynomial of $\mathfrak{k}[X_0, \ldots, X_n]$.

For $R_\bullet = \mathfrak{k}[X_0, \ldots, X_n]$, we have $R_0 = \mathfrak{k}$ and

$$p_{n,j} := \ell_{R_0}(R_j) = \dim_{\mathfrak{k}}\left(\mathfrak{k}[X_0, \ldots, X_n]_j\right),$$

the dimension of the space of homogeneous polynomials of degree j in $n+1$ variables. This is equal to the number of $(n+1)$-indices α with $|\alpha| = j$, or the number of ordered partitions of j into $n+1$ natural numbers.

Proof of the formula for the Hilbert Polynomial of $\mathfrak{k}[X_0, \ldots, X_n]$.

For $R_\bullet = \mathfrak{k}[X_0, \ldots, X_n]$, we have $R_0 = \mathfrak{k}$ and

$$p_{n,j} := \ell_{R_0}(R_j) = \dim_{\mathfrak{k}}(\mathfrak{k}[X_0, \ldots, X_n]_j),$$

the dimension of the space of homogeneous polynomials of degree j in $n+1$ variables. This is equal to the number of $(n+1)$-indices α with $|\alpha| = j$, or the number of ordered partitions of j into $n+1$ natural numbers. We thus have to show, for natural numbers n and j,

$$p_{n,j} = \binom{n+j}{n} \qquad (+)$$

Proof of the formula for the Hilbert Polynomial of $\mathfrak{k}[X_0, \ldots, X_n]$.

For $R_\bullet = \mathfrak{k}[X_0, \ldots, X_n]$, we have $R_0 = \mathfrak{k}$ and

$$p_{n,j} := \ell_{R_0}(R_j) = \dim_{\mathfrak{k}}(\mathfrak{k}[X_0, \ldots, X_n]_j),$$

the dimension of the space of homogeneous polynomials of degree j in $n+1$ variables. This is equal to the number of $(n+1)$-indices α with $|\alpha| = j$, or the number of ordered partitions of j into $n+1$ natural numbers. We thus have to show, for natural numbers n and j,

$$p_{n,j} = \binom{n+j}{n} \qquad (+)$$

We use induction on n, the case $n = 0$ being trivial

Proof of the formula for the Hilbert Polynomial of $\mathfrak{k}[X_0, \ldots, X_n]$.

For $R_\bullet = \mathfrak{k}[X_0, \ldots, X_n]$, we have $R_0 = \mathfrak{k}$ and

$$p_{n,j} := \ell_{R_0}(R_j) = \dim_{\mathfrak{k}}(\mathfrak{k}[X_0, \ldots, X_n]_j),$$

the dimension of the space of homogeneous polynomials of degree j in $n+1$ variables. This is equal to the number of $(n+1)$-indices α with $|\alpha| = j$, or the number of ordered partitions of j into $n+1$ natural numbers. We thus have to show, for natural numbers n and j,

$$p_{n,j} = \binom{n+j}{n} \qquad (+)$$

We use induction on n, the case $n = 0$ being trivial as both sides of $(+)$ are 1.

Proof of the formula for the Hilbert Polynomial of $\mathfrak{k}[X_0, \ldots, X_n]$.

For $R_\bullet = \mathfrak{k}[X_0, \ldots, X_n]$, we have $R_0 = \mathfrak{k}$ and

$$p_{n,j} := \ell_{R_0}(R_j) = \dim_{\mathfrak{k}}(\mathfrak{k}[X_0, \ldots, X_n]_j),$$

the dimension of the space of homogeneous polynomials of degree j in $n+1$ variables. This is equal to the number of $(n+1)$-indices α with $|\alpha| = j$, or the number of ordered partitions of j into $n+1$ natural numbers. We thus have to show, for natural numbers n and j,

$$p_{n,j} = \binom{n+j}{n} \qquad (+)$$

We use induction on n, the case $n = 0$ being trivial as both sides of $(+)$ are 1. Let $n > 0$ and the assertion shown with n replaced by $n-1$.

Proof of the formula for the Hilbert Polynomial of $\mathfrak{k}[X_0, \ldots, X_n]$.

For $R_\bullet = \mathfrak{k}[X_0, \ldots, X_n]$, we have $R_0 = \mathfrak{k}$ and

$$p_{n,j} := \ell_{R_0}(R_j) = \dim_{\mathfrak{k}}(\mathfrak{k}[X_0, \ldots, X_n]_j),$$

the dimension of the space of homogeneous polynomials of degree j in $n+1$ variables. This is equal to the number of $(n+1)$-indices α with $|\alpha| = j$, or the number of ordered partitions of j into $n+1$ natural numbers. We thus have to show, for natural numbers n and j,

$$p_{n,j} = \binom{n+j}{n} \qquad (+)$$

We use induction on n, the case $n = 0$ being trivial as both sides of $(+)$ are 1. Let $n > 0$ and the assertion shown with n replaced by $n-1$. We also use induction on j, the case $j = 0$ being trivial

Proof of the formula for the Hilbert Polynomial of $\mathfrak{k}[X_0, \dots, X_n]$.

For $R_\bullet = \mathfrak{k}[X_0, \dots, X_n]$, we have $R_0 = \mathfrak{k}$ and

$$p_{n,j} := \ell_{R_0}(R_j) = \dim_{\mathfrak{k}}(\mathfrak{k}[X_0, \dots, X_n]_j),$$

the dimension of the space of homogeneous polynomials of degree j in $n+1$ variables. This is equal to the number of $(n+1)$-indices α with $|\alpha| = j$, or the number of ordered partitions of j into $n+1$ natural numbers. We thus have to show, for natural numbers n and j,

$$p_{n,j} = \binom{n+j}{n} \qquad (+)$$

We use induction on n, the case $n = 0$ being trivial as both sides of $(+)$ are 1. Let $n > 0$ and the assertion shown with n replaced by $n-1$. We also use induction on j, the case $j = 0$ being trivial as both sides of $(+)$ equal 1 in this case.

Proof of the formula for the Hilbert Polynomial of $\mathfrak{k}[X_0, \ldots, X_n]$.

For $R_\bullet = \mathfrak{k}[X_0, \ldots, X_n]$, we have $R_0 = \mathfrak{k}$ and

$$p_{n,j} := \ell_{R_0}(R_j) = \dim_{\mathfrak{k}}(\mathfrak{k}[X_0, \ldots, X_n]_j),$$

the dimension of the space of homogeneous polynomials of degree j in $n+1$ variables. This is equal to the number of $(n+1)$-indices α with $|\alpha| = j$, or the number of ordered partitions of j into $n+1$ natural numbers. We thus have to show, for natural numbers n and j,

$$p_{n,j} = \binom{n+j}{n} \qquad (+)$$

We use induction on n, the case $n = 0$ being trivial as both sides of $(+)$ are 1. Let $n > 0$ and the assertion shown with n replaced by $n-1$. We also use induction on j, the case $j = 0$ being trivial as both sides of $(+)$ equal 1 in this case. Let $j > 0$ and the assertion be shown with j replaced by $j-1$.

Proof of the formula for the Hilbert Polynomial of $\mathfrak{k}[X_0, \ldots, X_n]$.

For $R_\bullet = \mathfrak{k}[X_0, \ldots, X_n]$, we have $R_0 = \mathfrak{k}$ and

$$p_{n,j} := \ell_{R_0}(R_j) = \dim_{\mathfrak{k}}(\mathfrak{k}[X_0, \ldots, X_n]_j),$$

the dimension of the space of homogeneous polynomials of degree j in $n+1$ variables. This is equal to the number of $(n+1)$-indices α with $|\alpha| = j$, or the number of ordered partitions of j into $n+1$ natural numbers. We thus have to show, for natural numbers n and j,

$$p_{n,j} = \binom{n+j}{n} \tag{+}$$

We use induction on n, the case $n = 0$ being trivial as both sides of $(+)$ are 1. Let $n > 0$ and the assertion shown with n replaced by $n-1$. We also use induction on j, the case $j = 0$ being trivial as both sides of $(+)$ equal 1 in this case. Let $j > 0$ and the assertion be shown with j replaced by $j-1$. The number of partitions of j into $n+1$ natural numbers of which the zeroth is zero is equal to $p_{n-1,j}$,

Proof of the formula for the Hilbert Polynomial of $\mathfrak{k}[X_0, \ldots, X_n]$.

For $R_\bullet = \mathfrak{k}[X_0, \ldots, X_n]$, we have $R_0 = \mathfrak{k}$ and

$$p_{n,j} := \ell_{R_0}(R_j) = \dim_{\mathfrak{k}}(\mathfrak{k}[X_0, \ldots, X_n]_j),$$

the dimension of the space of homogeneous polynomials of degree j in $n+1$ variables. This is equal to the number of $(n+1)$-indices α with $|\alpha| = j$, or the number of ordered partitions of j into $n+1$ natural numbers. We thus have to show, for natural numbers n and j,

$$p_{n,j} = \binom{n+j}{n} \qquad (+)$$

We use induction on n, the case $n = 0$ being trivial as both sides of $(+)$ are 1. Let $n > 0$ and the assertion shown with n replaced by $n-1$. We also use induction on j, the case $j = 0$ being trivial as both sides of $(+)$ equal 1 in this case. Let $j > 0$ and the assertion be shown with j replaced by $j-1$. The number of partitions of j into $n+1$ natural numbers of which the zeroth is zero is equal to $p_{n-1,j}$, the number of partitions of j into n natural numbers.

Proof of the formula for the Hilbert Polynomial of $\mathfrak{k}[X_0, \ldots, X_n]$.

For $R_\bullet = \mathfrak{k}[X_0, \ldots, X_n]$, we have $R_0 = \mathfrak{k}$ and

$$p_{n,j} := \ell_{R_0}\left(R_j\right) = \dim_{\mathfrak{k}}\left(\mathfrak{k}[X_0, \ldots, X_n]_j\right),$$

the dimension of the space of homogeneous polynomials of degree j in $n+1$ variables. This is equal to the number of $(n+1)$-indices α with $|\alpha| = j$, or the number of ordered partitions of j into $n+1$ natural numbers. We thus have to show, for natural numbers n and j,

$$p_{n,j} = \binom{n+j}{n} \qquad (+)$$

We use induction on n, the case $n = 0$ being trivial as both sides of $(+)$ are 1. Let $n > 0$ and the assertion shown with n replaced by $n-1$. We also use induction on j, the case $j = 0$ being trivial as both sides of $(+)$ equal 1 in this case. Let $j > 0$ and the assertion be shown with j replaced by $j-1$. The number of partitions of j into $n+1$ natural numbers of which the zeroth is zero is equal to $p_{n-1,j}$, the number of partitions of j into n natural numbers. By decreasing the zeroth summand by 1, the partitions of j into $n+1$ natural numbers with a positive zeroth summand are in bijection with the partitions of $j-1$ into $n+1$ natural numbers.

Proof of the formula for the Hilbert Polynomial of $\mathfrak{k}[X_0, \ldots, X_n]$.

For $R_\bullet = \mathfrak{k}[X_0, \ldots, X_n]$, we have $R_0 = \mathfrak{k}$ and

$$p_{n,j} := \ell_{R_0}(R_j) = \dim_{\mathfrak{k}}(\mathfrak{k}[X_0, \ldots, X_n]_j),$$

the dimension of the space of homogeneous polynomials of degree j in $n+1$ variables. This is equal to the number of $(n+1)$-indices α with $|\alpha| = j$, or the number of ordered partitions of j into $n+1$ natural numbers. We thus have to show, for natural numbers n and j,

$$p_{n,j} = \binom{n+j}{n} \qquad (+)$$

We use induction on n, the case $n = 0$ being trivial as both sides of $(+)$ are 1. Let $n > 0$ and the assertion shown with n replaced by $n-1$. We also use induction on j, the case $j = 0$ being trivial as both sides of $(+)$ equal 1 in this case. Let $j > 0$ and the assertion be shown with j replaced by $j-1$. The number of partitions of j into $n+1$ natural numbers of which the zeroth is zero is equal to $p_{n-1,j}$, the number of partitions of j into n natural numbers. By decreasing the zeroth summand by 1, the partitions of j into $n+1$ natural numbers with a positive zeroth summand are in bijection with the partitions of $j-1$ into $n+1$ natural numbers. Thus,

$$p_{n,j} = p_{n-1,j} + p_{n,j-1}$$

Proof of the formula for the Hilbert Polynomial of $\mathfrak{k}[X_0, \ldots, X_n]$.

For $R_\bullet = \mathfrak{k}[X_0, \ldots, X_n]$, we have $R_0 = \mathfrak{k}$ and

$$p_{n,j} := \ell_{R_0}(R_j) = \dim_{\mathfrak{k}}(\mathfrak{k}[X_0, \ldots, X_n]_j),$$

the dimension of the space of homogeneous polynomials of degree j in $n+1$ variables. This is equal to the number of $(n+1)$-indices α with $|\alpha| = j$, or the number of ordered partitions of j into $n+1$ natural numbers. We thus have to show, for natural numbers n and j,

$$p_{n,j} = \binom{n+j}{n} \tag{+}$$

We use induction on n, the case $n = 0$ being trivial as both sides of $(+)$ are 1. Let $n > 0$ and the assertion shown with n replaced by $n-1$. We also use induction on j, the case $j = 0$ being trivial as both sides of $(+)$ equal 1 in this case. Let $j > 0$ and the assertion be shown with j replaced by $j-1$. The number of partitions of j into $n+1$ natural numbers of which the zeroth is zero is equal to $p_{n-1,j}$, the number of partitions of j into n natural numbers. By decreasing the zeroth summand by 1, the partitions of j into $n+1$ natural numbers with a positive zeroth summand are in bijection with the partitions of $j-1$ into $n+1$ natural numbers. Thus,

$$p_{n,j} = p_{n-1,j} + p_{n,j-1} = \binom{j+n-1}{n-1} + \binom{j+n-1}{n}$$

Proof of the formula for the Hilbert Polynomial of $\mathfrak{k}[X_0, \ldots, X_n]$.

For $R_\bullet = \mathfrak{k}[X_0, \ldots, X_n]$, we have $R_0 = \mathfrak{k}$ and

$$p_{n,j} := \ell_{R_0}(R_j) = \dim_{\mathfrak{k}}(\mathfrak{k}[X_0, \ldots, X_n]_j),$$

the dimension of the space of homogeneous polynomials of degree j in $n+1$ variables. This is equal to the number of $(n+1)$-indices α with $|\alpha| = j$, or the number of ordered partitions of j into $n+1$ natural numbers. We thus have to show, for natural numbers n and j,

$$p_{n,j} = \binom{n+j}{n} \qquad (+)$$

We use induction on n, the case $n = 0$ being trivial as both sides of $(+)$ are 1. Let $n > 0$ and the assertion shown with n replaced by $n-1$. We also use induction on j, the case $j = 0$ being trivial as both sides of $(+)$ equal 1 in this case. Let $j > 0$ and the assertion be shown with j replaced by $j-1$. The number of partitions of j into $n+1$ natural numbers of which the zeroth is zero is equal to $p_{n-1,j}$, the number of partitions of j into n natural numbers. By decreasing the zeroth summand by 1, the partitions of j into $n+1$ natural numbers with a positive zeroth summand are in bijection with the partitions of $j-1$ into $n+1$ natural numbers. Thus,

$$p_{n,j} = p_{n-1,j} + p_{n,j-1} = \binom{j+n-1}{n-1} + \binom{j+n-1}{n} = \binom{j+n}{n}$$

Proof of the formula for the Hilbert Polynomial of $\mathfrak{k}[X_0, \ldots, X_n]$.

For $R_\bullet = \mathfrak{k}[X_0, \ldots, X_n]$, we have $R_0 = \mathfrak{k}$ and

$$p_{n,j} := \ell_{R_0}(R_j) = \dim_{\mathfrak{k}}(\mathfrak{k}[X_0, \ldots, X_n]_j),$$

the dimension of the space of homogeneous polynomials of degree j in $n+1$ variables. This is equal to the number of $(n+1)$-indices α with $|\alpha| = j$, or the number of ordered partitions of j into $n+1$ natural numbers. We thus have to show, for natural numbers n and j,

$$p_{n,j} = \binom{n+j}{n} \tag{+}$$

We use induction on n, the case $n = 0$ being trivial as both sides of $(+)$ are 1. Let $n > 0$ and the assertion shown with n replaced by $n-1$. We also use induction on j, the case $j = 0$ being trivial as both sides of $(+)$ equal 1 in this case. Let $j > 0$ and the assertion be shown with j replaced by $j-1$. The number of partitions of j into $n+1$ natural numbers of which the zeroth is zero is equal to $p_{n-1,j}$, the number of partitions of j into n natural numbers. By decreasing the zeroth summand by 1, the partitions of j into $n+1$ natural numbers with a positive zeroth summand are in bijection with the partitions of $j-1$ into $n+1$ natural numbers. Thus,

$$p_{n,j} = p_{n-1,j} + p_{n,j-1} = \binom{j+n-1}{n-1} + \binom{j+n-1}{n} = \binom{j+n}{n}$$

by the two induction assumtions

Proof of the formula for the Hilbert Polynomial of $\mathfrak{k}[X_0, \ldots, X_n]$.

For $R_\bullet = \mathfrak{k}[X_0, \ldots, X_n]$, we have $R_0 = \mathfrak{k}$ and

$$p_{n,j} := \ell_{R_0}(R_j) = \dim_{\mathfrak{k}}\big(\mathfrak{k}[X_0, \ldots, X_n]_j\big),$$

the dimension of the space of homogeneous polynomials of degree j in $n+1$ variables. This is equal to the number of $(n+1)$-indices α with $|\alpha| = j$, or the number of ordered partitions of j into $n+1$ natural numbers. We thus have to show, for natural numbers n and j,

$$p_{n,j} = \binom{n+j}{n} \qquad (+)$$

We use induction on n, the case $n = 0$ being trivial as both sides of $(+)$ are 1. Let $n > 0$ and the assertion shown with n replaced by $n-1$. We also use induction on j, the case $j = 0$ being trivial as both sides of $(+)$ equal 1 in this case. Let $j > 0$ and the assertion be shown with j replaced by $j-1$. The number of partitions of j into $n+1$ natural numbers of which the zeroth is zero is equal to $p_{n-1,j}$, the number of partitions of j into n natural numbers. By decreasing the zeroth summand by 1, the partitions of j into $n+1$ natural numbers with a positive zeroth summand are in bijection with the partitions of $j-1$ into $n+1$ natural numbers. Thus,

$$p_{n,j} = p_{n-1,j} + p_{n,j-1} = \binom{j+n-1}{n-1} + \binom{j+n-1}{n} = \binom{j+n}{n}$$

by the two induction assumtions and a well-known property of binomial coefficients (Pascal's triangle).