Graduate Seminar on Advanced Topology S4D4 (Master)

Sommersemester 2021

Generalized Homology and Cohomology Theories

Tuesdays, 14:15 – 16:00 Uhr (Zoom meetings)
Prof. Dr. Carl-Friedrich Bödigheimer

Preliminary and Organizational Meeting: Thursday, 11th February 2021, 18:00

Meeting-ID:	920 2508 8399
Kenncode:	815196

https://uni-bonn.zoom.us/j/92025088399?pwd=S0s0c05mT1JrdjBIck1HYXhhOFBTUT09

Registration procedure for seminars. For the summer semester 2021 it is sufficient that the lecturer submits after the preliminary (organizational) meeting a list of the participants; a signature of the students on this list is currently not required. However, the students *need to register* for the seminar via BASIS between 1st and 30th April 2021.

Content. Apart from singular homology H_* and cohomology H^* there are other homology and cohomology theories, like K-theory (real or complex), bordism theory (real or complex, oriented or unoriented) and stable homotopy theory. These were discovered or defined in the late fifties and early sixties by Atiyah, Bott, Milnor, Thom and others. Each theory has its own description.

Around the same time, G. Whitehead found a uniform description by so-called *spectra* or, almost equivalently, inifinte loop spaces. How to create a homology theory (and at the same time a cohomology theory) out of a spectrum is the topic of this seminar. We will study the examples above and construct a spectrum for each of them.

A spectrum is an astonishingly easy object: a sequence $E_0, E_1, E_2, \dots$ of spaces with structure maps $\sigma_n\colon \Sigma E_n \to E_{n+1}$. There are no diagrams required to commute. As an example, you can start with an arbitrary space E_0 and set $E_n = \Sigma^n E_0$ with $\sigma_n = \mathrm{id}$.

> *'And something as simple as that should create something as complicated as a homology theory? I don't believe it!' – You will see.*

Here are some examples.

- If we take $E_n = \mathbb{S}^n$ and $\sigma_n = \mathrm{id}$, we do obtain stable homotopy theory π_*^{stab}.
- If E_n are the infinite complex Graßmannian BU for n even, and to be the infinite unitary group U for n odd, and σ_n the adjoint of the Bott periodicity $BU \to \Omega U$ resp. of the canonical map $U \to \Omega BU$, we obtain complex K-Theory KU^*.
- If $E_n = K(\mathbb{Z}, n)$ are the Eilenberg–MacLane spaces for the group of integers, and σ_n the adjoint of the homotopy equivalence $K(\mathbb{Z}, n) \to \Omega K(\mathbb{Z}, n+1)$, we obtain singular homology theory H_* with integer coefficients.
- The Thom spaces of certain tautological bundles over real, complex, oriented or unoriented infinite Graßmannians give various bordism theories.
- Guess what you get, when E_n is arbitrary and σ_n null-homotopic.

A spectrum determines an infinite loop space $\Omega^\infty E := \mathrm{colim}(E_0 \to \Omega E_1 \to \Omega^2 E_2 \to \ldots)$ where we use the adjoints $\sigma_n' \colon E_n \to \Omega E_{n+1}$ of the structure maps σ_n. A connective spectrum and an infinite loop space are essentially equivalent notions. The easiest example of an infinite loop space is $\mathbb{S}^1 = K(\mathbb{Z}, 1) = \Omega K(\mathbb{Z}, 2) = \Omega^2 K(\mathbb{Z}, 3) = \ldots$, where $K(\mathbb{Z}, 2) = \mathbb{C}P^\infty$.

A turning point in algebraic topology is the representability theorem of Brown: *Any contravariant functor Φ from spaces to abelian groups, which is homotopy invariant and satisfies the Mayer–Vietoris axiom, is of the form $\Phi(X) \cong [X, B_\Phi]$ for some representing space B_Φ.* It says that all cohomology functors h^n are of this form and makes cohomology theory a part of homotopy theory. To a certain extend, at least as an example, the Dold–Thom theorem does this for singular homology: For a path-connected, based space X we have $\mathrm{H}_n(X; \mathbb{Z}) \cong \pi_n(\mathrm{SP}_\infty(X))$, writing singular homology groups as homotopy groups of the infinite symmetric products $\mathrm{SP}_\infty(X)$ of the space X; another example are the labelled configuration spaces $C(\mathbb{R}^\infty; X)$, giving the stable homotopy group of X as their own homotopy groups: $\pi_n^{\mathrm{stab}}(X) \cong \pi_n(C(\mathbb{R}^\infty; X))$. Moreover, $C(\mathbb{R}^\infty; X) \simeq \Omega^\infty \Sigma^\infty X$ is an example of an infinite loop space, again for connected X. In general, at least for a homology theory without negative homology groups, this can be achieved via Gamma spaces, following Segal.

Outline. We start with general consideration on homology theories and define spectra and infinite loop spaces; we learn how to associate a homology theory and a cohomology theory to a spectrum and see, that each cohomology theory is obtained from a spectrum. Then, using various techniques from topology and homotopy theory, we construct for each of our examples a spectrum. Later we study the question which additional structure on a spectrum is needed in order to define a product on the associated cohomology theory; this leads to the notion of a ring spectrum. Additionally, we want to know if a pair of homology and cohomology theory, associated to the same spectrum, permits something like Poincaré duality. The final talks consider operations on homology and cohomology theories, like the Steenrod operations on singular cohomology $\mathrm{H}^*(-; \mathbb{Z}/2)$ and the Adams operations on KU. At the end, we will study a representation of connected homology theories via Gamma spaces.

The seminar is a (very) steep journey into the world of generalized homology and cohomology theories, spectra and infinite loop spaces. We will mainly follow Switzer's book, but not all material can be found there. The material is advanced and many notions and proofs are hard and need all you have learned so far in algebraic topology, in particular in homotopy theory. Solid knowledge of these areas is necessary. Spectra and infinite loop spaces will also be topics in my lecture course *Algebraic Topology II* in the summer term 2021, but the main goal in the lecture course will be the classical theorems of Serre on the homotopy groups of spheres; spectral sequences (not to be mixed up with spectra) will be the main new method we will learn.

For all talks I have sketched the content, but the material to learn is certainly too much for a talk; so we need – after you have learned the entire material – to make a selection what to present in detail, what to report on, and what to skip.

Prerequisites. The lecture courses *Topologie I* and *Topologie II*, as well as the course *Algebraic Topology I* (homotopy theory).

Literature. We will be mostly using the book of R. Switzer [Switzer], but for most talks other sources are useful or necessary as well.

The talks are supposed to be 90 minutes long. That means, prepare around 70 minutes and allow time for questions during the talk. You should consult me about the talk at least two weeks before the day of the talk.

(1) **Homology and cohomology theories** 13.04.2021

Eilenberg–Steenrod axioms, reduced/unreduced versions. Mayer–Vietoris versus excision axiom. Coefficients of a homology/cohomology theory. Examples, as far as already known. Rolê of Puppe sequence. Additivity, wedge axiom, Milnor sequence. Natural transformations of homology/cohomology theories. Uniqueness theorems.

[Switzer, Chap. 7]

(2) **Spectra and their homology/cohomology theory** 20.04.2021

Definition of a spectrum, subspectrum. Maps between spectra. First examples. CW spectra and cells, filtrations. Maps and homotopies. Homotopy sets $[E, F]$, and homotopy groups of a spectrum. Exact sequences.

[Switzer, Chap. 8], [Gray, Chap. 18+19], [Adams-74, Part III.1–6].

(3) **Browns's Representability Theorem** 27.04.2021

Contravariant functors of the form $\Phi(X) = [X, B]$. Brown's Representability (in the version of Adams). Examples: (Principal) bundles over X with structure group G, and $B = BG$, classifying space of G.

[Switzer, Chap. 9], [Adams-71], [Hatcher, Appendix 4.E], [Spanier, 7.7]

(4) **Eilenberg–MacLane spaces and ordinary cohomology** 04.05.2021

Construction of Eilenberg–MacLane spaces $K(G, n)$ and the Eilenberg–MacLane spectrum HG (Switzer 6.39 − 6.46). Recollection of Hurewicz and Whitehead theorems (Switzer 10.23 − 10.29). Cohomotopy groups $\pi^q(X)$ and Hopf's isomorphism theorem (Switzer 10.32). Application: $[X, K(G, n)] \cong H^n(X; G)$.

[Switzer, Chap. 6+10], [Gray, Chap. 17], [Hatcher, Sect. 4.3], [Spanier, 8.1].

(5) **Symmetric products and ordinary homology theory** 11.05.2021

Finite symmetric products $\mathrm{SP}_n(X) = X^n/\mathfrak{S}_n$. For a based space X the infinite symmetric product $\mathrm{SP}_\infty(X) = \mathrm{colim}_n \mathrm{SP}_n(X)$. Quasifibrations. The theorem of Dold and Thom. (Generalized) Eilenberg–MacLane spaces as examples.

[Dold-Thom], [Hatcher, Appendix 4.K] [A-G-P, Chap. 5.2+5.3 +6].

(6) **Configuration spaces and stable homotopy theory** 18.05.2021

Ordered and unordered finite configuration spaces $\tilde{C}_n(M) \to C_n(M)$ of points in a manifold. Fadell–Neuwirth fibrations. Braid groups as examples, symmetric groups as examples. Infinite configuration spaces. Configuration spaces $C(M, M_0; X)$ of points in a manifold pair (M, M_0) and with labels in a based space X. Scanning maps and a theorem about section spaces. Finite loop spaces $\Omega^m\Sigma^m X$ and infinite loop spaces $\Omega^\infty\Sigma^\infty X$ as examples, if X is path-connected.

[F-H, Chap. I+II], [Boedigheimer].

(7) **Vector bundles and K-theories** 01.06.2021

Vector bundles, principal bundles. Classification theorem (Switzer 11.16). Representation theorem (Switzer 11.32+11.35). KO, KU and KSp defined by vector bundles. Bott periodicity (Switzer 16.47). Spectra for these K-theories.

[Switzer, Chap. 11] [Hatcher-K], [Gray, Chap. 29].

(8) **Manifolds and bordism theories** 08.06.2021

Steenrod problem: representability of homology classes by (embedded) manifolds (Switzer p. 224). Generalities about smooth maps, transversality. $\mathcal{X}$-structures Θ on manifolds. Thom spaces of vector bundles. Thom spectra $M\Theta$, in particular MO, MSO and MU. Thom's Theorem (Switzer 12.30, 12.35).

[Switzer, Chap. 12], [Gray, Chap. 30], [tom Dieck, Chap. 21].

(9) **Ring spectra and products** 15.06.2021

Smash products $E \wedge F$ of spectra. Ring spectra. Examples. Homology and cohomology cross product, Homology and cohomology slant product. Cap product and cap product. Kronecker product. Pontrjagin product for H-spaces. Small Künneth theorem.

[Switzer, Chap. 13], [Gray, Chap. 23].

(10) **Orientation and duality** 22.06.2021

Tangent bundles of differentiable manifolds and microbundles for topological manifolds. Definition of orientability with respect to a homology theory. Orientation class, Thom class. Poincaré–Lefschetz–Alexander duality. Spanier–Whitehead duality, S-duals of spaces and of spectra. Representation theorem for homology theories by spectra (Switzer 14.35+14.36).

[Switzer, Chap.14], [Gray, Chap. 26], [Adams-74, Part III.5+10].

(11) **Characteristic classes** 29.06.2021

G-bundles and characteristic classes. General classification (Switzer 16.1). Generalized Chern classes and Stiefel–Whitney classes (Switzer 16.2+16.3). Cohomology of infinite Graßmannians $B\mathrm{O}(n)$, $B\mathrm{U}(n)$ and $B\mathrm{Sp}(n)$, and of $B\mathrm{O}$, $B\mathrm{U}$ and $B\mathrm{Sp}$. Complex oriented spectra (Switzer 16.27). K-theory and bordism theory of $B\mathrm{U}$ (Switzer 16.33).

[Switzer, Chap. 16].

(12) **Cohomology operations and homology cooperations** 06.07.2021

Cohomology operations (additive, stable). Action of $E^*(E)$ on each $E^*(X)$. Commutative ring spectra (Switzer 17.8). The Hurewicz map $[X, Y] \to \mathrm{Hom}_{E_*(\mathbb{S}^0)}(E_*(X), E_*(Y))$ and the Boardman map $[X, Y] \to \mathrm{Hom}_{E^*(\mathbb{S}^0)}(E^*(Y), E^*(X))$. Examples.

[Switzer, Chap. 17], [Gray, Chap. 27+28].

(13) **Steenrod Algebra and its dual** 13.07.2021

Eilenberg–Zilber Theorem. $\cup_i$-products. Steenrod squares $\mathrm{Sq}^i\colon H^*(X; \mathbb{Z}/2) \to H^{*+i}(X; \mathbb{Z}/2)$. Main properties (Switzer 18.12). Steenrod algebra $\mathcal{A}^*$. Adem relations. $H^*(H)$ (Switzer 18.14+18.15). Dual Steenrod algebra $\mathcal{A}_*$ (Switzer 18.20). Hopf invariant as an application.

[Switzer, Chap. 18]

(14) **Segal's Gamma spaces** 20.07.2021

The category Γ and its dual. Definition of a Γ-space. Examples: $\mathrm{SP}_\infty(X)$ as he free abelian topological monoid generated by X; for any abelian group $\mathbb{A}$ more generally $\mathbb{A}(X)$ as the free abelian topological $\mathbb{A}$-module generated by X. $C(\mathbb{R}^\infty; X)$, the configuration space of finite subsets of $\mathbb{R}^\infty$ with labels in X, or free algebra over the little infinity cubes operad generated by X. $\mathrm{Gr}(\mathbb{R}^\infty; X)$, the Graßmann space of finite-dimensional linear subspaces of $\mathbb{R}^\infty$ with labels in X. Representability of any connective homology theory by a Γ-space.

[Segal], [Adams-78, Chap. 2].

Literatur

[Adams-71] **J. F. Adams**: *A variant of E. H. Brown's representability theorem.* Topology 10 (1971), 185–198.

[Adams-74] **J. F. Adams**: *Stable Homotopy and Generalized Homology.*
University of Chicago Mathematics Lecture Notes (1974).

[Adams-78] **J. F. Adams**: *Infinite Loop Spaces.*
Annals of Mathematics Studies vol. 90. Princeton University Press (1978).

[A-G-P] **M. Aguilar, S. Gitler, C. Prieto**: *Algebraic Topology from a Homotopical Viewpoint.*
Universitext, Springer Verlag (2002).

[Boedigheimer] **C. - F. Bödigheimer**: *Stable splittings of mapping spaces.*
in: Algebraic Topology, Proc. Seattle (1985), Springer LNM 1286 (1987), 174–187.

[tom Dieck] **T. tom Dieck**: *Algebraic Topology.*
Textbooks in Mathematics, European Mathematical Society (2008).

[Dold-Thom] **A. Dold - R. Thom**: *Quasifaserungen und unendliche symmetrische Produkte.*
Annals Math. 67 (1958), 239–281.

[F-H] **E. R. Fadell, S. Y. Husseini**: *Goemetry and Topology of Configuration Spaces.*
Springer Monographs in Mathematics, Sringer Verlag (2001).

[Gray] **B. Gray**: *Homotopy Theory.*
Academic Press (1975).

[Hatcher] **A. Hatcher**: *Algebraic Topology.*
Cambridge University Press (2002).

[Hatcher-K] **A. Hatcher**: *Vector Bundles and K-Theory.*
Unfinished book, available on the author's webpage.

[Segal] **G. Segal**: *Categories and cohomology theories.*
Topology 13 (1974), 293–312.

[Spanier] **E. H. Spanier**: *Algebraic Topology.*
McGraw-Hill (1966). Second reprint by Springer Verlag (1989)[2].

[Switzer] **R. Switzer**: *Algebraic Topology: Homotopy and Homology.*
Grundlehren der Math. Wissenschaften, vol. 212, Springer Verlag (1975). Reprinted in Classics in Mathematics (2002).