TAUTOLOGICAL CLASSES ON MODULI OF K3 SURFACES AFTER MARIAN–OPREA–PANDHARIPANDE

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Abstract. These notes are from my talk at the Bonn/Paris seminar on *Moduli of Hyperkähler manifolds*. My job was to introduce the so called tautological subring of the Chow ring on the moduli space of quasi-polarised K3 surfaces following [MOP17]. These notes benefitted a lot from discussions with Thorsten Beckmann, Mirko Mauri, Georg Oberdieck, and Johannes Schmitt and comments from Oliver Debarre, and Daniel Huybrechts. That said, any responsibility of errors lies with the author alone.

1. introduction

We recall a few notions introduced prior in this seminar.

Definition 1.0.1 (Quasi-polarised K3 Surfaces)**.** A quasi-polarised K3 surface (S, H) is a 2-dimensional simply connected proper complex variety with a unique symplectic 2-form $\sigma \in H^0(X, \Omega^2_X)$ and a primitive line bundle L on S, such that under the intersection pairing $q\colon H^2(X,\mathbb{Z}) \times H^2(X,\mathbb{Z}) \to \mathbb{Z}$, we have $q(v, v) > 0$ where $v = c_1(H)$ and $H \cdot C$ is non-negative.

Note that the above conditions does not necessarily imply that H is ample, rather H is big and nef or "birationally ample". The degree $q(v^2)$, henceforth written sometimes also as $H^2 = 2\ell$ for an integer ℓ. This integer 2ℓ is said to be the *degree* of the K3 surface S. Once such degree is fixed, we have seen that the moduli space $\mathcal{M}_\ell$ of quasi-polarised K3s of degree 2ℓ is 19-dimensional quasi-projective algebraic variety and is isomorphic to $D_\ell/O(\Lambda_\ell)$. Here $\Lambda_\ell \coloneqq v^\perp \subset H^2(X,\mathbb{Z})$. Since $H^2(X,\mathbb{Z})$ admits an isometry to the lattice $E_8^{\oplus 2} \oplus U^{\oplus 3}$ and for a basis (e, f) of U, $v \mapsto e + \ell f$, we deduce that

$$\Lambda_\ell \coloneqq E_8^{\oplus 2} \oplus U^{\oplus 2} \oplus \mathbb{Z}(-\ell)$$

Finally, recall that

$$D_\ell \coloneqq \{w \in H^2(X,\mathbb{Z}) | (w, w) = 0, (w, \overline{w}) > 0\} \cap \mathbb{P}(\Lambda_d \otimes_{\mathbb{Z}} \mathbb{C}).$$

1.1. **Noether–Lefschetz loci.** A general member (S, H) of the moduli space has picard rank 1, i.e. $\mathrm{Pic}_{\mathbb{Q}}(S) = \mathbb{Q} \cdot H$. It is known that Picard rank jumps along countably many divisors in $\mathcal{M}_\ell$. Such divisors are called Noether–Lefschetz divisors. These divisors $\mathcal{P}$

can be defined by rank 2 lattices (L, v) such that $v^2 = 2\ell$ and $L \hookrightarrow \mathrm{Pic}(S)$ for a general $S \in \mathcal{P}$. Such rank 2 lattices are in turn determined by integers d and h in the matrix

$$\begin{pmatrix} 2\ell & d \\ d & 2h-2 \end{pmatrix}$$

Note that if $L = (\beta, v)$ with β coming from an effective divisor, we have by the Hodge index theorem and $d \geq 0$ we have $(2h-2)^2 > 4\ell^2 \cdot d^2$.

Example 1.1.1. Let $\ell = 1$, i.e. consider the moduli space of quasi-polarised K3 of degree 2. Then Noether–Lefschetz divisors with β and $H - \beta$ effective are given by $(d, h) = (0, 0), (1, 0)$ corresponding to the matrices

$$\begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$$

Note that the Noether–Lefschetz divisor $\mathcal{P}$ corresponding to the second matrix parametrises degree 2 elliptic fibrations. Indeed, a K3 surface (S, H) is an elliptic fibration if and only if there exists $0 \neq H' \in \mathrm{Pic}(X)$ such that $H'^2 = 0$. If in addition we specify that β is primitive, by a Prop. 1.3 of O'Grady's thesis $\mathcal{P}$ and $\mathcal{S}$ are irreducible divisor. He further shows that $\mathrm{Pic}(S) = \mathbb{Q}[\mathcal{P}] \oplus \mathbb{Q}[\mathcal{S}]$.

1.2. **Tautological classes.** Tautological classes on $\mathcal{M}_\ell$ are roughly speaking, the chern classes of coherent sheaves arising canonically on the Moduli space. We have seen one example of such class in the previous talk that fell out of the Hodge line bundle. Let by

$$\pi \colon \mathcal{X} \to \mathcal{M}_\ell$$

denote the universal surface. Then, we immediately have choices of two canonically defined bundles namely the universal quasi-polarisation (unique up-to tensoring by a line bundle from $\mathcal{M}_\ell$) $\mathcal{H} \to \mathcal{X}$ and the relative cotangent bundle $\Omega^1_{\mathcal{X}/\mathcal{M}_d} \to \mathcal{X}$. We define respectively the following chern classes

$$\lambda \coloneqq c_1(\pi_* \Omega^2_{\mathcal{X}/\mathcal{M}_d}) \in A^1(\mathcal{M}, \mathbb{Q})$$

and

$$\kappa_{a,b} \coloneqq \pi_*(c_1(\mathcal{H})^a \cdot c_2(\mathcal{T}_\pi)^b) \in A^{a+2b-2}(\mathcal{M}_\ell).$$

Note that $c_1(\Omega^1_{\mathcal{X}/\mathcal{M}_d}) = \pi^*\lambda$. Indeed, $\pi_*\Omega^1_{\mathcal{X}/\mathcal{M}_d} = 0$, we can identify this with the pull-back of the Hodge line bundle $\mathbb{E} \coloneqq \pi_*\Omega^2_{\mathcal{X}/\mathcal{M}_d}$. Further chern classes like $c_i(R^1\pi_*\Omega_{\mathcal{X}/\mathcal{M}_\ell})$ were shown to all belong to $\mathbb{Q}[\lambda]$ in the previous talk.

It is a result of Maulik [Mau14] that λ is supported on the Noether–lefschetz locus. We have also seen that a result of [BLMM17] states that $\kappa_{3,0}, \kappa_{1,1} \in A^1(\mathcal{M}_\ell, \mathbb{Q}$ are also supported on the NL-divisors.

Aside from these kappa classes, one may also define similar classes using universal line bundles (also unique up-to tensoring by line bundles from $\mathcal{M}_\Lambda$, the Noether Lefschetz locus defined by the Picard lattice $\Lambda \subset \Lambda_\ell$) $\mathcal{H}_1, \cdots \mathcal{H}_r$. Define

$$\kappa_{a_1,\cdots,a_r,b} = \pi_*(c_1(\mathcal{H}_1)^{a_1} \cdots c_1(\mathcal{H}_r)^{a_r} \cdot c_2(\Omega^1_{\mathcal{X}/sM})^b) \in A^{\sum_i a_i + 2b - 2}(\mathcal{M}_\Lambda).$$

Definition 1.2.1. The tautological ring of cycle of the moduli space of K3 surfaces of degree 2ℓ is defined to be the subring

$$R^*(\mathcal{M}_\ell) \subseteq A^*(\mathcal{M}_\ell, \mathbb{Q})$$

generated by the pushforwards from the Noether–Lefschetz loci of all monomials in the κ classes.

By definition $NL^* \subseteq R^*$. It is a conjecture of Marian–Oprea–Pandharipande [MOP17, Conjecture 2] and was later shown by Pandharipande and Yin [PY20] that this is infact an equality.

1.3. **Relations.** We have seen last time that $\lambda^{18} = 0$ [vdGK05]. This time we will use the π relative Quot scheme $\mathcal{Q}^\pi_{H,\chi}(\mathbb{C}^2)$ on $\mathcal{X}$ to find relations in the tautological ring $R^*(\mathcal{M}_\ell)$. The idea is to use π-relative obsruction theory on $p\colon \mathcal{Q}^\pi_{H,\chi}(\mathbb{C}^2) \to \mathcal{M}_\ell$ and consider the push-forward

$$\pi_*\left(\gamma \cdot c_1(\mathcal{O}_{\mathcal{Q}^\pi_{H,\chi}(\mathbb{C}^2)}) \cdot [\mathcal{Q}^\pi_{H,\chi}(\mathbb{C}^2)]^{\mathrm{vir}}\right) \in A^*(\mathcal{M}_\ell, \mathbb{Q}).$$

This class is trivial since $c_1(\mathcal{O}_{\mathcal{Q}^\pi_{H,\chi}(\mathbb{C}^2)}) = 0$. Then the idea is to use the $\mathbb{C}^*$ action on $\mathcal{Q}^\pi_{H,\chi}(\mathbb{C}^2)$ to localise the trivial class above into classes over the fixed locus of the action. Each of this classes composed of kappa classes will then also be trivial, giving rise to non-trivial relations amongst the kappa classes involved. Let F_i denote the irreducible components of the fixed loci of $\mathcal{Q}^\pi_{H,\chi}$ then the virtual localisation formula is given by

$$0 = p_*\left(\gamma \cdot c_1(\mathcal{O}_{\mathcal{Q}^\pi_{H,\chi}(\mathbb{C}^2)}) \cdot [\mathcal{Q}^\pi_{H,\chi}(\mathbb{C}^2)]^{\mathrm{vir}}\right) = \sum_i \pi_*\left(\frac{\widetilde{\gamma}_{F_i} \cdot [F_i]^{\mathrm{red\,vir}}}{e(N_F)}\right)$$

where $[F_i]^{\mathrm{red\,vir}}$ denotes that *reduced virtual class* of F_i, N_F^{vir} is the normal bundle determined by the moving part of the obstruction theory and $e()$ is the *equivariant euler class*. We make these definitions precise in the following section.

Example 1.3.1 (Moduli of K3 of degree 2)**.** The kappa classes in $\mathcal{A}^1(\mathcal{M}_1, \mathbb{Q})$ are given by

$$\kappa_{3,0} \coloneqq \pi_*(c_1(\mathcal{H})^3), \kappa_{1,1} \coloneqq \pi_*(c_1(\mathcal{H}) \cdot c_2(\mathcal{T}_\pi))$$

and the Noether–Lefschetz classes $[\mathcal{S}]$ corresponding to the lattice $\left(\begin{smallmatrix} 2 & 0 \\ 0 & -2 \end{smallmatrix}\right)$, and $[\mathcal{P}]$ corresponding to the lattice $\left(\begin{smallmatrix} 2 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

By O'Grady's result [Gra86], we know that $A^1(\mathcal{M},\mathbb{Q}) = \mathbb{Q}[\mathcal{P}] + \mathbb{Q}[\mathcal{S}]$. The virtual localisation formula gives for instance the following two relations among the four kappa classes in codimension 1:

$$\kappa_{1,1} - 4\kappa_{3,0} + 18\lambda + 12[\mathcal{P}] = 0$$

and

$$\kappa_{1,1} - 4\kappa_{3,0} + \frac{9}{2}\lambda - \frac{24}{5}[\mathcal{P}] - \frac{3}{10} = 0.$$

MOP say that showing that the relations are non-trivial required non-trivial non-vanishing results in the intersection theory of π-relative Hilbert scheme.

2. Preliminaries

2.1. **Equivariant cycles.** Given an algebraic scheme X with a group G acting on it, one can define equivariant algebraic cycles on X. When the action is free, the equivariant Chow groups are precisely $A_*(X/G)$. However in general given $k \in \mathbb{Z}_{\geq 0}$ one finds a contractible algebraic scheme V with G acting freely outside a G-invariant closed subvariety S of codimension at least k and defines

$$A^i_G(X) \coloneqq A^i(X \times (V \setminus S)/G)$$

for $i < k$. A result of Totaro shows that this definition is independent of the choices of V and S. For computational purposed it is sometimes easier to take $V = BG$, the classifying space of G-torsors and consider the $A^k(BG)$-module structure on $A^k_G(X)$. This structure comes from the projection $X \times_G EG \to BG$, which is a fiber bundle with fiber X. Note also that the projection $X \times_G EG \to X$ is not a fiber bundle. Indeed, the fiber over a point is given by the classifying space of the stabiliser at that point. Hence over the fixed locus X^f, the fibers are BG.

Example 2.1.1. When $G = \mathbb{C}^*$ and $X = \text{pt}$, note that given k, V can be taken to be $\mathbb{C}^k$ and S to be the origin. We obtain

$$A^*_G(\text{pt}) = A^*(\mathbb{P}^k) \simeq \mathbb{Q}[t]/t^{k+1} \text{ for } * < k$$

for some variable t. Letting k go to infinity we obtain that

$$A^*_G(\text{pt}, \mathbb{Q}) = \mathbb{Q}[t].$$

In what follows we will exploit the action of t on $A^*_G(X)$ where X is an algebraic scheme with a G-action. This action is induced by the $A^*(B\mathbb{C}^*) \simeq \mathbb{Q}[t]$-module structure on $A^*_G(X)$, coming from the map $X \times EG/BG \to BG$.

2.2. **Localisation Formula.** The main reference for this part is [GP99]. Given a topological space X with a $\mathbb{C}^*$-action, the localisation is a method for reducing certain intersection calculations to the fixed locus. Although our goal is to exploit the opposite; namely, we intend to integrate $c_1(\mathcal{O}_X) = 0$. Clearly over X the answer is easy to compute and is 0. However the equivariant lift of 0 to the fixed loci is non-trivial and therefore gives non-trivial relation over the fixed loci. We will see later why that is the case.

We begin with assuming that X is a manifold and X^f the fixed submanifold of codimension c under the action of $\mathbb{C}^*$, the following composition

$$A_G^{\bullet-c}(X^f) \to A_G^{\bullet}(X) \to A_G^{\bullet}(X^f)$$

is defined by cupping with $e(N_{X^f})$. As we have seen before these are $\mathbb{C}[t]$ modules. Now it can be shown that [1] $e(N_{X^f})$ is invertible in

$$A_G^{\bullet}(X)_t \simeq A_G^{\bullet}(X) \otimes_{\mathbb{Q}[t]} \mathbb{Q}[t, \frac{1}{t}]$$

establishing an isomorphism

$$A_G^{\bullet}(X)_t \xrightarrow{i^*} A^{\bullet}(X^f) \otimes \mathbb{Q}(t)$$

Therefore given a $\mathbb{C}^*$-equivariant cycle class $\sigma \in A^*(X)$, i.e. σ admits a lift $\widetilde{\sigma}$ under the map $A_G^*(X) \to A_G^*(X)$ induced by the inclusion $X \hookrightarrow X \times EG/BG$ as a fibre of the aforementioned projection to BG, we have the formula

$$\widetilde{\sigma} = i_* \frac{i^* \widetilde{\sigma}}{e(N_{X^f})}.$$

Integrating we obtain

$$\int_X \sigma = \int_{X^f} \frac{i^* \widetilde{\sigma}}{e(N_{X^f})}.$$

2.3. **Virtual Localisation.** Given an algebraic scheme X and a perfect obstruction theory on it, namely a tangent data and an obastruction data, one could define a virtual fundamental class associated to X. This data is encoded in $D^b_{\mathrm{coh}}(X)$ via a two term complex

$$E_\bullet \coloneqq E_0 \to E_1$$

where the kernel is the tangent data and the cokernel is the obstruction data. Given a $G = \mathbb{C}^*$-action on X, with induced action on $E_\bullet$, a perfect obstruction theory on X induces a perfect obstruction theory on the scheme theoretic fixed point locus X^f (i.e. the closed subscheme defined by the $\mathbb{C}^*$-eigenfunctions with non-trivial eigenvalues when $X = \operatorname{Spec} A$. If $X^f = \bigcup_i X_i$, then we obtain perfect obstruction theories $E^f_{\bullet,i}$ on X_i by restricting $E_\bullet$ and considering the fixed part under the induced action.

[1]Even in cohomologies this is equivalent to showing $H^k_G(X \setminus X_f)$ is a torsion $\mathbb{Q}[t]$-module. This is done in Atiyah–Bott's original paper [AB84] and involves a bit of argument.

We need the following facts for the purpose of explicit calculations

(1) If X is smooth and $E_\bullet$ is a perfect obstruction theory, then $h^1(E^\bullet)$ is a locally free sheaf and $[X]^{\mathrm{vir}} \simeq e(h^1(E^\bullet))$ [BF97, Prop. 5.6].
(2) The virtual normal bundle N_i^{vir} to $[X_i]^{\mathrm{vir}}$ is given by the moving part $E^m_{\bullet,i}$ of the restriction of $E_\bullet$ to X_i.
(3) the equivariant virtual euler class $e(N_i^{vir})$ defined by $e(E^m_{1,i})/e(E^m_{0,i})$ is invertible in the equivariant Chow ring formally enhanced with the action of $\frac{1}{t}$, i.e. $A^k_G(X^f)_t := A^k(X^f) \otimes_{\mathbb{Q}} \mathbb{Q}[t, \frac{1}{t}]$.
(4) $A^k_G(X^f)_t \simeq A^k(X^f) \otimes \mathbb{Q}[t, \frac{1}{t}]$. Hence, we could describe an equivariant lift of $0 = c_1(\mathcal{O}_{X^f})$ as follows

$$e_G(\mathcal{O}_{X^f})) = e(\mathcal{O}_{X^f} \otimes \mathcal{O}_{\mathbb{P}^\infty}(1))_{t=1} = (c_1(\mathcal{O}_{X^f}) + t)_{t=1} = 1.$$

Example 2.3.1. Let (S, H) be a quasi-polarised K3 surface and let $X = \mathcal{Q}_{H,\chi}$ the Quot scheme parametrising short exact sequences

$$0 \to E \to \mathcal{O}_S^{\oplus 2} \to F \to 0$$

where $\chi(F) = \chi$ and $c_1(F) = [H]$. Note that by Serre duality

$$\mathrm{Ext}^2(E, F) \simeq \mathrm{Hom}^0(F, E \otimes \omega_S)^\vee = 0$$

since F is torsion and $E \otimes \omega_S$ is locally free. This shows that the obstruction theory on $\mathcal{Q}_{H,\chi}$ is determined by $\mathrm{Hom}(E, F)$ and $\mathrm{Ext}^1(E, F)$. Therefore the dimension of the virtual fundamental class of $\mathcal{Q}_{H,\chi}$ is given by

$$\dim \mathrm{Ext}^0(E, F) - \dim \mathrm{Ext}^1(E, F) = 2\chi(\mathcal{O}_S, F) - \chi(F, F) = 2\chi + (0, H, \chi)\cdot(0, H, \chi) = 2\chi + H^2$$

where $(0, H, \chi)$ is the Mukai vector associated to F.

Finally, the composition

$$\mathrm{Ext}^1(E, F) \to \mathrm{Ext}^2(F, F) \xrightarrow{\text{trace}} H^2(S, \mathcal{O}_S) \simeq \mathbb{C}$$

is surjective. Indeed, the trace map is already surjective and we have $\mathrm{Ext}^2(\mathcal{O}_X, F) = H^2(F) = 0$ as F is supported on a curve.

Definition 2.3.2 (Reduced obstruction theory)**.** The tangent data $\mathrm{Ext}^0(E, F)$ and the *reduced* obstruction data $\ker\colon \mathrm{Ext}^1(E, F) \to \mathbb{C}$ also give a perfect obstruction theory and is called as *reduced obstruction theory*.

Example 2.3.3 (Contd.)**.** For $\mathrm{Pic}(S) = \mathbb{Q}\langle H\rangle$ for a primitive class H, we consider the action of $\mathbb{C}^*$ on $\mathcal{Q}_{H,\chi}(\mathbb{C}^2)$ induced by $\mathbb{C}^2 \ni (x, y) \mapsto (x, ty) \in \mathbb{C}^2$. The fixed loci under this action are given by rank 1 sheaves E_i, $i = 1, 2$ mapping to first **or** the second component of $\mathcal{O}_S^{\oplus 2}$. Indeed, a diagonal sheaf invariant under the action described above will span all of $\mathcal{O}_S^{\oplus 2}$. Therefore we obtain a short exact sequence

$$0 \to E_1 \oplus E_2 \to \mathcal{O}_S^{\oplus 2} \to F_1 \oplus F_2 \to 0$$

with $c_1(E_1) + c_1(E_2) = -[H]$.[2] Since $E_1^\vee$ and $E_2^\vee$ are sheaves that admit global section and H is irreducible we obtain

$$E_1 = \mathcal{I}_Z, E_2 = \mathcal{I}_W \otimes \mathcal{O}_S(-H)$$

or the other way round with $Z, W \subset S$ are sets of points, defining the loci

$$F_{z,w} \coloneqq S^{[z]} \times S^{[w]} \times |H|.$$

Note that, Z is contained in the support of F_1 and W is contained in the support of F_2. Denoting the respective lengths by $\ell(Z) = z$ and $\ell(W) = w$, we have $\chi(F) = ch_2(F) + 2\,\mathrm{rk}(F) = \frac{c_1^2(F)}{2} - c_2(F) = \ell - (z + w)$. Hence,

$$z + w = \chi + \ell =: n.$$

Now note that except for the case $(z, w) = (n, 0)$ or $(0, n)$, we know that both $\mathrm{Ext}^1(E_i, F_i)$ for $i = 1, 2$ admit a trivial quotient $\mathbb{C}$ each. This implies that $[F_{z,w}]^{\text{red vir}} = 0$ as it would still contain one trivial quotient globally on $\mathcal{Q}_{H,\chi}(\mathbb{C}^2)$. Therefore, the fixed loci are given by

$$F^+ \simeq F^- \simeq S^{[n]} \times |H|.$$

We have the universal curve given by $\mathcal{C} \hookrightarrow S \times |H|$ such that $\mathcal{O}_{S\times|H|}(-\mathcal{C}) \simeq \mathcal{O}_S(-H) \otimes \mathcal{L}^{-1}$ where $\mathcal{L} \coloneqq \mathcal{O}_{|H|}(1)$. Furthermore we have the associated universal subsheaves on the universal surface $S \times F^+$ given by

$$\mathcal{E}_1 = \mathcal{O}_{S\times F^+} \text{ and } \mathcal{E}_2 = \mathcal{I}_{\mathcal{W}} \otimes \mathcal{O}_{S\times F^+}(\mathcal{C}) = \mathcal{I}_{\mathcal{W}} \otimes H^{-1} \otimes \mathcal{L}^{-1}.$$

In what follows we denote by $\mathcal{O}$ the structure sheaf of $S \times F^+$. The torus fixed part of the reduced obstruction data is given by the difference of the tangent and the obstruction sheaf with one trivial factor taken off. Furthermore $\mathrm{Ext}^\bullet(\mathcal{E}_1, \mathcal{F}_1) = 0$ as $\mathcal{F}_1 = 0$. Hence we need to calculate

$$(\mathrm{Tan} - \mathrm{redObs})^f = \pi_*(\mathcal{E}xt^0(\mathcal{E}_2, \mathcal{F}_2) - \mathcal{E}xt^1(\mathcal{E}_2, \mathcal{F}_2) + \mathcal{E}xt^2(\mathcal{E}_2, \mathcal{F}_2)) + \mathcal{O}_{F^+}$$

under the projection $\pi \colon S \times F^+ \to F^+$. In what follows, we will analyse the fibers of this sheaf at each point of F^+.

As explained in great detail in MOP, on can deduce

$$(\mathrm{Tan} - \mathrm{redObs})^f = \mathcal{T}_{|H|\times X^{[n]}} - \mathcal{L} \otimes \left((H^{-1})^{[n]}\right)^\vee$$

[2] Since we will be taking equivariant lifts soon, it is good to keep in mind that since the action on E_1, F_1 is trivial and the action on E_2, F_2 is by multiplication by t, one should think of E_2, F_2 as $E_2 \otimes \mathcal{O}_{\mathbb{P}^\infty}(1)$ and F_2 as $F_2 \otimes \mathcal{O}_{\mathbb{P}^\infty}(1)$.

Hence,

$$[F^+]^{\mathrm{vir}} = [F^-]^{\mathrm{vir}} = e(\mathcal{L} \otimes \left((H^{-1})^{[n]}\right)^{\vee}.$$

We are now going to calculate the equivariant euler class of the normal bundle, which is given by the moving part of the tangent obstruction data. Following MOP we compute the sheaf fiberwise

$$\begin{aligned}
\mathrm{Ext}^\bullet(\mathcal{O}, \mathcal{F}_2 \otimes \mathcal{O}_{\mathbb{P}^\infty}(1)) + \mathrm{Ext}^\bullet(\mathcal{I}_W \otimes H^{-1} \otimes \mathcal{L}^{-1}, 0)& \\
= H^\bullet(\mathcal{O}_S) - H^\bullet(\mathcal{I}_W \otimes H^{-1}) \otimes \mathcal{L}^{-1}) \otimes \mathcal{O}_{\mathbb{P}^\infty}(1)& \\
= (\mathcal{O} + \mathcal{O} - H^\bullet(H^{-1}) \otimes \mathcal{L}^{-1} + H^\bullet(\mathcal{O}_W \otimes H^{-1}) \otimes \mathcal{L}^{-1}) \otimes \mathcal{O}_{\mathbb{P}^\infty}(1)& \\
= (\mathcal{O} + \mathcal{O} - \mathbb{C}^{\ell+2} \otimes \mathcal{L} + (H^{-1})^{[n]} \otimes \mathcal{L}^{-1}) \otimes \mathcal{O}_{\mathbb{P}^\infty}(1)&
\end{aligned}$$

Therefore, $e(N^+) = \left(\frac{t \cdot t \cdot e((H^{-1})^{[n]} \otimes \mathcal{L}^{-1} \otimes \mathcal{O}_{\mathbb{P}^\infty}(1))}{e(\mathbb{C}^{\ell+2} \otimes \mathcal{L} \otimes \mathcal{O}_{\mathbb{P}^\infty}(1))}\right)_{t=1}$. Given a bundle F and a line bundle $\mathcal{L}$ we have $e(F \otimes \mathcal{L}) = \sum_{i=0}^{\mathrm{rk}(F)} c_i(F) \cdot c_1(\mathcal{L})^{\mathrm{rk}(F)-i}$. We obtain

$$\frac{1}{e(N)} = \frac{(1-\zeta)^{\ell+2}}{c\left((H^{-1})^{[n]} \otimes \mathcal{L}^{-1}\right)}.$$

Remark 2.3.4. Note that when $\mathrm{rk}\,\mathrm{Pic}(S) > 1$ and $[H] = [C_1] + [C_2]$ we have $\chi = z + w + \chi(\mathcal{O}_{C_1}) + \chi(\mathcal{O}_{C_2})$. Since $\frac{H^2}{2} = \frac{C_1^2 + C_2^2}{2} + C_1 \cdot C_2$, we have $\chi + \ell - C_1 \cdot C_2 = z + w$. Hence in the relative setting there are finitely many such Noether–Lefschetz divisors and fixed loci over them.

3. RELATIONS IN THE TAUTOLOGICAL RING

In the relative setting there are many more fixed loci that we need to consider, i.e. the contributions of the fixed loci to the localisation formula are not trivial. This is because in this setting, the Obstruction data $\mathcal{E}xt(\mathcal{E}, \mathcal{F})$ admit a quotient given by the Hodge bundle $\mathbb{E}$ whose euler class is not trivial. The fixed loci are given by

$$\mathcal{X}^{[z]} \times_{\mathcal{M}} \mathcal{X}^{[w]} \times_{\mathcal{M}} \mathbb{P}(\mathbb{V})$$

where we use the notation $\mathbb{V} = \pi_* \mathcal{H}$ with $\mathcal{H}$ the relative qusi-polarisation, and $\pi\colon \mathcal{X} \to \mathcal{M}$ is the universal surface over the moduli space of K3 surfaces of degree 2ℓ.

We write down the final formulæ over the generic K3 suface $(S, H) \in \mathcal{M}_1$. On the fixed loci $\mathcal{F}^{z,w} := \mathcal{X}^{[z]} \times_{\mathcal{M}} \mathcal{X}^{[w]} \times \mathbb{P}(\mathbb{V})$ the reduced obstruction and the reduced virtual class are determined by the negative part of the

$$(\tan - \mathrm{red\,obs})^f := \mathrm{Ext}^\bullet(\mathcal{E}_1, \mathcal{F}_1) + \mathrm{Ext}^\bullet(\mathcal{E}_2, \mathcal{F}_2) + \mathbb{C}$$

just as before[3]. The final answer is

$$[\mathcal{F}^{z,w}]^{\text{red vir}} = e(\mathbb{E}^\vee + \mathbb{E}^\vee \otimes \left(\mathcal{O}^{[z]}\right)^\vee + \mathbb{E}^\vee \otimes \mathcal{L} \otimes \left((\mathcal{H}^{-1})^{[w]}\right)^\vee)$$

where $\mathcal{L} = \mathcal{O}_{\mathbb{P}(\mathbb{V})}(1)$. When $(z,w) = (n,0)$, we have as before

$$[F^+]^{\text{red vir}} = e_G(\mathbb{E} \otimes \mathcal{L}^{-1} \otimes (\mathcal{H}^{-1})^{[n]})$$

and

$$N^+ = (\mathbb{C} + \mathbb{E}^{-1} + \mathcal{L}^{-1} \otimes (\mathcal{H}^{-1})^{[n]} - \mathcal{L}^{-1} \otimes \mathbb{V}^\vee \otimes \mathbb{E}^{-1})[1] \text{ and } \frac{1}{e_G(N^+)} = \frac{c_-(\mathcal{L} \otimes \mathbb{V} \otimes \mathbb{E})}{(1-\lambda) \cdot c_+(\mathcal{L}^{-1} \otimes (\mathcal{H}^{-1})^{[n]})}$$

Example 3.0.1 (degree 2, $\chi = 0$)**.** In this case, as already noted above the kappa classes in $\mathcal{A}^1(\mathcal{M}_1, \mathbb{Q})$ are given by

$$\kappa_{3,0} \coloneqq \pi_*(c_1(\mathcal{H})^3), \kappa_{1,1} \coloneqq \pi_*(c_1(\mathcal{H}) \cdot c_2(\mathcal{T}_\pi))$$

and the Noether–Lefschetz classes $[\mathcal{S}]$ corresponding to the lattice $\left(\begin{smallmatrix} 2 & 0 \\ 0 & -2 \end{smallmatrix}\right)$, and $[\mathcal{P}]$ corresponding to the lattice $\left(\begin{smallmatrix} 2 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Since $n = 1$, the fixed loci over the generic K3 looks like $\mathcal{X} \times \mathbb{P}(\mathbb{V})$ and there are two of these. The total contribution is given by

$$2\kappa_{3,0} - 4c_1(\mathbb{V}).$$

We calculate $c_1(\mathbb{V})$ using the Riemann–Roch theorem and that $c_1(\mathcal{T}_\pi) = -\pi^*\lambda$

$$\begin{aligned} c_1(\mathbb{V}) &= \text{degree 1 piece of } \pi_*\big((1 + c_1(\mathcal{H}) + \frac{c_1(\mathcal{H})^2}{2} + \frac{c_1(\mathcal{H})^3}{6} + \cdots) \\ &\qquad \cdot (1 - \frac{1}{2}\pi^*\lambda + \frac{\pi^*\lambda^2 + c_2(\mathcal{T}_\pi)}{12} - \frac{\pi^*\lambda \cdot c_2(\mathcal{T}_\pi)}{24} + \cdots)\big) \\ &= -\frac{1}{2}\lambda + \frac{\kappa_{3,0}}{6} + \frac{\kappa_{1,1}}{12} - \lambda\frac{\pi_* c_2(\mathcal{T}_\pi)}{24} \\ &= -\frac{3}{2}\lambda + \frac{\kappa_{3,0}}{6} + \frac{\kappa_{1,1}}{12} \end{aligned}$$

The contribution from the NL divisors. In case of $[\mathcal{P}]$, one has the classes of effective curves $[C_1]$ and $[C_2] = [H - C_1]$ with

$$C_1 \cdot C_2 = 1, \quad C_1^2 = 0 = C_2^2.$$

Here $z + w = n - C_1 \cdot C_2 = 0$ hence the fixed locus is given by two identical copies of $F = |C_1| \times |C_2| = \mathbb{P}^1 \times \mathbb{P}^1$.

The fibers of the reduced obstruction sheaf is then given by

$$H^0(C_1, \mathcal{O}_{C_1}(C_1)) \otimes \mathbb{E}^\vee + H^0(C_2, \mathcal{O}_{C_2}(C_2) \otimes \mathbb{E}^\vee - p^*\mathbb{E}^\vee.$$

[3] As pointed out by Georg to me the moduli $\mathcal{M}_\ell$ of quasi-polarised K3s may not be separated and hence $\mathcal{X}^{[z]}$ the relative Hilbert scheme of points on the universal surface $\mathcal{X}$ may not be well-behaved. Nonetheless the Behrend–Fantechi result (see Remark 1) seem to apply if I understand [?] correctly.

Hence $[F]^{\text{red vir}} = e(\mathbb{E}^\vee)$. Furthermore, [Gra86, Lemma 1.2] states that if we have a family of K3 $p\colon \mathcal{X} \to F$ such that $\mu(F) \subseteq \mathcal{P}$ under the moduli map μ and if the generic K3 does not have any automorphism then $c_1(\mu^*[\mathcal{P}]) = \mu^* e(\mathbb{E}^\vee)$. Hence we have $[F]^{\text{red vir}} = q^* c_1([\mathcal{P}])$ where $q\colon \mathcal{Q}_{H,0}(\mathbb{C}^2) \to \mathcal{M}_1$.

Now the normal bundles of the fixed loci are determined by the relative line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ corresponding to C_1 and C_2. On a fixed surface the piece coming from $\mathcal{E}_1 = \mathcal{O}_{C_1}$ and $\mathcal{F}_2 = \mathcal{O}_{C_2}$ is given by

$$H^0(\mathcal{O}_{C_2}(C_1)) - H^1(\mathcal{O}_{C_2}(C_1)) + H^2(\mathcal{O}_{C_2}(C_1)) = H^\bullet(\mathcal{O}_F(C_1)) - H^\bullet(\mathcal{O}_F(C_1 - C_2))$$

Hence the total contribution toward normal bundle of F of this piece is given by $N_1 = (\mathbb{C}^{\oplus \chi(\mathcal{O}_S(C_1))} \otimes \mathcal{L}_1 - \mathbb{C}^{\oplus \chi(\mathcal{O}_S(C_1 - C_2))} \otimes \mathcal{L}_1 \otimes \mathcal{L}_2^{-1})) \otimes \mathcal{O}_{\mathbb{P}^\infty}(1)$. Similarly computing the contribution from $\mathcal{E}_2$ and $\mathcal{F}_1$ we obtain

$$N = \mathbb{C}^2 \otimes \mathcal{L}_1[1] + \mathbb{C}^2 \otimes \mathcal{L}_2[-1] - \mathcal{L}_1 \otimes \mathcal{L}_2^{-1}[1] - \mathcal{L}_2 \otimes \mathcal{L}_1^{-1}[-1]$$

where $[\pm 1]$ denotes twist by $\mathcal{O}_{\mathbb{P}^\infty}(\pm 1)$. We now calculate the contribution of this fixed locus to the Localisation formula.

$$\begin{aligned}
q_*\left(\frac{q^*([\mathcal{P}])}{e_G(N)}\right) &= \left(\int_{\mathbb{P}^1 \times \mathbb{P}^1} \frac{1}{e_G(N)}\right) \cdot [\mathcal{P}] \\
&= \int_{\mathbb{P}^1 \times \mathbb{P}^1} \frac{(1 + \zeta_1 - \zeta_2) \cdot (-1 - \zeta_1 + \zeta_2)}{(1 + \zeta_1)^2 \cdot (-1 + \zeta_2)^2} \cdot [\mathcal{P}] \\
&= -(\int_{\mathbb{P}^1} \frac{1}{(1 - \zeta_2)^2} + \int_{\mathbb{P}^1} \frac{1}{(1 + \zeta_1)^2} \int_{\mathbb{P}^1} \frac{\zeta_2^2}{(1 - \zeta_2)^2}) \cdot [\mathcal{P}] \\
&= -\int_{\mathbb{P}^1} 2\zeta_2 \cdot [\mathcal{P}] \\
&= -2[\mathcal{P}]
\end{aligned}$$

Finally on $[\mathcal{S}]$, we have $C_1^2 = -2$, $[C_2] = [H - C_1]$ with $C_2^2 = 0$ and $C_1 \cdot C_2 = 2$ we obtain $z + w = n - C_1 \cdot C_2$. So $[\mathcal{S}]$ does not contribute to the localisation formula.

Putting everything together we obtain

$$2\kappa_{3,0} - c_1(\mathbb{V}) - 4[\mathcal{P}] = -\kappa_{1,1} + 4\kappa_{3,0} - 18\lambda - 12[\mathcal{P}] = 0.$$

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