University Bonn — WS 2017/18	Ursula Hamenstädt
Advanced Geometry I	
Exercises 7	21. November 2017

7.1. Recall that two Riemannian manifolds $(M,g),(N,h)$ are isometric if there is a diffeomorphism $\Phi : M \to N$ so that $\Phi^* h = g$. Show that for any Kähler manifold (M,g), there are uncountably many Kähler manifolds which are diffeomorphic but not isometric to M.
Hint:
1) Consider first the case of the standard Hermitian inner product h on $\mathbb{C}^n$.
2) Try to remember how one can locally change a Kähler metric.
3) Observe that the diffeomorphism group of a manifold acts on the space of metrics.
4) (Harder- for this it is easiest to use some standard Riemannian geometry that you may or may not know- if you don't know enough geometry, don't try this part) Try to find invariants of isometric metrics that can locally be modified.

7.2. A line bundle $L \to M$ on a complex manifold is *positive* if it admits a Hermitian metric with positive curvature form. Let M be a closed (compact without boundary) complex manifold which admits a positive line bundle. Let $E \to M$ be a holomorphic vector bundle of rank at least two. Show that the bundle $N \to M$ whose fibre over x equals the projectivization of the fibre of E over x admits a positive line bundle.

7.3. Let M be a manifold which admits a positive line bundle. Show that the blowup of M at any finite number of points admits a positive line bundle.

7.4. Let M be any n-dimensional complex manifold. The *canonical bundle* K_M of M is the is the n-th exterior product of the holomorphic cotangent bundle of M.

(a) Show that the canonical bundle of M is a holomorphic line bundle. Its curvature defines a class in $H^2_{dR}(M,\mathbb{R})$.

(b) Construct a compact Kähler manifold such that the class in $H^2_{dR}(M,\mathbb{R})$ defined by the K'ahler form is not any multiple of the class defined by the canonical bundle.

7.5. (This was left over from the previous exercise sheet) The canonical embedding $\mathbb{C}^n \to \mathbb{C}^{n+1}$ induces an embedding $\mathbb{C}P^{n-1} \to \mathbb{C}P^n$. Show that the restriction to $\mathbb{C}P^{n-1}$ of the Fubini Study metric on $\mathbb{C}P^n$ is the Fubini Study metric on $\mathbb{C}P^{n-1}$.